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Analytical Method for Solving the Pollution for A River System

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Abstract
Water pollution is a very serious threat to our environment. The water pollution is possible to be monitored with the use of differential equation to know the amount of the pollution in the river. This study focuses on river problem and interested in knowing the level of pollution in the river. A mathematical model of differential equation of river pollution was investigated. The model was modelled based on the condition of the river. Laplace Transform (LT) method was introduced to solve the model of the river pollution. The use of LT helps to solve the system of river pollution and applying the inverse LT, exact solution of this problem obtained. The methodology that used in this study can be used with little modification for the river pollution model to obtain the result. Therefore, the mathematical model and the result analysis can be used to solve the river pollution problem by comparing the result with the previous research of river pollution.

1 Introduction

Water pollution is a serious threat to our environment. In 2014, The Department of Environment state that 39% of the river in Malaysia slightly contaminated and 9% contaminated [1]. The first step to save the environment is by monitoring the pollution. The water pollution is possible to be monitored withthe use of differential equation to know the amount of the pollution in the river. This study will explore how to apply the system of differential equation to know the amount of pollution in the river. The mathematical model that used in this study involved system of differential equation which will be solved using a LT. This method is introduced to solve model of pollution of a river system. A mathematical model using differential equation has the ability to predict the amount pollution of a river [2]. By applying this LT and solving the system an exact solution can be obtain [3].

2 Mathematical model

Figure 1 shows the schematic diagram that is used to model the river system.

Figure 1. System of Mamut River

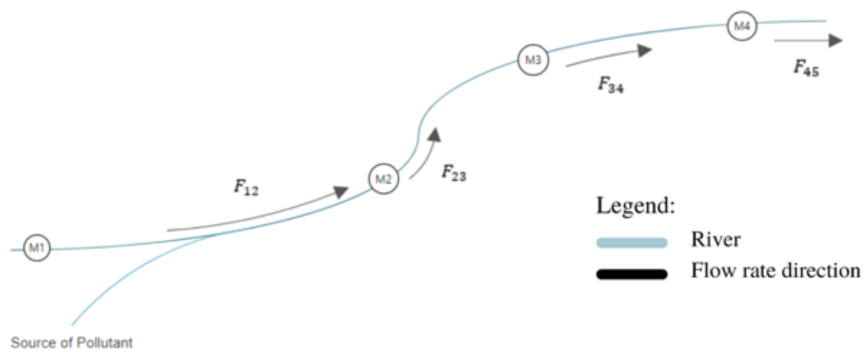

Figure 2 Schematic Diagram of System of River Pollution

The amount of pollution, $x_i(t)$ in *i* at any time $t \geq 0$ where *i=2,3,4* will be calculated. From Figure 1, the amount of pollution in M2, M3, and M4 will be calculated. We assume that the volume of water, V_i in *i* or M2, M3, and M4 remains constant. The concentration of the pollutant in *i* at any time is given by,

$$concentration, c_i(t) = \frac{amount\ of\ pollution, x_i(t)}{volume\ of\ water, V_i}$$

Initially, the river assumed to be free from any contaminant. The flow rate, F_{ij} from *i* to *j* is aconstant. The flux of pollutant flowing from *i* into *j* at any time, denoted by $r_{ij}(t)$, is defined by

$$r_{ij} = F_{ij}c_i(t) = F_{ij}\frac{x_i(t)}{V_i}$$

We also observe that,

rate of change of pollutant = input rate – output rate.

The volume of the riverremains constant, and we can assume that the flow rate of the river at M2, M3, and M4 must be equal the flow in and out the river. Therefore, we obtained the following condition:

River at section $M2$: F_{12} = F_{23},
$M3$: F_{23} = F_{34},
$M4$: F_{34} = F_{45}.
Thus, F_{12} = F_{23} = F_{34} = F_{45}.

Then we will obtain the following first order ordinary differential equation for the system,

$$\frac{dx_2}{dt} = F_{12} + \rho(t) - \frac{F_{23}}{V_2} x_2(t) \tag{1}$$

$$\frac{dx_3}{dt} = \frac{F_{23}}{V_2} x_2(t) - \frac{F_{34}}{V_3} x_3(t) \tag{2}$$

$$\frac{dx_4}{dt} = \frac{F_{34}}{V_3} x_3(t) - \frac{F_{45}}{V_4} x_4(t) \tag{3}$$

with the initial condition $x_i(0) = 0, \;\; i = 2,3,4$

$\rho(t)$ is defined on interval $0 \leq t \leq b$ and $F_{12}, F_{23}, F_{34}, F_{45}, V_2, V_3, \text{and } V_4$ are constant while $x_2(t), x_3(t), \text{and } x_4(t)$ are the amount of pollutant in river sections, M2, M3, and M4. The left-hand side of the equation represent the time rate of change of amount of pollutant in each section, M2, M3, and M4.

Generally, in Equation (1) the first term is the water that flows from river M1 into river M2. The second term, *ρ(t)* indicates the source of pollutant enters river M2. The last term indicates the mixture of water with the pollutant flows out from river M2 entering river M3. In Equation (2), the mixture of polluted water from river M2 entering river M3 is represented by the first term on the right-hand side of the equation. The last term on the right-hand side representing the mixture of polluted water flows out from river M3. Then, the mixture flows into river M4 which is represented by the first term on the right-hand side of Equation (3). The last term on the right-hand side of Equation (3) represents the polluted water that flows out from river M4 to another river section.

In this study, we assume that the pollutant is introduced from the source of pollutant to section M2 with a linear concentration. Thus *ρ(t)=ct*, where c is a constant. Then, we let that *c*=100.

2.1 Analytical Solution

Definition 1. [4] The Laplace transform *f(s)* is defined as,

$$\mathcal{L}[f(x)] = \int_0^\infty f(t)e^{-st}dt,$$

Where s can be either real or complex.

Theorem 1. [5] The Laplace Transform of the derivative defined as,

$$\mathcal{L}\left[f^{(n)}(t)\right] = s^nF(s) - s^nF(s) - s^{n-1}f(0) - s^{n-2}f(0) - \cdots - f^{(n-1)}(0).$$

The Laplace Transform applied into Equation (1), Equation (2), and Equation (3) and we obtained these following equations,

$$\mathcal{L}\left[\frac{dx_2}{dt}\right] = \mathcal{L}[F_{12}] + \mathcal{L}[\rho(t)] - \frac{F_{23}}{V_2}\mathcal{L}[x_2(t)],$$
$$\mathcal{L}\left[\frac{dx_3}{dt}\right] = \frac{F_{23}}{V_2}\mathcal{L}[x_2(t)] - \frac{F_{34}}{V_3}\mathcal{L}[x_3(t)]$$
$$\mathcal{L}\left[\frac{dx_3}{dt}\right] = \frac{F_{34}}{V_3}\mathcal{L}[x_3(t)] - \frac{F_{45}}{V_4}\mathcal{L}[x_4(t)]$$

Then, by using the differential property of Laplace Transform, we get,

$$s\,\mathcal{L}[x_2] - x_2(0) = \mathcal{L}[F_{12}] + \mathcal{L}[\rho(t)] - \frac{F_{23}}{V_2}\mathcal{L}[x_2(t)],$$
$$s\,\mathcal{L}[x_3] - x_3(0) = \frac{F_{23}}{V_2}\mathcal{L}[x_2(t)] - \frac{F_{34}}{V_3}\mathcal{L}[x_3(t)],$$
$$s\,\mathcal{L}[x_4] - x_4(0) = \frac{F_{34}}{V_3}\mathcal{L}[x_3(t)] - \frac{F_{45}}{V_4}\mathcal{L}[x_4(t)]$$

The unknown function $x_i(t)$, $i = 2,3,4$ can obtain by solving the above equation and applying inverse Laplace Transform. We assume that,

$F_{12} = F_{23} = F_{34} = F_{45} = 10m^3$
$V_2 = 500m^3$,
$V_3 = 600m^3$, and
$V_4 = 550m^3$.

Then, the system of first order ordinary differential equation becomes,

$$\frac{dx_2}{dt} = 10 + 100t - \frac{10}{500}x_2(t)$$
$$\frac{dx_3}{dt} = \frac{10}{500}x_2(t) - \frac{10}{600}x_3(t)$$
$$\frac{dx_4}{dt} = \frac{10}{600}x_3(t) - \frac{10}{550}x_4(t)$$

By using Laplace Transform and we get,

$$x_2(s) = \frac{-249500}{s} + \frac{5000}{s^2} + \frac{12475000}{50s+1}$$

$$x_3(s) = -\frac{659400}{s} + \frac{6000}{s^2} + \frac{129384000}{60s+1} - \frac{74850000}{50s+1}$$

$$x_4(s) = -\frac{906950}{s} + \frac{5500}{s^2} + \frac{686125000}{50s+1} + \frac{1423224000}{60s+1} - \frac{2009477250}{55s+1}$$

By applying the inverse Laplace Transform, we obtain,

$$x_2(t) = -249500 + 5000t + 249500e^{-\left(\frac{t}{50}\right)}$$
$$x_3(t) = -659400 + 6000t + 2156400e^{-\left(\frac{t}{60}\right)} - 1497000e^{-\left(\frac{t}{50}\right)}$$
$$x_4(t) = -906950 + 5500t + 13722500e^{-\left(\frac{t}{50}\right)} + 23720400e^{-\left(\frac{t}{60}\right)} - 36535950e^{-\left(\frac{t}{55}\right)}$$

Therefore, the exact amount of pollution at any time, $t \geq 0$ can be calculated.

3. Result and Discussion

The amount of pollution in the river is obtained by solving the ordinary differential equation of the river system using Laplace Transform and Inver Laplace Transform. Table 1 shows the amount of pollution at time, $t \leq 40$.

Table 1. Amount of Pollution at Time, $t \leq 40$.

	Amount of Pollution (m^3)		
t (day)	x_2	x_3	x_4
1	59.56899004	0.429081	0.001922
2	216.9650685	3.008668	0.026065
3	470.2511293	9.624218	0.123261
4	817.5284235	22.09198	0.37399
5	1256.9358	42.16091	0.886719

6	1786.648961	71.51449	1.796302
7	2404.879732	111.7726	3.262439
8	3109.875347	164.4931	5.468189
9	3899.917747	231.1739	8.618546
10	4773.322893	313.2542	12.93906
11	5728.440092	412.1165	18.67451
12	6763.651336	529.0879	26.08765
13	7877.370658	665.442	35.45797
14	9068.043493	822.3999	47.0805
15	10334.14606	1001.132	61.26473
16	11674.18475	1202.76	78.3335
17	13086.69553	1428.358	98.62191
18	14570.24335	1678.951	122.4764
19	16123.4216	1955.522	150.2538
20	17744.85149	2259.009	182.3203
21	19433.18154	2590.307	219.0505
22	21187.08706	2950.271	260.8271
23	23005.26955	3339.713	308.0391
24	24886.45626	3759.41	361.0822
25	26829.3996	4210.097	420.3569
26	28832.87672	4692.477	486.2689
27	30895.68897	5207.212	559.2274
28	33016.66143	5754.935	639.6451

2 9	35194.6 4246	6336 .241	727. 9376
3 0	37428.5 0321	6951 .695	824. 5222

	Amount of Pollution (m^3)		
t (day)	x_2	x_3	x_4
31	39717.13718	7601.831	929.818
32	42059.4598	8287.149	1044.245
33	44454.40796	9008.123	1168.224
34	46900.9396	9765.198	1302.175
35	49398.0333	10558.79	1446.519
36	51944.68786	11389.28	1601.674
37	54539.92192	12257.05	1768.058
38	57182.77354	13162.42	1946.087
39	59872.29982	14105.71	2136.176
40	62607.57655	15087.22	2338.735

The data from Table 1 is calculated for 40 days for all sections, M1, M2, and M3 for the river. Figure 2, Figure 3, and Figure 4. show the amount of pollution in each section of the river.

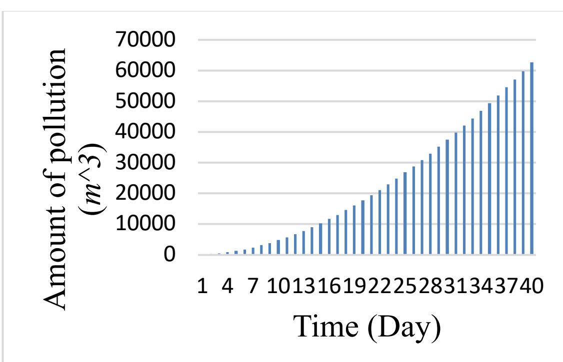

Figure 2. Amount of Pollution at M2

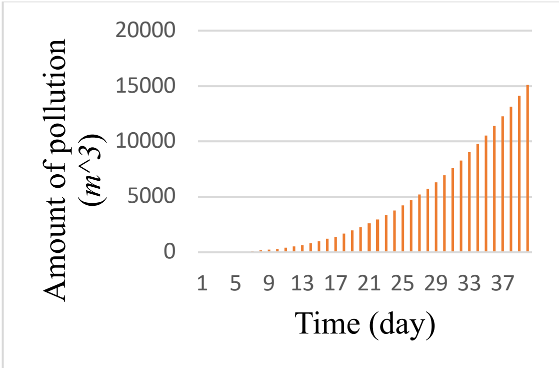

Figure 3. Amount of Pollution at M3

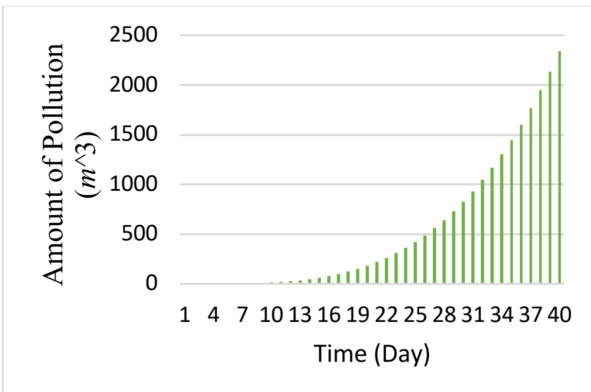

Figure 4. Amount of Pollution at M4

Figure 2 shows that the amount of pollution for section M2 is increases from 56.569m^3 on day 1 to more than 60000m^3 on day 40. Figure 3 and 4 also shows that the amount of pollution for section M3 and M4 increase linearly with the time.

Based on the result from Table 1, it shows that M2 has the highest value of amount of pollution compared to M3 and M4. Tthe pollution enters section M2 at linear rate, thus the pollution concentration in section M2 increases linearly as time increases. As there are flow from section M2 into sections M3 and M4, this pollution concentration will also enter these sections. It is seen that there are linear increase of pollution concentration in sections M3 and M4 as time increases, but at a slower rate. On day 1, 56.569m^3 pollution concentration enters section M2. 0.72% of the pollution concentration from section M2 enters section M3. Only 0.03% enters section M4. In section M2, it is predicted that the pollution concentration will be over than 1000m^3 in five days. For section M3, it will take 15 days for the pollution to increase to that value. Fortunately, the pollution at section M4 will not reach that value within 30 days. The pollution at section M4 it will take 32 days for the pollution increase over than 1000m^3.

4 Conclusion

Many rivers in Malaysia are polluted and this problem can be solved with the use of Ordinary Differential Equation which helps to calculate the exact amount of pollution in the river. The conclusion of the findings of this study are based on the result obtained. To obtain the result, mathematical model for river pollution is modelled based on the condition of Mamut river, Sabah. The river divided into three sections which are M2, M3, and M4 and the amount of pollution will be calculated for each section.

Laplace Transform and Inverse Laplace Transform used to solve the mathematical model for the river pollution. By solving the mathematical model using Laplace Transform and Inverse Laplace Transform, the exact solution or the result is obtained. M2 has the highest amount of pollution followed by M3 and M4. This is because, the distance between the source of pollutant and M2 was closer compared to M3 and M4. Therefore, the mathematical model that formulated in this study is suitable to solve the problem of river pollution. This method easily implemented, and it is simple to calculate the exact solution for the system.

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