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Numerical Simulation of Burgers Equation Using Explicit and Implicit Schemes of Classical and Exponential Finite Difference Methods

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Abstract
This paper solves one-dimensional nonlinear partial differential equation (PDE), the Burgers equation using classical and exponential finite difference methods (CFDM and EFDM) in both explicit and implicit schemes. The accuracy of numerical methods is evaluated by comparing numerical approximations with analytical solutions computed using the method of separation of variables (SOV). Hopf-Cole transformation is first implemented to linearize the problem to solve it using analytical and explicit numerical methods. The implicit numerical schemes require an iterative procedure to solve the system of nonlinear equations. Overall, both CFDM and EFDM show good accuracy in approximating the exact solution of Burgers equation.
Keywords Burgers equation; classical finite difference method; exponential finite difference method

1 Introduction

The partial differential equation (PDE) represents the relationship between the multivariable function and its partial derivatives. It has a vital role in science and engineering. Since it often takes a considerable effort to solve nonlinear PDE analytically to obtain its exact solutions, the numerical approach plays a crucial role in approximating solutions of nonlinear PDE with less effort in computation cost and time. Burgers equation is a notable nonlinear PDE contributing to the study in different fields such as turbulence, traffic flow, heat conduction, acoustics, etc. Burgers equation can be considered a hyperbolic or parabolic problem with nonlinear propagation effects and diffusive effects, depending on its diffusion coefficients [1]. It is chosen in this study as it is a fundamental problem in various fields. Besides, it can be solved using analytical and numerical methods, making it easier to evaluate the accuracy of the numerical methods.

Classical and exponential finite difference methods (CFDM and EFDM) are applied in the paper to approximate exact solutions of the Burgers equation. The finite difference method (FDM) is a well-known numerical approach in solving ordinary and partial differential equations. Bhattacharya [2] introduced EFDM to solve the one-dimensional unsteady state heat conduction problem. FDM has explicit and implicit schemes that differ in terms of computation effort and stability. The explicit technique can be applied to linear problems but is only conditionally stable.

The implicit scheme can be applied directly to nonlinear problems and is numerically stable [3]. Therefore, Hopf-Cole transformation is used to linearize the Burgers equation before applying the explicit scheme. Linearization is also required to solve the problem analytically using the method of separation of variables (SOV). Implicitly solving nonlinear problems will lead to a system of equations that need to be solved using an iterative procedure, in this case, the Newton's method. This paper aims to obtain numerical approximations of one-dimensional nonlinear Burgers equation using CFDM and EFDM in both explicit and implicit schemes to evaluate their accuracy and compare their performance. The performance of CFDM and EFDM is assessed by generating the error norms of the methods.

2 Literature Review

2.1 Burgers Equation and Its Previous Work

The equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2}$$

is first proposed by Bateman [4] to represent fluid motion which is approaching the inviscid limit. u, v, x, t refer to fluid velocity, kinematic viscosity, spatial coordinate, and temporal coordinate, respectively. Burgers [5] later analyzed the model for turbulent flow; the equation is then known as Burgers equation to appreciate his contribution. When ν is nonzero, the equation is known as the viscous Burgers equation, which generally gives a smooth solution.

Independent studies by Hopf and Cole proposed the transformation of the nonlinear Burgers equation into a linear heat equation. Hopf [6] evaluated the one-dimensional Burgers equation by introducing a new dependent variable to the equation, whereas Cole [7] studied the linear transformation of Burgers equation with given initial conditions in the shock wave and turbulence theory. The technique is then known as Hopf-Cole transformation, as expressed below.

$$u(x,t)=-2\nu\frac{\theta_X}{\theta}$$

Burgers equation can then be solved analytically as a linear problem to obtain its exact solution. SOV is an analytical approach to solve the Burgers equation. The method is used in problems with constant coefficients, which have a finite domain [8].

Many numerical techniques have been used to solve the Burgers equation [9-11]. The well-known finite difference scheme is also applied to approximate solutions of Burgers equation. Exact-explicit FDM is proposed to solve the one-dimensional Burgers equation and proved its good accuracy under refined grid size [12]. Besides, Yokus and Kaya [13] highlighted that FDM is stable and can solve the time-fractional Burgers equation.

2.2 Classical Finite Difference Method and Exponential Finite Difference Method

Euler first implemented CFDM around 1768 to solve differential equations by numerical approximations [14]. There are two finite difference schemes used in solving heat equation, namely Forward Time, Centered Space (FTCS) approximation and Backward Time, Centered Space (BTCS) approximation [15]. FTCS is explicit by applying the forward difference to the time derivative, while BTCS is implicit by using backward time difference to the time derivative. Since the explicit scheme is only conditionally stable, the Von Neumann stability analysis can be applied to ensure the stability of the FTCS scheme [16]. The implicit finite difference scheme requires an iterative approach to solve nonlinear systems of equations. Newton's method is an iterative procedure that can be applied to obtain converging approximations [17]. CFDM has been used in solving many differential equation problems, and it can be further modified to solve problems with different properties [18-20].

Bhattacharya [2] presented EFDM as an explicit and conditionally stable finite difference approach in solving one-dimensional unsteady heat conduction problems in cartesian coordinates. The method is then extended to various linear and nonlinear problems [21]. Other than the general finite difference schemes mentioned earlier, EFDM implements a discrete operator in its algorithm. The method is used to solve the one-dimensional Helmholtz equation for the electromagnetic response of layered earth [22]. Explicit EFDM is used to solve the Burgers-Huxley equation [23-24]. EFDM in implicit and fully implicit schemes are applied to solve the Newell-Whitehead-Segel equation to test their consistency and convergence [25]. In general, satisfactory results were obtained using EFDM in terms of accuracy, consistency, and convergence.

3 Methodology

3.1 Formulation of Burgers Equation and its Exact Solutions

In this paper, Burgers equation with the following initial and boundary conditions is considered

$$\begin{aligned} &\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}=\nu\frac{\partial^2 U}{\partial x^2}, && \nu>0 \\ &U(x,0)=\sin(\pi x), && 0<x<1 \\ &U(0,t)=U(1,t)=0, && t>0 \end{aligned} \tag{1}$$

where $U(x,t)$, ν, t and x represent one-dimensional fluid speed, kinematic viscosity, temporal coordinate, and spatial coordinate, respectively. The domain is discretized into x_i and t_j for $i=1, 2, 3, \dots, N$, $j=1, 2, 3, \dots, M$ where $x_{i+1}=x_i+\Delta x$ and $t_{j+1}=t_j+\Delta t$. Δx and Δt are the space step and time step. Different coefficients of ν are applied in the study to test the performance of numerical methods. The physical behavior of the speed of fluid will be affected by changing the value of ν. In this study, three values of ν at 1.0, 0.1 and 0.01 are considered.

Problem (1) is linearized into a linear heat equation with Neumann boundary conditions shown in (3) using Hopf-Cole transformation in (2) to apply explicit numerical methods and SOV.

$$U=-2\nu\left(\frac{\varphi_x}{\varphi}\right) \tag{2}$$

$$\begin{aligned} &\frac{\partial\varphi}{\partial t}=\nu\frac{\partial^2\varphi}{\partial x^2} \\ &\varphi(x,0)=\exp\left(\frac{1}{2\pi\nu}(\cos\pi x-1)\right) \\ &\frac{\partial\varphi(0,t)}{\partial x}=\frac{\partial\varphi(1,t)}{\partial x}=0 \end{aligned} \tag{3}$$

Using SOV, the exact solutions for $U(x,t)$ is given by

$$U_n(x,t)=2\nu\pi\frac{\sum_{n=1}^{\infty}D_n n\sin(n\pi x)\exp(-n^2\pi^2\nu t)}{D_0+\sum_{n=1}^{\infty}D_n\cos(n\pi x)\exp(-n^2\pi^2\nu t)},$$

where $D_0=\int_0^1\varphi(x,0)\,dx$ and $D_n=2\int_0^1\varphi(x,0)\cos(n\pi x)\,dx$.

3.2 Numerical Methods

3.2.1 Classical Finite Difference Method (CFDM)

Explicit CFDM (e-CFDM) is applied to problem (3). The equation to solve φ is expressed as

$$\varphi_i^{j+1} = r\varphi_{i+1}^j + (1-2r)\varphi_i^j + r\varphi_{i-1}^j , \tag{4}$$

where $r = \frac{\nu \Delta t}{(\Delta x)^2}$. Substituting (4) into (2), the numerical approximation of U is obtained by the equation below.

$$u_i^j = -\nu \frac{\varphi_{i+1}^j - \varphi_{i-1}^j}{\Delta x \, \varphi_i^j} \tag{5}$$

Implicit CFDM (i-CFDM) is applied directly to problem (1). The equation to approximate U is expressed below.

$$u_i^{j+1} = u_i^j - u_i^{j+1} \frac{\Delta t}{2\Delta x}\left(u_{i+1}^{j+1} - u_{i-1}^{j+1}\right) + r\left(u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}\right) \tag{6}$$

3.2.2 Exponential Finite Difference Method (EFDM)

Explicit EFDM (e-EFDM) is applied to problem (3). The equation to solve φ is expressed as

$$\varphi_i^{j+1} = \varphi_i^j \exp\left(\left(\frac{r}{\varphi_i^j}\right)\varphi_{i+1}^j - 2\varphi_i^j + \varphi_{i-1}^j\right), \tag{7}$$

where $r = \frac{\nu \Delta t}{(\Delta x)^2}$. Substituting (7) into (2), numerical approximation of U is obtained by (5).

Implicit EFDM (i-EFDM) and fully implicit EFDM (fi-EFDM) are applied directly to problem (1). Equations to approximate U using i-EFDM and fi-EFDM are shown in (8) and (9), respectively.

$$u_i^{j+1} = u_i^j \exp\left(\frac{\Delta t}{u_i^j}\left(\nu \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{(\Delta x)^2} - u_i^j \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2\Delta x}\right)\right) \tag{8}$$

$$u_i^{j+1} = u_i^j \exp\left(\frac{\Delta t}{u_i^j}\left(\nu \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{(\Delta x)^2} - u_i^{j+1} \frac{u_{i+1}^{j+1} - u_{i-1}^{j+1}}{2\Delta x}\right)\right) \tag{9}$$

Von Neumann stability analysis is applied to the explicit numerical methods to ensure e-CFDM and e-EFDM are stable. Δt and Δx have to be restricted to satisfy the stability condition shown in (10).

$$r = \nu \frac{\Delta t}{\Delta x^2} \le \frac{1}{2} \tag{10}$$

3.2.3 Newton's method

Solving problem (1) using i-CFDM, i-EFDM, and fi-EFDM as shown in equations (6), (8), and (9) leads to a set of nonlinear equations that are then solved using the Newton's method. The algorithm of the Newton's method is listed below.

1. Form $F(u) = 0$ from the set of nonlinear equations where $F = \left[f_1, f_2, \ldots, f_{N-1}\right]^T$ and $u = \left[u_1^{j+1}, u_2^{j+1}, \ldots, u_{N-1}^{j+1}\right]^T$.
2. Set the initial guess $u^{(0)}$.
3. Begin the iteration with the formula given as

$$u^{(k+1)} \approx u^{(k)} - \left[J(u^{(k)})\right]^{-1} F(u^{(k)}),\ k = 0, 1, 2, ...$$

where $J(u^{(k)})$ is the Jacobian matrix.

The Newton's method is applied repeatedly at every time step to compute the final solutions. At each time step, the initial guess is taken from the solution at the previous time step. In this case, a stopping criterion is set, that is when $\| F(u^{(k)}) \|_\infty < \varepsilon$, the process is terminated. For instance, ε, the tolerance is set to be a sufficiently small number, 10^{-5}, to ensure the accuracy of the solution and avoid unnecessary computation time.

3.3 Performance Evaluation

The performance of numerical approaches is analyzed based on their accuracy compared to the exact solution. Three measures are considered: relative error (RE), Euclidean norm, L_2, and maximum norm, L_∞. The formulas are shown below.

$$RE = \frac{\left|u_i^j - U_i^j\right|}{\left|U_i^j\right|}$$

$$L_2 = \left\|u - U\right\|_2 = \left(\Delta x \sum_{i=0}^{N} \left|u_i - U_i\right|^2\right)^{\frac{1}{2}}$$

$$L_\infty = \left\|u_i^j - U_i^j\right\|_\infty = \max\left|u_i - U_i\right|,\ 0 \le i \le N$$

4 Results and Discussion

4.1 Numerical Output of e-CFDM and e-EFDM

Problem (1) is reduced to (3) and solved using e-CFDM and e-EFDM for $v = 1.0$ at $t = 0.5$ and $\Delta t = 0.000025$ for $N = 20$, 40, 60, and 80. Error norms of the methods are tabulated in Table 1 below. The error norms decrease significantly as N increases.

Table 1: L_2 and L_∞ norm of explicit methods for $v = 1.0$ and $\Delta t = 0.000025$ at $t = 0.5$

N	e-CFDM		e-EFDM	
	L_2	L_∞	L_2	L_∞
20	2.88E-05	4.07E-05	2.88E-05	4.07E-05
40	5.81E-06	8.22E-06	5.81E-06	8.21E-06
50	3.06E-06	4.33E-06	3.05E-06	4.32E-06
80	7.65E-08	1.08E-07	7.06E-08	1.00E-07

Next, the problem is solved with $\Delta t = 0.000025$ and $N = 50$ ($\Delta x = 0.02$) at different values of t.

Error norms of the methods are summarized in Table 2. Figures 1 and 2 illustrate the exact solutions at different t. The curves take the general form of sine waves with maximum speed located at the center of x. The speed of the fluid is maximum at the initial time $t = 0$ and decreases as time increases.

Table 2: L_2 and L_∞ norm of explicit methods for $v = 1.0$, $\Delta t = 0.000025$ and $N = 50$

t	e-CFDM		e-EFDM	
	L_2	L_∞	L_2	L_∞

0.05	1.33E-04	1.88E-04	1.33E-04	1.88E-04
0.10	5.46E-05	7.73E-05	5.49E-05	7.76E-05
0.15	2.26E-05	3.23E-05	2.24E-05	3.19E-05
0.25	5.75E-06	8.17E-06	5.68E-06	8.05E-06
1.00	5.90E-08	8.35E-08	5.90E-08	8.34E-08
1.20	1.15E-08	1.63E-08	1.15E-08	1.63E-08

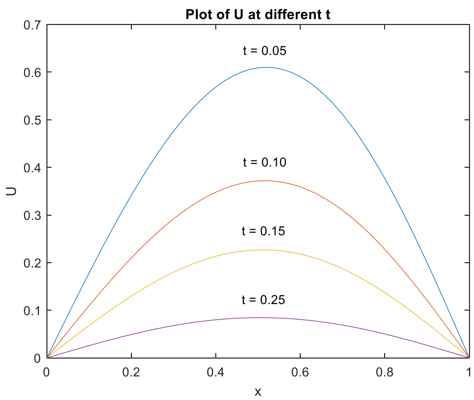

Figure 1: Exact solutions for $v=1.0$ at $t=0.05$, 0.10, 0.15, 0.25

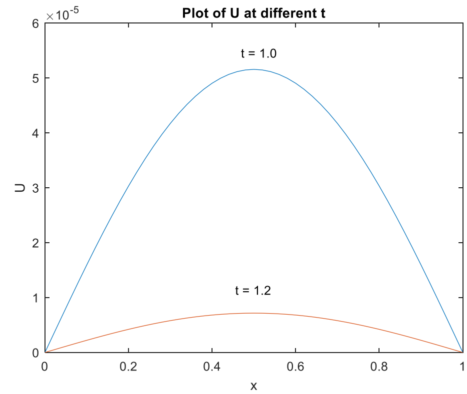

Figure 2: Exact solutions for $v=1.0$ at $t=1.0$, 1.2

The problem is solved with $\Delta t=0.000025$ and $N=50$ at different t, considering v for 0.1 and 0.01. Error norms of the methods are summarized in Table 3.

Table 3: L_2 and L_∞ norm of explicit methods for $v=0.1$ and 0.01 at $\Delta t=0.000025$ and $N=50$

t	$v=0.1$				$v=0.01$			
	e-CFDM		e-EFDM		e-CFDM		e-EFDM	
	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.5	1.51E-04	2.11E-04	1.55E-04	2.20E-04	3.97E-02	1.05E-01	3.96E-02	1.04E-01
1.0	6.42E-05	9.44E-05	6.67E-05	9.87E-05	4.79E-03	1.15E-02	4.74E-03	1.14E-02
1.5	1.87E-05	2.76E-05	2.00E-05	2.91E-05	1.41E-03	5.20E-03	1.41E-03	5.26E-03

e-CFDM and e-EFDM have similar performance in solving linear heat equation (3) at different t and v. The differences between their error norms are generally small; neither of the two methods shows dominance by having the smallest error norms at all conditions tested.

4.2 Numerical Output of i-CFDM, i-EFDM, and fi-EFDM

Application of the Newton's method at each time step drastically increases computation time of implicit methods. Since the technique is always stable, a larger time step could approximate solutions since there is no stability condition to fulfil. Problem (1) applied directly into implicit methods for $v=1.0$ at $t=0.5$ and $\Delta t=0.00025$ for $N=20$, 40, 60, and 80. L_2 norm of the methods is tabulated in Table 4. The error norms decrease as N increases.

Table 4: L_2 and L_∞ norm of implicit methods for $v=1.0$ and $\Delta t=0.00025$ at $t=0.5$

N	i-CFDM	i-EFDM	fi-EFDM

	L_2	L_∞	L_2	L_∞	L_2	L_∞
20	8.32E-05	1.18E-04	1.14E-04	1.62E-04	1.14E-04	1.62E-04
40	4.40E-05	6.22E-05	7.49E-05	1.06E-04	7.50E-05	1.06E-04
50	3.93E-05	5.56E-05	7.02E-05	9.93E-05	7.03E-05	9.95E-05
80	3.42E-05	4.84E-05	6.51E-05	9.21E-05	6.52E-05	9.23E-05

Then, the problem is solved for $v=1.0$, 0.1 and 0.01 with $\Delta t=0.00025$ and $N=50$ at different values of t. Error norms of the methods are tabulated in Tables 5, 6, and 7, respectively. Figures 3, 4, and 5 illustrate the approximations using i-EFDM as time increases for $v=1.0$, 0.1 and 0.01 with $\Delta t=0.00025$ and $N=50$. The behavior of the physical state of the fluid changes as the kinematic viscosity changes. As v decreases, the curve of u no longer follows the general form of the sine wave by having its highest speed at the center of x. The peak of u moves towards the right as v decreases.

Table 5: L_2 and L_∞ norm of implicit methods for $v=1.0$, $\Delta t=0.00025$ and $N=50$

t	i-CFDM		i-EFDM		fi-EFDM	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.05	3.38E-04	4.83E-04	5.94E-04	8.39E-04	6.02E-04	8.55E-04
0.10	4.11E-04	5.81E-04	7.28E-04	1.03E-03	7.32E-04	1.04E-03
0.15	3.75E-04	5.31E-04	6.65E-04	9.43E-04	6.69E-04	9.47E-04
0.25	2.32E-04	3.28E-04	4.13E-04	5.84E-04	4.14E-04	5.86E-04
1.00	5.67E-07	8.01E-07	1.02E-06	1.44E-06	1.02E-06	1.44E-06
1.20	9.46E-08	1.34E-07	1.70E-07	2.40E-07	1.70E-07	2.41E-07

Table 6: L_2 and L_∞ norm of implicit methods for $v=0.1$, $\Delta t=0.00025$ and $N=50$

t	i-CFDM		i-EFDM		fi-EFDM	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.5	3.85E-04	7.32E-04	3.95E-04	7.69E-04	4.31E-04	7.93E-04
1.0	3.12E-04	5.46E-04	3.28E-04	5.81E-04	3.56E-04	6.12E-04
1.5	2.02E-04	3.19E-04	2.16E-04	3.44E-04	2.35E-04	3.68E-04

Table 7: L_2 and L_∞ norm of implicit methods for $v=0.01$, $\Delta t=0.00025$ and $N=50$

t	i-CFDM		i-EFDM		fi-EFDM	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
0.5	2.42E-02	1.66E-01	2.41E-02	1.65E-01	2.43E-02	1.66E-01
1.0	1.34E-02	8.36E-02	1.34E-02	8.36E-02	1.34E-02	8.37E-02
1.5	5.72E-03	2.80E-02	5.71E-03	2.80E-02	5.73E-03	2.80E-02

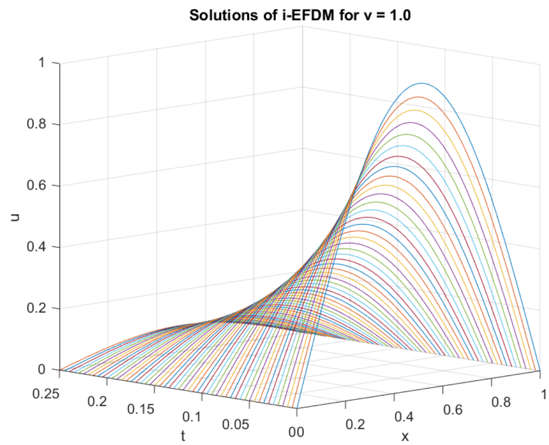

Figure 3: Solutions of i-EFDM for $v = 1.0$, $N = 50$ and $\Delta t = 0.00025$

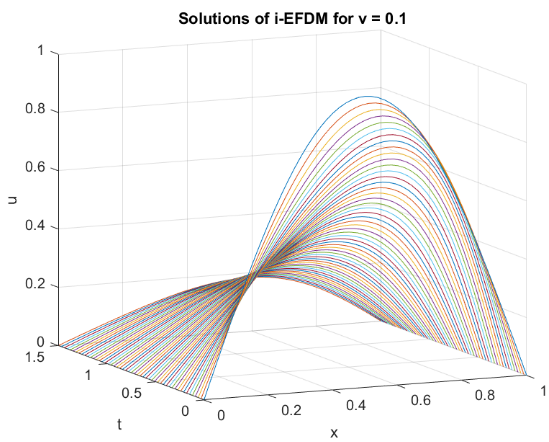

Figure 4: Solutions of i-EFDM for $v = 0.1$, $N = 50$ and $\Delta t = 0.00025$

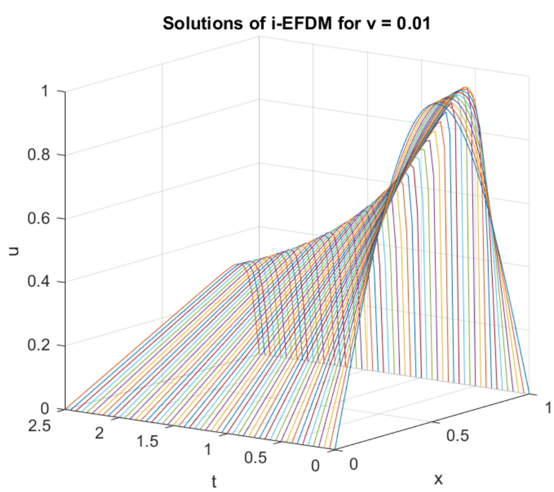

Figure 5: Solutions of i-EFDM for $v = 0.01$, $N = 50$ and $\Delta t = 0.00025$

i-CFDM gives the lowest error norms when v is set at 1.0 which outperforms i-EFDM and fi-EFDM. Then, when v is decreased to 0.1 and 0.01, the difference between error norms of i-CFDM, i-EFDM, and fi-EFDM also reduces. Overall, i-CFDM outperforms in this study. i-EFDM and fi-EFDM have similar accuracy, where the i-EFDM has the better approximations with smaller deviations compared to that of fi-EFDM in all conditions tested.

5 Conclusion

Both CFDM and EFDM in explicit and implicit schemes have good performance with consistent accuracy at different conditions being tested. The explicit scheme can generate results in a much shorter time when compared to the implicit scheme. There are two requirements to implement the explicit scheme: linearization of the problem and stability analysis. When these two requirements are met, the explicit scheme is preferred over the implicit scheme as it can generate approximations with tiny errors in seconds. However, an implicit scheme would be another alternative to directly apply to the nonlinear problem when the abovementioned requirements are unmet. There is no restriction in deciding time step and space step since the method is numerically stable. Still, a bigger mesh size or time step would affect the accuracy of the results. Also, the choice of time step and space step significantly affects the computation time. A wise decision has to be made to optimize both accuracy and computational effort.

In future work, more test problems of different nature can be considered. More advanced problems with higher dimensions or more terms can be considered to be applied in the numerical methods. Efforts to modify the CFDM and EFDM to fit more complex problems would contribute to numerical analysis in science and engineering. There may be a possibility to alter the methods with other numerical approaches to improve its accuracy.

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