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Analytical Solution of Temporarily Dependent Dispersion Along Uniform Flow of One Dimensional Advection-Diffusion

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Abstract Analytical solutions are obtained for a one-dimensional advection–diffusion equation with variable coefficients in a longitudinal finite initially solute free domain. The first investigates temporally dependent solute dispersion in a homogeneous domain along a uniform flow with uniform continuous input. The second investigate temporally dependent solute dispersion in a homogeneous domain along a uniform flow with input condition of increasing nature. The Laplace transformation technique is used to obtain analytical solutions. Throughout the process, a new time variable is introduced. With the support of graphs obtained using MATLAB, the effects of dispersion dependence on time on solute transport are investigated separately.

Keywords: Analytical solutions; advection; diffusion equation; Laplace transformation

1. Introduction

Diffusion is the mechanism by which molecules move across a concentration gradient, and it is a process that occurs in all living things. Diffusion occurs in liquids and gases because molecules can travel freely. As a result, concentration gradients regulate diffusive transport is, (Mahajan, 2010). $\bar{j_d} = D'_d \nabla C$, where j_d denotes the diffusive flux, D'_d the effective diffusion coefficient which accounts for porosity and tortuosity, and ∇C the concentration gradient. In the earlier analytical solutions, the solute dispersion parameter and velocity have been considered constant in a homogeneous medium. The basic approach was to reduce the advection-diffusion equation, into a diffusion equation by eliminating the advection term. It was done either by introducing moving coordinates (A.F. Van et al., 2018). Laplace transform is used for solving differential and integral equations. The solution of the advection diffusion equation is a long-standing problem and many analytical and numerical methods have been introduced to model accurately the interaction between advective and diffusive processes. Either analytical or numerical solutions, aid in understanding the contaminant or pollutant concentration distribution behavior through an open medium such as air, rivers, lakes, and a porous medium such as an aquifer, based on which remedial processes to reduce or eliminate damages can be implemented (Kumar et al., 2009).

The major cause of deterioration of the hydro-environment in surface water sources and aquifers is the immiscible solute or tracer particles of contaminants. The origins of such contaminants derive from the actions of humans on earth. Besides that, due to diffusion and advection that degrades the hydro-environment, there is rising concern in understanding and assessing the contaminants solvent particles transport along the medium. It is important to solve the equation of advection diffusion in real cases.

In that order, for a general case of temporally dependent dispersion along temporally dependent flow, this one-dimensional equation is solved where dispersion is not proportional to the velocity, with respect to a homogeneous first type initial state, non-homogeneous first and third type input conditions, and homogeneous flux type condition at the far end of the semi-infinite medium type condition. The alternatives are obtained as basic instances for other cases of temporally dependent dispersion along uniform flow, uniform dispersion along unstable flow and uniform dispersion along uniform flow. The analytical solutions are contrasted with each other in each of the four cases (Kumar et al., 2010). The advection diffusion equation is a transport equation that combines advective and diffusive processes in species transfer. Advection occurs when a solute is carried along with the fluid's bulk flow (Fulford et al., 2002).

Assume the mass flux J to be due to both diffusion and advection, so J=Jd+Ja. x is the rate of movement of mass per unit time per unit area through the cross section at x. Let $v(x,t)$ denote the fluid velocity. Solute particles will move at the same velocity as the mixture when diffusion is absence. The total mass of solute that is transported through the cross-section is the volume of mixture moving past the cross-section in a time Δx multiplied concentration. This volume is $vA\Delta t$. Thus, the mass flux due solely to advection is given by, $\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial X} = D\,\frac{\partial^2 c}{\partial X^2}$

The objectives of this study are to solve analytically the temporally dependent dispersion in non-uniform flow of uniform input and input condition of increasing nature. Finally, interpret the analytical solution of the temporally dependent dispersion of non-uniform flow in diffusion–advection into graph.

2. Temporally dependent dispersion of uniform continuous input.

Diffusion–advection equation in one dimension with variable coefficients is,

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left[[D(x,t)]\frac{\partial C}{\partial x} - u(x,t)C\right], \tag{1}$$

For temporally dependent dispersion along uniform flow,

$$D(x,t) = D_0 f(mt), \tag{2}$$

$$u(x,t) = u_0 \,. \tag{3}$$

Substitute equation (2) and (3) into equation (1)

$$\frac{\partial C}{\partial t} = [D_0 f(mt)]\,\frac{\partial^2 C}{\partial x^2} - u_0\frac{\partial C}{\partial x}\,. \tag{4}$$

Along with the initial condition and boundary condition as below,

$$\begin{aligned} &C(x,t) = 0, \quad 0 \le x \le L, \quad t = 0, \\ &C(x,t) = c_0, \quad x = 0 \;\; t > 0 \\ &\frac{\partial C(x,t)}{\partial x} = 0, \;\; x = L, \;\; t \ge 0. \end{aligned} \tag{5}$$

The steps start with introducing a new independent variable by a transformation.

$$X = \int \frac{dx}{f(mt)} . \tag{6}$$

Differentiation on both LHS and RHS of equation (6) to get equation (7),

$$\frac{dX}{dx} = \frac{1}{f(mt)}. \tag{7}$$

By applying chain can be obtained $\frac{\partial C}{\partial X}$, and $\frac{\partial C}{\partial X}$. Substitute it into equation (4) to get,

$$f(mt)\frac{\partial C}{\partial t} = D_0 \frac{\partial^2 C}{\partial X^2} - u_0 \frac{\partial C}{\partial X} . \tag{8}$$

The initial and boundary value problem in new space variable may be expressed as:

$$C(X,t) = 0, \quad 0 \le X \le X_0, \quad t = 0; \quad X_0 = \frac{L}{f(mt)},$$

$$C(X,t) = c_0, \quad X = 0 \quad t > 0,$$

$$\frac{\partial C(x,t)}{\partial X} = 0, \; X = X_0, \; t \ge 0.$$

To get rid of the time dependent coefficient, the following transformation is used,

$$T = \int_0^t \frac{dt}{f(mt)} . \tag{9}$$

Differentiation on both LHS and RHS of equation (9) yield,

$$\frac{dT}{dt} = \frac{1}{f(mt)}. \tag{10}$$

Applying chain rule, to find $\frac{\partial C}{\partial t}$, and then we gey,

$$\frac{\partial C}{\partial T} = D_0 \frac{\partial^2 C}{\partial X^2} - u_0 \frac{\partial C}{\partial X} \tag{11}$$

The initial and boundary condition value problems are below,

$$C(X,T) = 0, \quad 0 \le X \le X_0, \quad T = 0; \quad X_0 = \frac{L}{f(mt)}, \tag{12}$$

$$C(X,T) = C_0, \quad X = 0 \quad T > 0 \tag{13}$$

$$\frac{\partial C(X,T)}{\partial X} = 0, \; X = X_0, \; T \ge 0. \tag{14}$$

The most basic method for solving the advection-diffusion equation is to convert equation (11) into a diffusion equation by removing the advection term. This is accomplished by including a new dependent variable (Ogata & Banks, 1961).

$$C(X,T) = K(X,T)\exp\left[\frac{u_0 X}{2D_0} - \frac{{u_0}^2 T}{4D_0}\right]. \tag{15}$$

Now, differentiate equation (15) with respect to T and X. It will give us,

$$\frac{\partial K}{\partial t} = D_0 \frac{\partial^2 K}{\partial X^2}. \tag{16}$$

Based on equation (12) and (15), the initial condition for this problem is,

$$K(X,T) = 0, \quad T = 0, \; X \ge 0. \tag{17}$$

Method of Laplace transform will be used to solve the diffusion equation. Applying the Laplace transform into equation (16) and the general solution for the diffusion equation may be obtained as,

$$\widehat{K}(X,s) = C_1 \, exp\left[-\sqrt{\frac{s}{D_0}}X\right] + C_2 \, exp\left[\sqrt{\frac{s}{D_0}}X\right]. \quad (18)$$

Substitute both C_1 and C_2 values into equation () yields the solution of this boundary value problem,

$$\widehat{K}(X,s) = \frac{C_0}{(s-\alpha^2)}\left(\left[1 - \frac{\left(\sqrt{\frac{s}{D_0}} - \frac{u_0}{2D_0}\right)}{\left(\sqrt{\frac{s}{D_0}} - \frac{u_0}{2D_0}\right) + \left(\sqrt{\frac{s}{D_0}} + \frac{u_0}{2D_0}\right)\left(e^{2\sqrt{\frac{s}{D_0}}X}\right)}\right] \exp\left[-\sqrt{\frac{s}{D_0}}X\right] + \exp\left[\sqrt{\frac{s}{D_0}}X\right]\right) \quad (18)$$

Applying the inverse Laplace transformation on this equation we will get the desired analytical solution, $\mathcal{L}^{-1}\left[\widehat{K}(X,s)\right]$,

$$\mathcal{L}^{-1}\left[\frac{C_0}{(s-\alpha^2)}\left(\left[1 - \frac{\left(\sqrt{\frac{s}{D_0}} - \frac{u_0}{2D_0}\right)}{\left(\sqrt{\frac{s}{D_0}} - \frac{u_0}{2D_0}\right) + \left(\sqrt{\frac{s}{D_0}} + \frac{u_0}{2D_0}\right)\left(e^{2\sqrt{\frac{s}{D_0}}X}\right)}\right] \exp\left[-\sqrt{\frac{s}{D_0}}X\right] + \exp\left[\sqrt{\frac{s}{D_0}}X\right]\right)\right]$$

Applying inverse Laplace transformation on it, will give $K(X,T)$ which hence we can get $C(X,T)$. Based on (Van Genuchten et al., 1982) and (Atul Kumar et al., 2009) the final solution of temporally dependent dispersion of uniform continuous input is,

$$C(X,T) = C_0 A(X,T), \quad (19)$$

where A (X, T) is,

$$\frac{1}{2}\,\text{erfc}\left(\frac{X-u_0T}{2\sqrt{D_0T}}\right) + \frac{1}{2}\,\exp\left(\frac{u_0X}{D_0}\right)\text{erfc}\left(\frac{X+u_0T}{2\sqrt{D_0T}}\right) + \frac{1}{2}\left[2 + \frac{u_0(2X_0-X)}{D_0} + \frac{u_0^2T}{D_0}\right]\exp\left(\frac{u_0X_0}{D_0}\right)\text{erfc}\left(\frac{(2X_0-X)+u_0T}{2\sqrt{D_0T}}\right) - \left(\frac{u_0^2T}{\pi D_0}\right)^{1/2}\exp\left[\frac{u_0X}{D_0} - \frac{(2X_0-X+u_0T)^2}{4D_0T}\right],$$

$$X = \frac{x}{f(mt)}, \quad X_0 = \frac{L}{f(mt)},$$

3 Temporally dependent dispersion of input Condition of Increasing Nature

The mathematical formulation is the same as in (1) to (5) but, the input condition of increasing nature introduced at the origin of the domain will make the boundary condition (6) becomes,

$$-D_0 f(m,t)\frac{\partial C}{\partial x} + u_0 C = u_0 C_0. \quad (20)$$

Substitute equation (7) into equation (20),

$$-D_0\frac{\partial C}{\partial X}+\ u_0C\ =\ u_0C_0\quad X=0\,,T>0\qquad. \tag{21}$$

The condition above in equation (12) is reduced into new dependent variable K using equation (15) and (17), here we get,

$$-D_0\left(\frac{\partial K}{\partial X}\exp\left[\frac{u_0x}{2D_0}-\frac{{u_0}^2t}{4D_0}\right]+\frac{u_0}{2D_0}K\exp\left[\frac{u_0x}{2D_0}-\frac{{u_0}^2t}{4D_0}\right]\right)+u_0\left(K\exp\left[\frac{u_0x}{2D_0}-\frac{{u_0}^2t}{4D_0}\right]\right)=u_0C_0 \tag{22}$$

Factorize equation (17) and substitute value of $X = 0$ and the boundary condition to get,

$$\left(-D_0\frac{\partial K}{\partial X}-\frac{u_0}{2}K\right)\ =\ u_0C_0e^{\alpha^2t}\ ;\quad t>0,\ X=0, \tag{23}$$

where, $\alpha^2\ =\frac{{u_0}^2t}{4D_0}$.

Applying Laplace transformation into the equation (23)

$$\mathcal{L}\ \left(-D_0\frac{\partial K}{\partial X}-\frac{u_0}{2}K\right)\ =\ \mathcal{L}\left(u_0C_0e^{\alpha^2t}\right). \tag{24}$$

While the second boundary condition $\frac{\partial C}{\partial x}\ =\ 0;\ X\rightarrow\ X_0,t\geq0$ is same as case 1 Hence, it becomes,

$$\frac{\partial K}{\partial X}\ +\ \left(\frac{u_0}{2D_0}\right)K\ =\ 0;\quad X=X_0,t\geq0. \tag{25}$$

Applying Laplace transformation into the equation (25)

$$\mathcal{L}\ \left[\frac{\partial K}{\partial X}\ +\ \left(\frac{u_0}{2D_0}\right)K\ \right]=\ \mathcal{L}(0). \tag{26}$$

will reduce both Laplace equation (24) and (26) to an ordinary second order boundary value problem, which comprises of the following two equations,

$$\left(-D_0\frac{\partial\widehat{K}}{\partial X}-\frac{u_0}{2}\widehat{K}\right)\ =\ \frac{u_0C_0}{s-\alpha^2};\ \ X=0, \tag{27}$$

$$\left(\frac{\partial\widehat{K}}{\partial X}\ +\ \frac{u_0}{2D_0}\widehat{K}\right)\ =\ 0;\ \ X\rightarrow\ X_0. \tag{28}$$

Same as case 1, where $C3$ and $C4$ are arbitrary constants that can be specifies using equations The boundary condition at $X\rightarrow\ X_0$ leads to,

$$\widehat{K}(X,s)\ =\ C_3\exp\left[-\sqrt{\frac{s}{D_0}}X\right]\ +\ C_4\exp\left[\sqrt{\frac{s}{D_0}}X\right]. \tag{29}$$

Hence, substitute C_3 and C_4 values that obtained in equation (28) and the particular solution of this boundary value problem may obtained as,

$$\widehat{K}(X,s)=\left(\frac{2u_0C_0\left(\sqrt{\frac{s}{D_0}}+\frac{u_0}{2D_0}\right)}{(s-\alpha^2)\left(3s+\frac{u_0s\sqrt{D_0}}{2}+\frac{u_0}{2}\sqrt{\frac{s}{D_0}}-\frac{3u_0^2}{4D_0}\right)}\right)\exp\left[-X\sqrt{\frac{s}{D_0}}\right]$$
$$+\left(\frac{2u_0C_0\left(\sqrt{\frac{s}{D_0}}-\frac{u_0}{2D_0}\right)}{(s-\alpha^2)\left(\exp\left[2X\sqrt{\frac{s}{D_0}}\right]\right)\left(\frac{u_0\sqrt{sD_0}}{2}-\frac{u_0}{2}\sqrt{\frac{s}{D_0}}-\frac{u_0^2}{4D_0}-s+u_0\right)}\right)\exp\left[X\sqrt{\frac{s}{D_0}}\right].$$

Applying the inverse Laplace transformation on this equation we will get the desired $K(X,T)$

$$\mathcal{L}^{-1}\left(\widehat{K}(,s)\right) = \mathcal{L}^{-1}\left[\left(\frac{2u_0C_0\left(\sqrt{\frac{s}{D_0}}+\frac{u_0}{2D_0}\right)}{(s-\alpha^2)\left(3s+\frac{u_0 s\sqrt{D_0}}{2}+\frac{u_0}{2}\sqrt{\frac{s}{D_0}}-\frac{3u_0^2}{4D_0}\right)}\right)\exp\left[-X\sqrt{\frac{s}{D_0}}\right]+\left(\frac{2u_0C_0\left(\sqrt{\frac{s}{D_0}}-\frac{u_0}{2D_0}\right)}{(s-\alpha^2)\left(\exp\left[2X\sqrt{\frac{s}{D_0}}\right]\right)\left(\frac{u_0\sqrt{sD_0}}{2}-\frac{u_0}{2}\sqrt{\frac{s}{D_0}}-\frac{u_0^2}{4D_0}-s+u_0\right)}\right)\exp\left[X\sqrt{\frac{s}{D_0}}\right]\right].$$

Applying inverse Laplace transformation on it, will give $K(X,T)$ which hence we can get $C(X,T)$. based on (Van Genuchten et al., 1982) and (Atul Kumar et al., 2009) the final solution of temporally dependent dispersion of input condition of increasing nature,

$$C(X,T) = C_0 A(X,T) \tag{29}$$

where A (X, T) is,

$$\frac{1}{2}\,erfc\left(\frac{X-u_0T}{2\sqrt{D_0T}}\right) + \left(\frac{u_0^2T}{\pi D_0}\right)^{1/2} exp\left(-\frac{(X+u_0T)^2}{4D_0T}\right) - \frac{1}{2}\left(1+\frac{u_0X}{D_0}+\frac{u_0^2T}{D_0}\right) exp\left(\frac{u_0X}{D_0}\right) erfc\left(\frac{X-u_0T}{2\sqrt{D_0T}}\right) + \left(\frac{4u_0^2T}{\pi D_0}\right)^{1/2}\left[1+\frac{u_0}{4D_0}(2X_0-X+u_0T)\right] exp\left[\frac{u_0X_0}{D_0}-\frac{(2)}{4D_0T}\right] - \left(\frac{u_0}{D_0}\right)\left[(2X_0-X)+\frac{3u_0^2T}{2}+\frac{4u_0}{D_0}(2X_0-X+u_0T)^2\right]\left[exp\left(\frac{u_0X_0}{D_0}\right)\,erfc\,\left(\frac{(2X_0-X)-u_0T}{2\sqrt{D_0T}}\right)\right].$$

$$X = \frac{x}{f(mt)},\quad X_0 = \frac{L}{f(mt)},$$

4 RESULTS AND DISCUSSION

As a result, the findings for both situations produced using MATLAB by doing the coding to construct the graphs.

4.1 Temporally dependent dispersion of uniform continuous input.

The curves in Figure 1, Figure 2, and Figure 3 are drawn for the concentration values are evaluated in a finite domain $0 \le x \le 1$ with L=1.0 km for input values $C_0 = 1.0$, $u_0 = 0.11(\text{km/year})$, and $a = 1.0(\text{ km})^{-1}$ is taken. The value of $m(\text{year})^{-1} = 0.1$ is chosen for all the curves. Figure 1 curves drawn with manipulating times t (years), Figure 2 drawn by manipulating D values while figure 3 drawn by manipulating D_0 values.

Besides that, we were able to analyze that, at lower time period, the decrease in the concentration value is greater when the distance increases while at higher time period, the decrease in the concentration value is relatively smaller as the distance increases from Figure 1. Moreover, at lower *D*, the decrease in the concentration value is greater when the distance increases while at higher *D*, the decrease in the concentration value is relatively smaller as the distance increases from Figure 2. At last, at lower D_0, the decrease in the concentration value is greater when the distance increases while at higher D_0, the decrease in the concentration value is relatively smaller as the distance increases from figure 3.

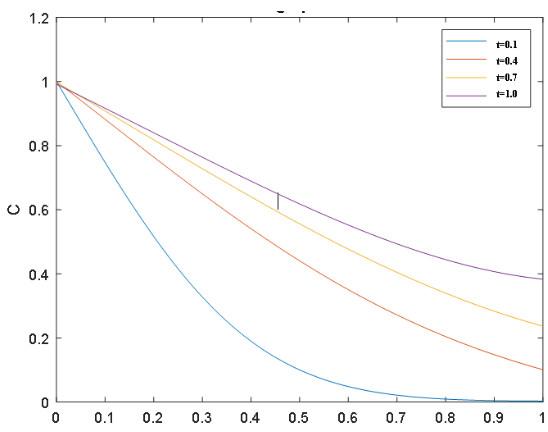

Figure 1: Temporally dependent solute dispersion along uniform flow of uniform input with t (years) =0.1, 0.4, 0.7, and 1.0.

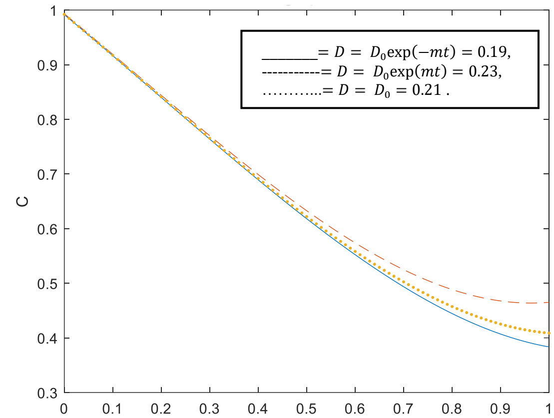

Figure 2: Temporally dependent solute dispersion along uniform flow of uniform input with $D = D_0\exp(-mt)$, $D_0\exp(mt)$, and D_0

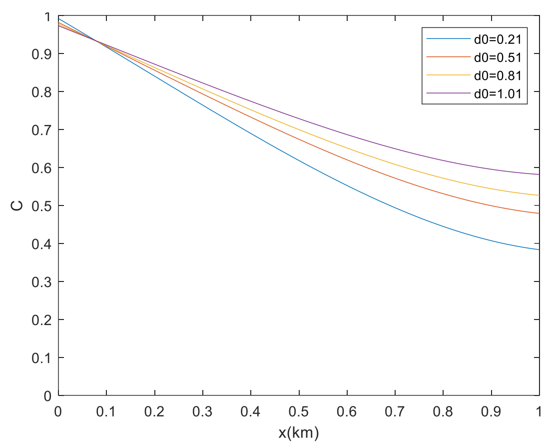

Figure 3: Temporally dependent solute dispersion along uniform flow of uniform input with $D_0 = 0.21$,

4.2 Temporally dependent dispersion of uniform continuous input.

The curves in Figure 4, Figure 5, and Figure 6 are drawn for the concentration values are evaluated in a finite domain $0 \leq x \leq 1$ with L=1.0 km for input values $C_0 = 1.0$, $u_0 = 0.11(\text{km/year})$, and $a = 1.0(\text{ km})^{-1}$ is taken. The value of $m(\text{year})^{-1} = 0.1$ is chosen for all the curves. Figure 4 curves drawn with manipulating times t (years), Figure 5 drawn by manipulating D values while figure 6 drawn by manipulating D_0 values.

Moreover, we were able to analyze that, at lower time period, the initial concentration value is smaller while at higher time period, the initial concentration value is larger from Figure 4. We can analyze that, at lower D, the decrease in the concentration value is greater when the distance increases while at higher D, the decrease in the concentration value is relatively smaller as the distance increases. At 0.4214km the concentration of both curve is equal at 0.1007 from Figure5. At last, we were able to analyze that, at lower D_0, the decrease in the concentration value is greater when the distance increases while at higher D_0, the decrease in the concentration value is relatively smaller as the distance increases from Figure 6.

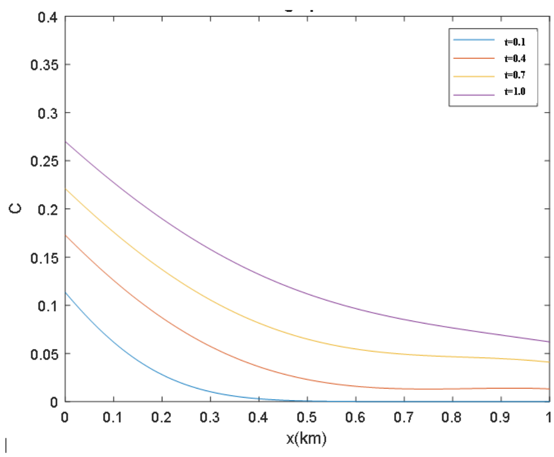

Figure 4: Temporally dependent solute dispersion along uniform flow of input of increasing nature with t (years) =0.1, 0.4, 0.7, and 1.0.

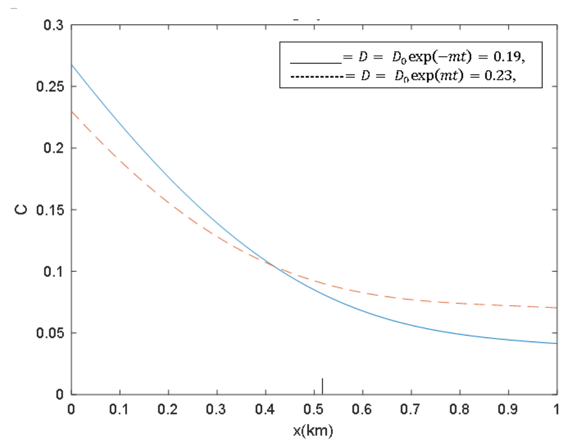

Figure 5: Temporally dependent solute dispersion along uniform flow of input of increasing nature with $D = D_0 \exp(-mt)$, and $D_0 \exp(mt)$

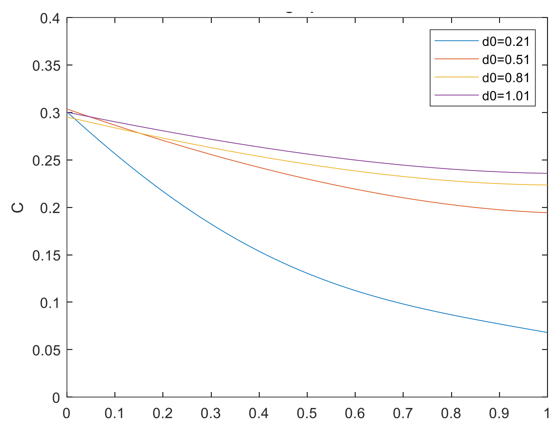

Figure 3: Temporally dependent solute dispersion along uniform flow of input of increasing nature with $D_0 = 0.21,$
$0.51,\ 0.81,\ \text{and}\ 1.01$.

5 Conclusion

In temporally dependent dispersion along uniform, analytical solutions to a one-dimensional diffusion–advection equation with variable coefficients and two sets of boundary conditions (one set of input condition is uniform, while the other is of increasing nature, and the second condition in each set is homogeneous of flux type) have been obtained in an initially solute free finite domain. The application of a new transformation to the diffusion–advection equation, which introduces another spatial variable, allows the Laplace transformation approach to be used to obtain analytical solutions. A two-level explicit finite difference approach was also used to get numerical solutions. The corresponding analytical and numerical answers were also compared, and there was a lot of agreement between them. The numerical solution of the same problem with dispersion varying with velocity was compared to the analytical solution of uniform input. Such analytical solutions could be used to validate numerical solutions in more realistic dispersion problems, making it easier to assess the transport of pollutants' solute concentrations away from their source through soil, aquifers and oil reservoirs.

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