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Conjugate Gradient Method vs Gaussian Elimination for solving linear system in the first order Polarization Tensor

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Abstract

This paper highlights the computation of the first order polarization tensor (PT) which includes the linear element integration coupled with the conjugate gradient method (CG method) in order to solve system of equations for the first PT. Previously, Gaussian Elim-ination (GE) method was used which resort to lower upper decomposition. The iterative technique of CG Method in this paper is more stable with user defined stopping criteria in order to solve system which tends to singular form. By implementing this iterative tech-nique, the result of this polarization tensor is more accurate compared to GE method. In this paper, the first order PT for sphere is computed which represent the most basic geometry. The results obtained with CGM were validated with published results. Here, we used MATLAB with Netgen Mesh Generator is used in order to generate the surface element of this basic geometry for surface integration.

Keywords : Numerical Integration; Matlab; Netgen Mesh Generator; Singular Matrix; System of equations.

Introduction

Generalized Polarization Tensor (GPT) is a building blocks of an infinitely numbers of a sequence of tensor where, the first order Polarization Tensor (PT) is the lowest order of the building blocks of GPT (Ammari et al., 2005; Khairuddin and Lionheart, 2016; Ammari et al., 2018; Ammari et al., 2014). The concept of GPT was firstly arised from Pólya-Szegö's studies of hydrodynamics and potential theory where, the author firstly introduced this polarization tensor as the virtual mass. The virtual mass is velocity dependent and zero in its conductivity (Pólya, 1947; Schiffer and Szego, 1949).

In this paper, we are going to implement linear element numerical integration tech-nique in conjunction with Conjugate Gradient Method (CG Method) instead of using Gauss Elimination Method (GE Method) that has been used by researchers in (Khairuddin and Lionheart, 2013) for computing the first order PT. Our purpose of this study is to actually increase our understanding about PT as well as to show that CG Method can be used to accelerate the convergence of computing the first order PT.

The background concept of polarization tensor will be firstly stated in the first section of the paper. Next, the comparison between the numerical results of first order PT using

both methods will be presented using graphical representation. These result is developed by using MATLAB software.

Background of the Polarization Tensor

The study of polarization tensor has been discussed by many researcher for example in (Ammari and Kang, 2007; Yunos et al., 2017; Khairuddin and Lionheart, 2014. Suppose that we have a Lipschitz bounded domain B where its origin is inside the domain, $O \in B$. H is the harmonic function in $\Re^3$

$$\begin{cases} \nabla \cdot ((1+(k-1)\chi(B)\nabla(u)) = 0 \text{ in } \Re^3 \\ u(x) - H(x) = O(1/|x|^2) \text{ as } |x| \to +\infty \end{cases} \tag{1}$$

where u is the solution of the problem and is defined as far field expansion as below

$$(u-H)(x) = \sum_{|i|,|j|=1}^{+\infty} \frac{(-1)^{|i|}}{i!j!}\, \partial_x^i\, \Gamma(x)\, M_{ij}(k,B)\, \partial^j\, H(0) \text{ as } |x| \to +\infty \tag{2}$$

or equation (2) can also be written as

$$(u-H)(x) = \sum_{|i|,|j|=1}^{+\infty} \frac{(-1)^{|i|}}{i!j!} \partial_x^i \Gamma(x) \partial^j H(0) \int_{\partial B} (\lambda I - K_B^*)^{-1}(v_x \cdot \nabla x^i)(y) y^j d\sigma(y) \tag{3}$$

as $|x| \to +\infty$.

$i = (i_1, i_2, i_3)$ and $j = (j_1, j_2, j_3)$ is multi-indices while k is the conductivity where $0 < k \neq 1 < +\infty$. When $k = 0$, this polarization tensor is known as virtual mass. Γ is the fundamental solution of the Laplacian while $M_{ij}(k,B)$ is the Generalized Polarization Tensor (GPT). The Laplacian,Γ, is defined as below

$$\Gamma(x) = \begin{cases} \dfrac{1}{2\pi} \ln|x| & d = 2, \\ \dfrac{1}{(2-d)w_d}|x|^{2-d} & d \geq 3; \end{cases} \tag{4}$$

while

$$M_{ij} = \int_{\partial B} y^j \phi_i(y)\, d\sigma(y). \tag{5}$$

Here, $\phi_i(y)$ is given by

$$\phi_i(y) = (\lambda I - K_B^*)^{-1}(v_x \cdot \nabla x^i)(y), \quad y \in \partial B \tag{6}$$

where, $\lambda = (k+1)/2(k-1)$. v_x is the outer unit normal vector to the boundary ∂B at x and

K_B^* is the singular integral operator where it can be defined as

$$K_B^*\phi(x) = \frac{1}{4\pi} P.V. \int_{\partial B} \frac{\langle x-y, v_x \rangle}{|x-y|^3} \phi(y) d\sigma(y). \tag{7}$$

In the next part of this paper, the theoretical computation in order to obtain first order PT will be explained.

First Order PT

Sphere Triangularization using Netgen Mesh Generator

Netgen Mesh Generator was used during this study in order to aid the computation process of first order PT. Different radius will generate different meshes since the software generate the meshes automatically.

By using different meshing option, which are, `coarse`, `moderate`, `fine`, `very fine` and `user defined`, the meshes of the sphere with different radius will be generated. For sphere with radius 0.01, the generated meshes of $44, 72, 118, 230, 620$ and 1888 triangles as shown in Figure 1.

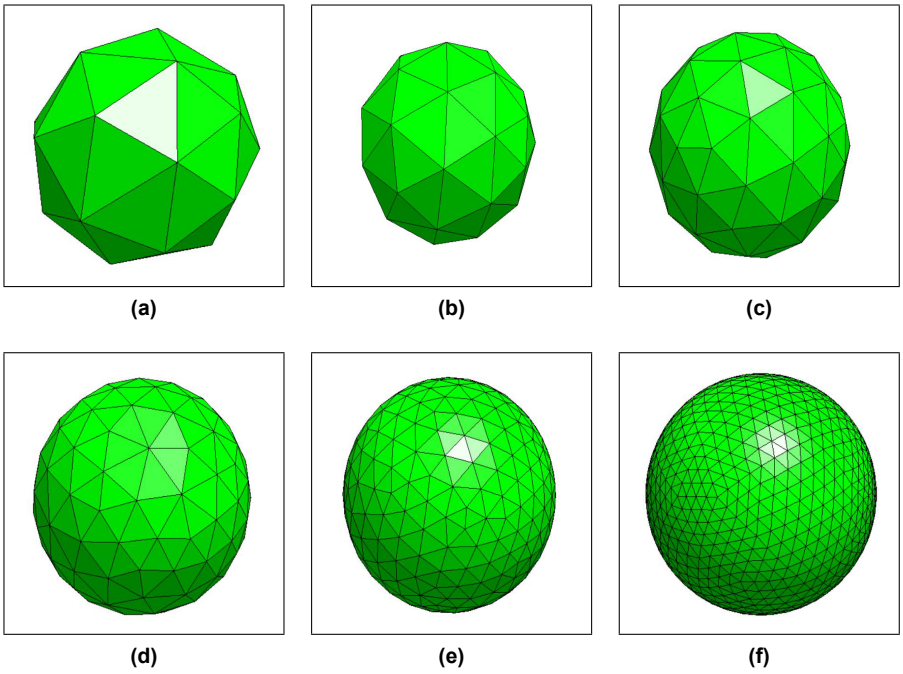


Figure 1: Mesh generated from Netgen Mesh Generator Software (a) 44 triangles (b) 72 triangles (c) 118 triangles (d) 230 triangles (e) 620 triangles (f) 1888 triangles

Since Netgen Mesh Generator will provide the information about the number of nodes

and the value of each points of x, y and z, we then proceed to use this information and called this information in MATLAB software. From this, the first order PT computed using the method will be explained in the next section.

Analytical Formula of First Order PT for Sphere

In (Ammari and Kang, 2007), the researcher had already derived the analytical formula of first order PT at conductivity k. This analytical formula that has been derived by (Ammari and Kang, 2007) can be used to compute the exact solution for sphere.

The representation of the analytical formula is

$$M(k,B) = (k-1)|B| \begin{bmatrix} \frac{3}{2+k} & 0 & 0 \\ 0 & \frac{3}{2+k} & 0 \\ 0 & 0 & \frac{3}{2+k} \end{bmatrix}. \tag{8}$$

Here, $|B|$ is the volume of the sphere, $|B| = 4/3\pi r^3$ while k is the conductivity. From this analytical formula, we will going to compute the relative error for both methods that we going to use.

Calculating K_B^* for Linear Element Integration

From equation (7) $P.V.$ is the Cauchy Principle Value. By using linear element integration as mentioned in (Khairuddin and Lionheart, 2013), the singular integral operator of K_B^* will then become,

$$K_B^*\phi(x) = \frac{1}{4\pi}\left(w_t \frac{\langle x_s - y_t, v_{x_s}\rangle}{|x_s - y_t|^3}\right)\phi(y_t), \tag{9}$$

where $s = 1, \cdots, N$ and $t = 1, \cdots, N$ and N is the number of meshes. Then, equation (9) can be expressed as in matrix form is

$$K_B^*\phi(x) = \frac{1}{4\pi}\begin{bmatrix} w_1K(x_1,y_1) & \cdots & w_NK(x_1,y_N) \\ \vdots & \ddots & \vdots \\ w_1K(x_N,y_1) & \cdots & w_NK(x_N,y_N) \end{bmatrix}. \tag{10}$$

In the next section, we will provide the theory on method of CG Method in order to calculate $\phi(y)$ for first order of PT.

Calculating $\phi(y)$ using CG Method

The CG Method can be used to solve linear system as in equation (6). It was firstly proposed in 1952 by Hestenes and Steifel in their study in (Hestenes and Stiefel, 1952). Hence in this paper, we intend to implement the CG Method to calculate first order PT . Given the

solution to the system of equations to calculate first order PT as below

$$\phi_i(y) = (\lambda I - K_B^*)^{-1}(v_{xt}), \tag{11}$$

where the equation in (11) can also be written as

$$(\lambda I - K_B^*)\ \phi_i(y) = v_{xt}. \tag{12}$$

CG Method can be used to compute system having a symmetric and positive definite system where we can check whether the equation of system is positive definite system or negative definite using the equation

$$\phi(y)^{\mathsf{T}}\ (\lambda I - K_B^*)\ \phi(y) > 0. \tag{13}$$

This iterative method will terminates the calculating process as the stopping criteria has been achieved. Throughout this study, we set the stopping criteria equal to 0.01. Since we are using the iterative method instead of direct method, hence, it is not required for us to state the value of the initial guess for the solution $\phi(y)$. Here, we denote the initial guess of $\phi(y)$ as ϕ_m. Firstly, we let,

$$p = r = v_{xt} - (\lambda I - K_B^*)\ \phi_i(y). \tag{14}$$

Based on equation (15), ϕ_{m+1} is the end result of $\phi(y)$ that we ought to obtain while ϕ_m is the initial term before the first iteration start.

$$\phi_{m+1} = \phi_m + (\alpha_m)(p_m), \tag{15}$$

where

$$\alpha_m = \frac{(r_m^{\mathsf{T}})(r_m)}{(p_m^{\mathsf{T}})(\lambda I - K_B^*)(p_m)}.$$

Then, we calculate the new r by using equation (16) as below

$$\begin{aligned} r_{m+1} &= v_x - (\lambda I - K_B^*)\phi_{m+1}, \\ &= r_m - \alpha_m(\lambda I - K_B^*)p_m. \end{aligned} \tag{16}$$

We will stop the iteration as $||r_{m+1}||_2/||r_m||_2 < 0.01$. We will continue the calculation if the condition does not satisfy by calculating new p.

$$p_{m+1} = r_{m+1} + D_m p_m, \tag{17}$$

where

$$D_m = \frac{(r_{m+1}^T)(r_{m+1})}{(r_m^T)(r_m)}.$$

Next, the results of error for both GE Method and CG Method are presented in graphical representation.

Results and Discussion

We start our computation of first order polarization tensor for sphere geometry with conductivity $k = 100$. Then we increased it to $k = 500, k = 1000, k = 5000,$ and lastly for $k = 10000$. The computation are based from the system of equation $\phi(y)$ using the method mentioned in Section .

From the results obtained from CG Method, we compared it with GE Method by finding its error. Here, we ought to obtain the relative error which is less than 0.05. Throughout this study, radius of the sphere is set be equal to $r = 0.01, 0.1$ and 1 .

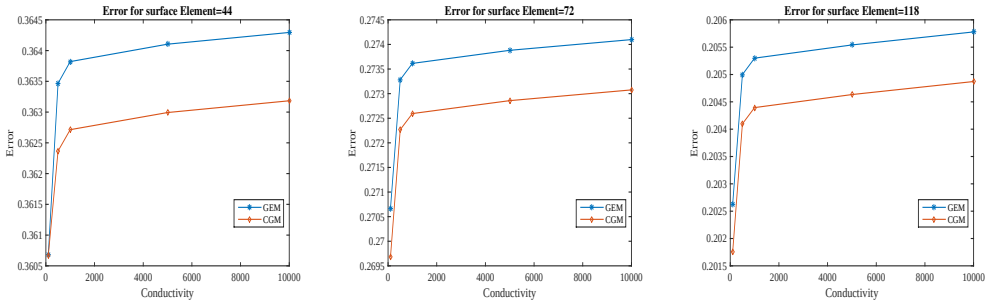
According to (Ammari and Kang, 2007), as $k \to +\infty$, the system of equation of $\lambda I - K_B^*$ will lead to singularity. This is true since as we can see from Figure 2, Figure 3 and Figure 4, as we increased the conductivity starting from $k = 100$ up until $k = 10000$, the error will be increased despite of its radius. It can be observe that, as the number of surface elements N increased, the results for both methods will eventually tend to zero.

Despite that, it can seen that for CG Method, the error for first order polarization tensor resulted to small relative error. As we used CG Method, the relative error of first order PT showed faster convergence instead of using direct method which is GE Method. It is adequate enough to use the number of iteration $N = 1$ since the error between those two methods show differences. The accuracy can also be increased by decreased the stopping criteria of $||r_{m+1}||_2/||r_m||_2$ where in this study, we set the stopping criteria to 0.01.

Since we used Netgen Mesh Generator to generate the surface elements N, therefore as mentioned in Section , we cannot simply set the number of surface element. It will automatically generate the number of surface element itself.

Hence, for sphere of radius 0.01, we used surface elements shown in Figure 1. We can observed that, the smallest error for GE Method is at surface element of 1888 triangles where the relative error is approximately 0.0414 while as we used CG Method, the error is 0.0412. The difference between those two errors is 0.002. It is sufficient to generate the meshes and do the computation up until 1888 triangles since the relative error for both methods is already less than 0.05.

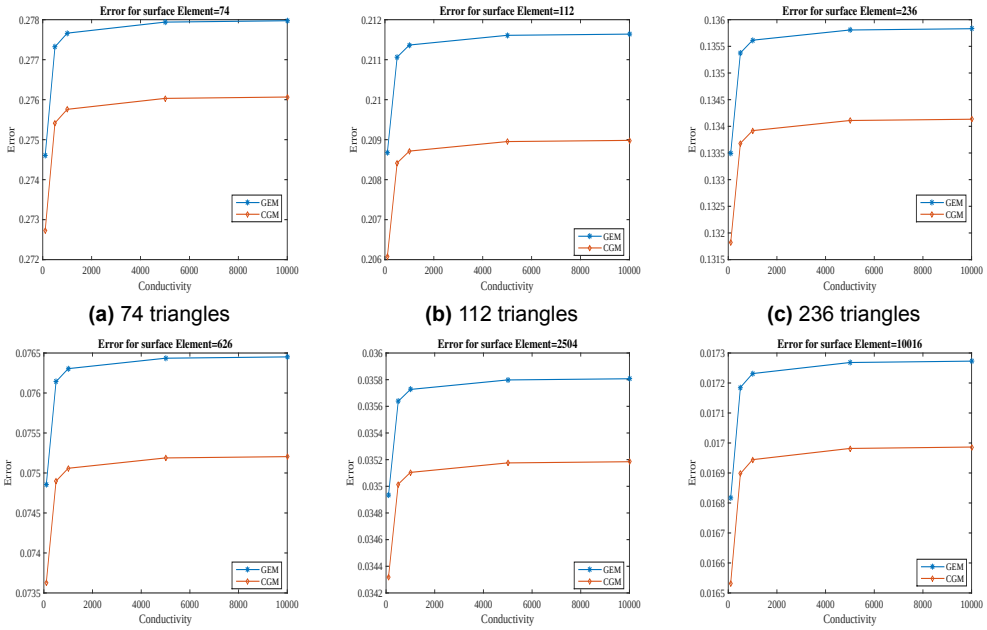
As shown in Figure 3, for sphere of radius 0.1 we use surface element of 74, 112, 236, 626, 2504 and 10016 triangles with same conductivity as sphere with radius 0.01.



(a) 44 triangles **(b)** 72 triangles **(c)** 118 triangles

(d) 230 triangles **(e)** 620 **(f)** 1888

Figure 2: Error comparison for first order PT for sphere with radius 0.01 using GE Method and CG Method.



(a) 74 triangles **(b)** 112 triangles **(c)** 236 triangles

(d) 626 triangles **(e)** 2504 triangles **(f)** 10016 triangles

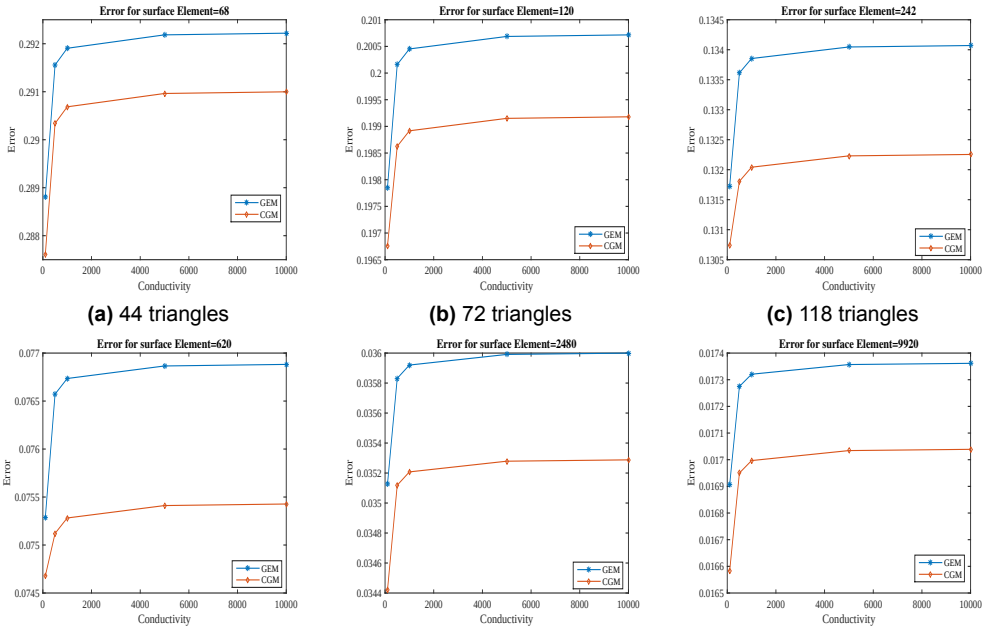
Figure 3: Error comparison for first order PT for sphere with radius 0.1 using GE Method and CG Method

For sphere of radius 1, the smallest relative error is approximately 0.0166 using CG Method with 9920 triangles as shown in Figure 4.

Conclusion and Recommendations

In this paper, we have proposed to implement the method of Conjugate Gradient in order to solve the linear system of first order polarization tensor. We show that as CG Method is implemented, the graph of error depicts that, faster convergence can be achieved rather than using direct method of Gauss Elimination Method. Not even that, as the number of meshes of the sphere were increased, it will improves the convergences of the error graph. Different conductivity, k were used in order to show that, as we increased the conductivity, the result of first order PT will be unbounded.

Furthermore, for future research, we suggest the researcher to use this method in order to deal with real life problem such as humanitarian de mining. Since PT can be used to indicate the shape of objects and can be used in humanitarian mining as in (Dekdouk et al., 2015), hence, it is crucial to have an accurate results of PT. It is too costly to remove land mines in specific area, therefore, accurate results of PT can be used in order to reduce the cost in the de mining process. We also would like to suggest that, instead of using linear element numerical integration, researchers can use the quadratic element numerical integration and coupled with method as mentioned above.



(a) 44 triangles **(b)** 72 triangles **(c)** 118 triangles

(d) 230 triangles **(e)** 620 triangles **(f)** 1888 triangles

Figure 4: Error comparison for first order PT for sphere with radius 1 using GE Method and CG Method

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