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Analysis of Three Species Lotka-Volterra Food Web Models with and without Omnivore

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Abstract
Lotka-Volterra prey-predator equation is a pair of first-order nonlinear differential equations that are used to explain the dynamics of some biological systems. In this study, we use three species Lotka-Volterra food web model with omnivore as the top predator. We work on two models which are three species food chain model and three species food web model with omnivore. Equilibrium points of each model will be determined and the stability of all the equilibrium points are analyzed. Then, the biological interpretations will be explained. The numerical solutions are shown using Matlab software.
Keywords: prey-predator equations; Lotka-Volterra; stability; equilibrium points

1 Introduction

Food network models of three organisms are important building blocks of large-scale ecosystems. It is important to understand the interacting dynamics of three species of food web models to explain the local or global and short-term or long-term behavior of ecosystems. While competition is likely to occur frequently in nature between species exploiting common prey populations, little theoretical work has been done on such systems [3]. Models have commonly considered only two species with 1 predator-1 prey system or dynamics.

The dynamics of three species of predator-prey systems have been studied since the 1970s with some interesting and impressive results [3, 4]. But it is still in its infancy to establish a population theory of the omnivore in the model [1]. A few investigators have compared trophic interactions between three or more species in the absence and presence of omnivorous feeding links. They are Pimm and Lawton in 1978, Matsuda in 1986, Polis 1989, and the latest by Polis and Holt in 1992.

In this paper, we will be using two Lotka-Volterra prey-predator equations with three species with and without the presence of the omnivore as the top predator. The study focuses on three predator-prey style food web models of species with an omnivorous. Omnivorous top predator describes a feeding at more than one trophic stage, where it is common among the marine or terrestrial food web ecological systems. The purpose of this study is to analyse the three species Lotka-Volterra food web models with omnivore by Sze-Bi Hsu, Shigui Ruan and

Ting-Hui Yang then reproduce the results that have been obtained by their research journal. We are going to analyse the dynamics of interaction between three different species in an ecosystem with the presence of omnivore as a top predator from its equilibrium points.

2 Food Chain Model

The food chain model is the model without the existence of omnivore as the top predator. Figure 1 below illustrates the interaction between three species.

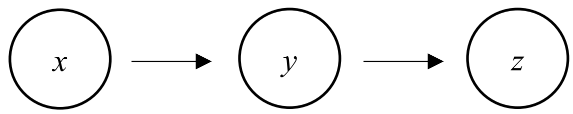

Figure 1: The feeding relationship in a food chain without omnivore. Arrows go from prey to predators.

The ordinary differential equation will be analysed using Lotka-Volterra prey-predator equations, frequently used to describe the dynamics of biological systems which interact a prey and a predator. In this study, we are modelling three species food web model using the basic Lotka-Volterra prey-predator equation into three systems. These equations have three of first-order nonlinear differential equations. So, they will be an interaction between a prey and two predators which the top predator. The populations change through time according to each species. The differential equations for this system without omnivore are:

$$\frac{dx}{dt} = x(1 - x - y),$$
$$\frac{dy}{dt} = y(-d_1 + \alpha x - \beta z), \tag{1}$$
$$\frac{dz}{dt} = z(-d_2 + \delta y).$$

Where x, y and z denote as the prey, intermediate predator and top predator respectively. The parameters d_1 and d_2 are the death rate of intermediate and top predator. Symbol α and β give meaning as the contribution rate to intermediate predator from resource and top predator respectively. Whereas parameter δ denotes as the birth rate of top predator contribute by resource and the intermediate predator. There is no relation between species *x* and species *z*.

After that, equilibrium point will be calculated. All the equilibrium points will be classifying as an equilibrium point if it fulfils $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0$ and $\frac{dz}{dt} = 0$. The stability of equilibrium points can obtain by computing a linearization using partial derivatives and Jacobian matrix will be using to solve it. The Jacobian matrix, ***J*** can be written as

$$\boldsymbol{J}(x, y, z) = \begin{bmatrix} 1 - 2x - y & -x & 0 \\ \alpha y & -d_1 + \alpha x - \beta z & -\beta y \\ 0 & \delta z & -d_2 + \delta y \end{bmatrix}. \tag{2}$$

Then, the stability of the systems can be observed using all the equilibrium points that had been found before. A stable equilibrium point only happen if and only if all the eigenvalues are negative values. From that, we can evaluate the biological interpretation of all the stable equilibrium points.

3 Omnivorous Food Web Model

The schematic diagrams of the interaction between three predator-prey species and the omnivore system are shown in Figure 2 below:

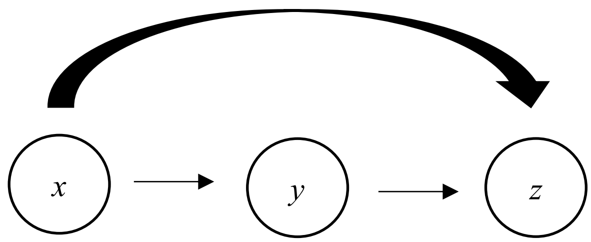

Figure 2: The feeding relationship in a food chain with omnivore. Arrows go from prey to predators.

The ordinary differential equation will be analysed using three species Lotka-Volterra prey-predator equations. So, we are modelling three species food web model using the basic Lotka-Volterra prey-predator equation into three systems. The populations change through time according to each species. The differential equations for this system are:

$$\begin{aligned} \frac{dx}{dt} &= x(1 - x - y - \mu z), \\ \frac{dy}{dt} &= y(-d_1 + \alpha x - \beta z), \\ \frac{dz}{dt} &= z(-d_2 + \gamma x + \delta y). \end{aligned} \tag{3}$$

where x, y and z denote as the resource, intermediate predator and top predator respectively. The parameters d_1 and d_2 are the death rate of intermediate and top predator. Symbol μ indicates the contribution of resource to the top predator while α and β give meaning as the contribution rate to intermediate predator from resource and top predator respectively. Whereas parameter γ and δ
imply the birth rate of top predator contribute by resource and the intermediate predator correspondingly.

After that, equilibrium point will be calculated. All the equilibrium points must satisfy $\frac{dx}{dt} = 0, \frac{dy}{dt} = 0$ and $\frac{dz}{dt} = 0$. The stability of equilibrium points can obtain by computing a linearization using partial derivatives and Jacobian matrix will be using to solve it. The Jacobian matrix, J can be written as

$$\boldsymbol{J}(x,y,z)=\begin{bmatrix}1-2x-y-\mu z & -x & -\mu x\\ \alpha y & -d_1+\alpha x-\beta z & -\beta y\\ \gamma z & \delta z & -d_2+\gamma x+\delta y\end{bmatrix}. \quad (4)$$

The stability of the systems can be observed using all the equilibrium points that had been found before. A stable equilibrium point only happen if and only if all the eigenvalues are negative values. Then, we can observe the biological interpretation each of the species using all the stable equilibrium points.

4 Results and Discussion

4.1 Food Chain Model

The model is going to be used is the model from Equation (1) since there is no omnivore involves in the system for this section. So, there is no contribution of omnivore as the top predator to the growth rate of prey and no contribution of prey to the growth rate of the top predator. Figures below illustrate the numerical simulation of all the stable equilibrium points of the system:

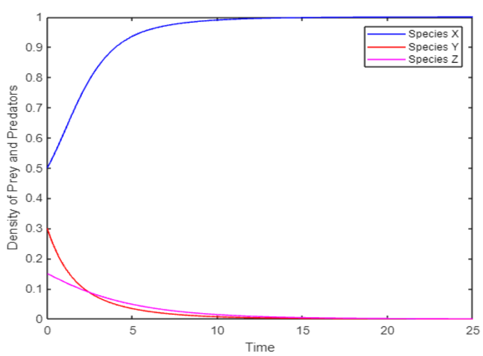

Figure 3: Numerical solution when (1,0,0) is a stable equilibrium point where $d_1 = 0.8, \alpha = 0.5, \beta = 0.2, \ d_2 = 0.25\ and\ \delta = 0.2$.

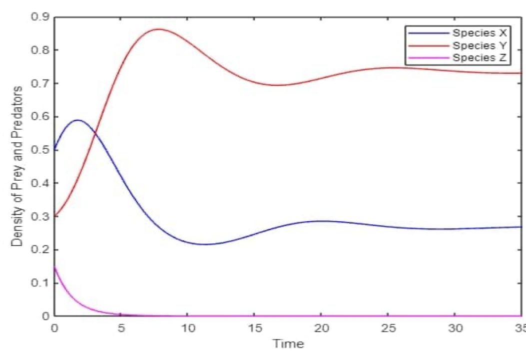

Figure 4: Numerical solution when $(0.2667, 0.7333, 0)$ is a stable equilibrium point where $d_1 = 0.2, \alpha = 0.75, \beta = 0.45, d_2 = 0.8\ and\ \delta = 0.2$.

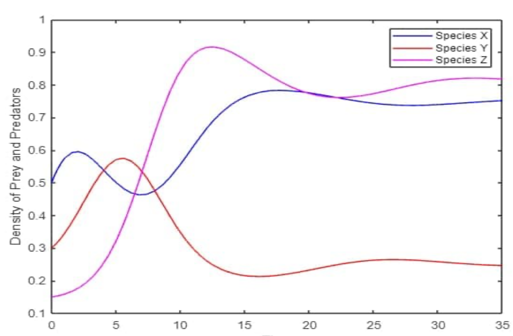

Figure 5: Numerical solution when $(0.75, 0.25, 0.8056)$ is a stable equilibrium point where $d_1 = 0.2, \alpha = 0.75, \beta = 0.45, d_2 = 0.2$ and $\delta = 0.8$.

Based on the Figure 3, the two species which are the intermediate predators and the top predators are extinct, leaving only the preys in the ecosystem. Decreasing the death rate of intermediate predator in Figure 4 will make the density of intermediate predator rises whereas the top predator that has large death rate will be extinct in the ecosystem. When there are no more top predators in that area, the intermediate predators are no longer being hunt and their number getting larger from time to time. However, in Figure 5, the survival of all species in the ecosystem is shown. By using the same death rate of intermediate predator and top predator, these two species could survive together with small different in their density at the same habitat, same goes to the species *x* which is the food source for intermediate predators.

4.2 Omnivorous Food Web Model

This section will evaluate the stability of food web model with the existence of omnivores as the top predators. Instead of eating intermediate predators only, omnivores also consume species *x* at the same time. The model used in this section is from equation (3). Figures below demonstrate the result from numerical simulation of all the stable equilibrium points of the system:

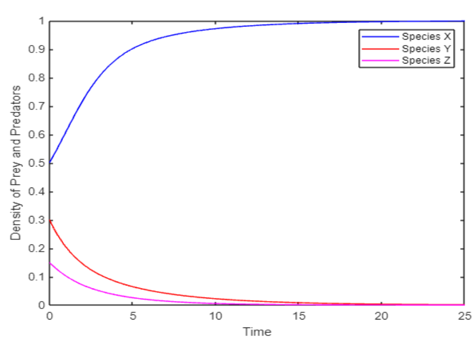

Figure 6: Numerical solution when (1,0,0) is a stable equilibrium point where $\mu = 0.1, d_1 = 0.64, \alpha = 0.45, \beta = 0.2, d_2 = 0.5, \gamma = 0.2$ and $\delta = 0.14$.

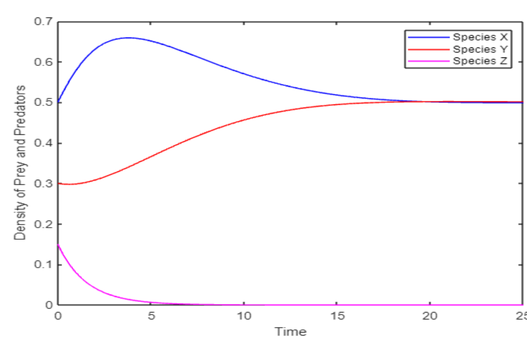

Figure 7: Numerical solution when $(0.5, 0.5, 0)$ is a stable equilibrium point where $\mu = 0.1, d_1 = 0.2, \alpha = 0.4, \beta = 0.2, d_2 = 0.8, \gamma = 0.2$ and $\delta = 0.14$.

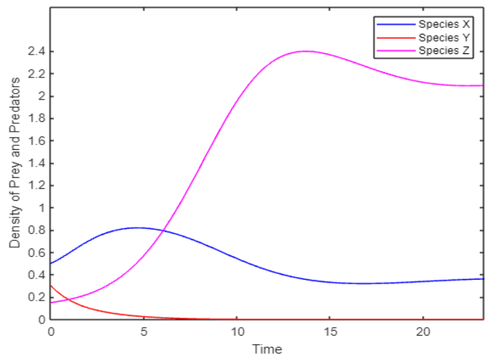

Figure 8: Numerical solution when $(0.3571, 0, 2.1429)$ is a stable equilibrium point where $\mu = 0.3, d_1 = 0.8, \alpha = 0.5, \beta = 0.2, \ d_2 = 0.25, \gamma = 0.7 \ and \ \delta = 0.2.$

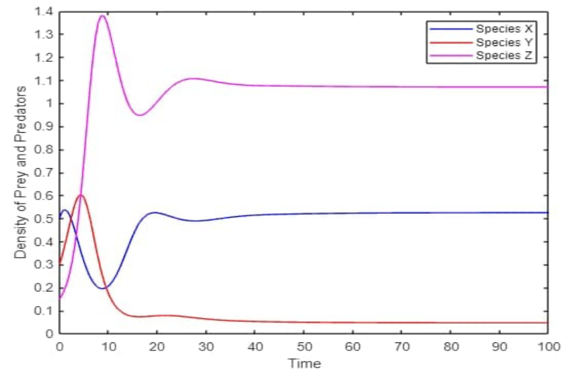

Figure 9: Numerical solution when $(0.5251, 0.04682, 1.0702)$ is a stable equilibrium point where $\mu = 0.4, d_1 = 0.1, \alpha = 0.7, \beta = 0.25, d_2 = 0.3, \gamma = 0.5 \ and \ \delta = 0.8.$

As shown in the Figure 6, only species *x* which is the prey only stay alive in the long term while the other two species will go extinct. By decreasing the death rate of intermediate predator, we can see in Figure 7, the species *x* and the intermediate predators will have equal density at the same time, but the omnivore will be extinct in the ecosystem. In Figure 8 used the lowest death rate of omnivore from others but resulting the highest density of species in the ecosystem. At the same time, intermediate predators reduce from time to time and extinct. Omnivore will only depend on species *x* as the food source in the ecosystem whereas the intermediate predators are no longer exist in the same habitat. Meanwhile, in Figure 9, there is coexistence happened in the ecosystem. All species depend on each other to maintain their life in an ecosystem. Omnivores have the highest density in the ecosystem since they have variety food sources compared to other species.

5 Conclusion

In this study, we can see how omnivores as the top predators act in an ecosystem. They have two food sources at the same time, so they have options if one of the species from their food source going extinct. Inversely for food chain without omnivore, the top predators only consume intermediate predators to stay alive. If the intermediate predators are no longer exist, the top predator will not have food source. Other than that, we can see the density of species depends on the death and birth rate of the species. The lowest death rate will result the highest density of species in an ecosystem and vice versa.

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