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Solving Complex Riccati Equations and its Application

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Abstract This paper is focused on studying complex Riccati equation and its application. Several methods to solve Riccati equations are discussed and it is observed that complex Riccati equation is equivalent to complex linear second order ODE under a transformation. This paper also discusses the geometrical application of complex Riccati equation in solving Frenet-Serret equation problem.

Keywords Complex Riccati equation; Frenet-Serret equation.

1 Introduction

Equation involving derivatives of functions are called differential equation. Differential equation is ordinary differential equation (ODE) if and only if it constitutes of only functions with respect to only one independent variable, else it is called partial differential equation (PDE) [1]. Complex number is the number with representation $x + iy$ where x is the real part, y is the imaginary part and i is the imaginary number $\sqrt{-1}$. René Descartes coined the term "imaginary" in imaginary number as a nonexistent number during his time [2].

Complex analysis is defined as a study of complex functions involving derivatives, integrals, series and any relevant theories. Analytic function is function that is differentiable over a domain. Hence, in complex analysis, analytic function refers to differentiable function over the complex domain [3]. Complex differential equation (CDE) is the differential equation with solutions in complex domain.

The first order ODE where the unknown function is quadratic is called Riccati equation. For a case where the coefficient for the constant term of the equation is zero, it is called Bernoulli equation. Meanwhile, Riccati equation that is analytic in complex domain is called complex Riccati equation. This research is carried out to seek relationship between the complex Riccati equation and the complex linear second order ODE, solving some of the complex Riccati equations and learning application of the complex Riccati equation.

2 Complex Riccati Equations

2.1 General Theory

Complex Riccati equation of the second order is defined as [4]

$$y'(z) = f_0(z) + f_1(z)y(z) + f_2(z)\big(y(z)\big)^2, \tag{2.1}$$

where f_0, f_1 and f_2 are analytic functions of z in the complex domain.

Now, consider a complex linear second order ODE

$$w''(z) + P(z)w'(z) + Q(z)w(z) = 0, \tag{2.2}$$

where P and Q are analytic in the complex domain.

It can be shown that complex Riccati equation (2.1) is equivalent to the complex linear second order ODE (2.2) under a transformation. First, consider the equation

$$y(z) = \frac{w'(z)}{f_2(z)w(z)}. \tag{2.3}$$

Equation (2.3) is differentiated both sides to yield

$$\begin{aligned} y'(z) &= \frac{w'(z)}{-f_2(z)}\left(\frac{1}{w(z)}\right)' + \frac{1}{w(z)}\left(\frac{w'(z)}{-f_2(z)}\right)' \\ y'(z) &= \frac{(w'(z))^2}{f_2(z)(w(z))^2} - \frac{w''(z)}{w(z)f_2} + \frac{w'(z)}{w(z)(f_2(z))^2 {f_2}'(z)}. \end{aligned} \tag{2.4}$$

Equations (2.3) and (2.4) are then substituted back into (2.1) to obtain

$$\begin{aligned} \frac{(w'(z))^2}{f_2(z)(w(z))^2} - \frac{w''(z)}{w(z)f_2} + \frac{w'(z)}{w(z)(f_2(z))^2 {f_2}'(z)} &= f_0(z) - \frac{f_1(z)w'(z)}{f_2(z)w(z)} + \frac{(w'(z))^2}{f_2(z)(w(z))^2} \\ -\frac{w''(z)}{w(z)f_2} + \frac{w'(z)}{w(z)(f_2(z))^2 {f_2}'(z)} &= f_0(z) - \frac{f_1(z)w'(z)}{f_2(z)w(z)}. \end{aligned} \tag{2.5}$$

Multiply both sides of (2.5) with $(f_2(z)w(z))^2$, we get

$$\begin{aligned} & w(z)w'(z){f_2}'(z) - w(z)f_2(z)w''(z) \\ &= \big(f_2(z)\big)^2\big(w(z)\big)^2 f_0(z) - f_1(z)f_2(z)w(z)w'(z) \\ & w''(z) + \left(-\frac{f_2'(z)}{f_2(z)} - f_1(z)\right)w'(z) + (f_0(z)f_2(z))w(z) = 0. \end{aligned} \tag{2.6}$$

Equation (2.6) is the complex linear second order ODE generated from (2.1) under transformation (2.3).
This is the equation of the form (2.2) where

$$P(z) = -\frac{f_2'(z)}{f_2(z)} - f_1(z)$$

and

$$Q(z) = f_0(z) f_2(z).$$

Another transformation can be made to obtain complex Riccati equation from complex linear second order ODE (2.2). Consider the transformation

$$y(z) = -\frac{w'(z)}{w(z)}. \tag{2.7}$$

Differentiate (2.7), gives

$$y'(z) = \frac{(w'(z))^2 - w(z)w''(z)}{(w(z))^2}$$

$$(y(z))^2 - y'(z) = \frac{w''(z)}{w(z)}. \tag{2.8}$$

Now, divide both sides of (2.2) with $w(z)$ to yield

$$\frac{w''(z)}{w(z)} + P(z)\frac{w'(z)}{w(z)} + Q(z) = 0. \tag{2.9}$$

Finally, substitute (2.7) and (2.8) into (2.9) to obtain

$$(y(z))^2 - y'(z) - P(z)y(z) + Q(z) = 0$$

$$y'(z) = Q(z) - P(z)y(z) + (y(z))^2. \tag{2.10}$$

Equation (2.10) is the obtained complex Riccati equation from (2.2) by transformation (2.7). Since both complex Riccati equation and complex linear second order ODE have relationships under certain transformations, it has been shown that complex Riccati equation is equivalent to the complex linear second order ODE.

2.2 Bernoulli Equation

General Bernoulli equation is defined as

$$y'(z) = f_1(z)y(z) + f_2(z)\big(y(z)\big)^n, \tag{2.11}$$

where f_1 and f_2 are analytic in complex domain and $n \neq 0,1$.

Let a transformation exist, that is

$$u(z) = (y(z))^{1-n}\,. \tag{2.12}$$

Then, it follows that

$$u'(z) = (1-n)\big(y(z)\big)^{-n}y'(z)$$

and

$$y'(z) = \frac{u'(z)(y(z))^n}{1-n}. \tag{2.13}$$

Equation (2.11) is substituted with (2.12) and (2.13) to obtain

$$\frac{u'(z)(y(z))^n}{1-n} = f_1(z)y(z) + f_2(z)(y(z))^n$$

$$u'(z) = (1-n)\big(f_2(z) + f_1(z)u(z)\big). \tag{2.14}$$

Equation (2.14) is the complex linear first order ODE. Let an initial condition $u(z_0) = u_0$ exists where $u_0 = {w_0}^{1-n}$ and $w(z_0) = w_0$ under transformation (2.12) then a solution exist and it is written in the form [4]

$$u(z) = e^{\int_{z_0}^{z} f_1(s)\,ds}\left(u_0 + \int_{z_0}^{z} f_2(t)e^{\int_{z_0}^{z} f_1(s)\,ds}dt\right). \tag{2.15}$$

By rearranging the transformation (2.12) in this form

$$w(z) = (u(z))^{\frac{1}{1-n}}\,,$$

it is then substituted with (2.15) to obtain the final solution

$$w(z) = \left(e^{\int_{z_0}^{z} f_1(s)\,ds}\left({w_0}^{1-n} + \int_{z_0}^{z} f_2(t)e^{\int_{z_0}^{z} f_1(s)\,ds}dt\right)\right)^{\frac{1}{1-n}}. \tag{2.16}$$

Therefore, equation (2.16) is the solution for the Bernoulli equation in (2.11).

Complex Riccati equation (2.1) with $f_0 = 0$ is a form of Bernoulli equation with $n = 2$. It then follows that the solution for this kind of equation is given in the form

$$w(z) = \left(e^{\int_{z_0}^{z} f_1(s)\, ds} \left(w_0{}^{-1} + \int_{z_0}^{z} f_2(t) e^{\int_{z_0}^{z} f_1(s)\, ds} dt \right) \right)^{-1}. \tag{2.17}$$

3 Several Examples on Solving Complex Riccati Equation

Example 3.1 Find the general solution to the equation [4, Exercise 4.1, p. 106]

$$-y' = \frac{y}{z(z^2-1)} + \frac{1}{2} y^2. \tag{2.18}$$

Solution Rearrange the equation (2.18) to match with the general form in (2.1),

$$y' = -\frac{y}{z(z^2-1)} - \frac{1}{2} y^2.$$

Then, it is found that $f_0 = 0$, $f_1 = -(z(z^2-1))^{-1}$ and $f_2 = -\frac{1}{2}$. This is a form of Bernoulli equation with $n = 2$, hence the general solution (2.17) can be used.

$$w(z) = \left(e^{\int_{z_0}^{z} -\frac{1}{s(s^2-1)} ds} \left(w_0{}^{-1} - \int_{z_0}^{z} \frac{1}{2} e^{\int_{z_0}^{z} -\frac{1}{s(s^2-1)} ds} dt \right) \right)^{-1}.$$

Solving this will yield

$$w(z) = \left(K \left(\frac{z\sqrt{z_0{}^2 - 1}}{z_0\sqrt{z^2-1}} \right) \left(w_0{}^{-1} - \frac{1}{2} K (z - z_0) \left(\frac{z\sqrt{z_0{}^2 - 1}}{z_0\sqrt{z^2-1}} \right) \right) \right)^{-1} \tag{2.19}$$

where K is written as

$$K = e^{i \operatorname{Arg} z - \frac{1}{2}(\operatorname{Arg}(z-1) + \operatorname{Arg}(z+1)) - \left(i \operatorname{Arg} z_0 - \frac{1}{2}(i \operatorname{Arg}(z_0 - 1) + i \operatorname{Arg}(z_0 + 1)) \right)}.$$

Therefore, equation (2.19) is the general solution of (2.18).

Example 3.2 The equation [4, Exercise 4.1, p. 106]

$$y' = 1 - y^2$$

has a solution that approaches infinity when $z = 0$. Does the equation have finite solutions?

Solution This is a complex Riccati equation with $f_0 = 1$, $f_1 = 0$ and $f_2 = -1$. Then, transformation (2.3) can be employed to convert this equation into this form of complex linear second order ODE with constant coefficients

$$w'' - w = 0.$$

The general solution for this equation is

$$w = c_1 e^z + c_2 e^{-z}.$$

Then, the solution is transformed back into the form

$$y = \frac{c_1 e^z + c_2 e^{-z}}{-c_1 e^z + c_2 e^{-z}},$$

where c_1 and c_2 are arbitrary constants that can be solved for some initial conditions.

A limit exists when z approaches 0, that is

$$\lim_{z \to 0} y = \frac{c_1 + c_2}{-c_1 + c_2}.$$

Set $c_1 = c_2$ and the solution will be inifinte, else it will be finite. Hence, finite solutions exist.

4 Geometrical Application: Frenet-Serret Equation

Consider a finite curve Γ in three-dimensional space with parameters of arc length s, radius of curvature $R(s)$ and radius of torsion $T(s)$. X, Y and Z are the running coordinates where each axis involved move along the positive axes on the right-hand screw. Let a moving trihedral of the unit vectors $\mathbf{x}_1(s), \mathbf{x}_2(s)$ and $\mathbf{x}_3(s)$ along the tangent, principle normal and binormal vectors of Γ respectively exists [4]. Then the moving trihedral along the curve is represented by Figure 4.1 [8].

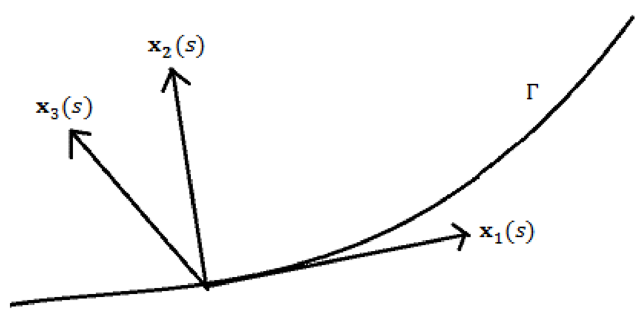

Diagram 4.1: Moving trihedral along the curve Γ.

Define Γ by the equation

$$\mathbf{x} = \mathbf{g}(s),$$

then $\mathbf{x_1} = \mathbf{g}'(s)$ is the tangent vector and $\mathbf{x_1}' = \mathbf{g}''(s)$ is the curvature vector.

Radius of curvature is written as [6]

$$R(s) = \frac{1}{\kappa},$$

where κ is the nonzero curvature $|\mathbf{g}''(s)|$ and the radius of torsion is given by

$$T(s) = \frac{1}{\tau},$$

where τ is the torsion.

Torsion τ is written as [7]

$$\tau = -\mathbf{N} \cdot \mathbf{B}',$$

where $\mathbf{N}$ is the principle normal vector and $\mathbf{B}$ is the binormal vector. Since $\mathbf{x_2}(s) = \mathbf{N}$ and $\mathbf{x}_3(s) = \mathbf{B}$, so this can also be written as

$$\tau = -\mathbf{x_2} \cdot \mathbf{x}_3{}'. \tag{4.1}$$

The normal vector is defined by [4]

$$\mathbf{x}_2 = \frac{\mathbf{g}''(s)}{|\mathbf{g}''(s)|} \tag{4.2}$$

and the binormal vector is a cross product of tangent and principle normal vectors, that is

$$\mathbf{x}_3 = \mathbf{x}_1 \times \mathbf{x}_2. \tag{4.3}$$

Since $\mathbf{x_1}(s), \mathbf{x_2}(s)$ and $\mathbf{x}_3(s)$ are mutually orthogonal, then $\mathbf{x_1} \cdot \mathbf{x_2} = \mathbf{x_1} \cdot \mathbf{x}_3 = \mathbf{x_2} \cdot \mathbf{x}_3 = 0$ and $\mathbf{x_1} \cdot \mathbf{x_1} = \mathbf{x_2} \cdot \mathbf{x_2} = \mathbf{x}_3 \cdot \mathbf{x}_3 = 1$.

From (4.1), $\mathbf{x}_3{}'$ can be determined, that is

$$\begin{gathered} -\mathbf{x_2} \cdot \mathbf{x}_3{}' = \tau \\ (-\mathbf{x_2} \cdot -\mathbf{x_2}) \cdot \mathbf{x}_3{}' = -\mathbf{x_2} \cdot \tau \\ \mathbf{x}_3{}' = -\tau \cdot \mathbf{x_2} \\ \mathbf{x}_3{}' = -\frac{\mathbf{x_2}}{T(s)}. \end{gathered}$$

Since $\mathbf{x_1}' = \mathbf{g}''(s)$, then substitute this into (4.2) to get

$$\mathbf{x}_1{}' = \frac{\mathbf{x}_2}{R(s)}.$$

$\mathbf{x}_1{}'$ and $\mathbf{x}_3{}'$ are then used to find $\mathbf{x}_2{}'$. From (4.3),

$$\begin{aligned}
\mathbf{x}_3 &= \mathbf{x}_1 \times \mathbf{x}_2 \\
-\mathbf{x}_2 &= \mathbf{x}_3 \times \mathbf{x}_1 \\
-\mathbf{x}_2{}' &= \mathbf{x}_3 \times \mathbf{x}_1{}' + \mathbf{x}_1 \times \mathbf{x}_3{}' \\
-\mathbf{x}_2{}' &= \mathbf{x}_3 \times \frac{\mathbf{x}_2}{R(s)} + \mathbf{x}_1 \times -\frac{\mathbf{x}_2}{T(s)} \\
\mathbf{x}_2' &= -\frac{\mathbf{x}_1}{R(s)} + \frac{\mathbf{x}_3}{T(s)}.
\end{aligned}$$

Frenet-Serret equation is the system generated by $\mathbf{x}_1{}', \mathbf{x}_2{}'$ and $\mathbf{x}_3{}'$. Consider

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \text{ and } \mathbf{x}' = \begin{bmatrix} \mathbf{x}_1{}' \\ \mathbf{x}_2{}' \\ \mathbf{x}_3{}' \end{bmatrix}.$$

Then the Frenet-Serret equation is written as

$$\mathbf{x}' = \begin{bmatrix} 0 & \frac{1}{R(s)} & 0 \\ -\frac{1}{R(s)} & 0 & \frac{1}{T(s)} \\ 0 & -\frac{1}{T(s)} & 0 \end{bmatrix} \mathbf{x}. \tag{4.4}$$

Suppose the vector components $\mathbf{x}_i \in \mathbf{x}$ be $\mathbf{x}_{ij}$, then the unitary matrix $\mathcal{U}(s)$ is written as

$$\mathcal{U}(s) = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}. \tag{4.5}$$

Multiplying both sides of (4.4) with arbitrary matrix is possible as long as the original equation is valid. Similarly, it is possible to obtain a unitary system of scalar ODEs for the first column of (4.5), that is

$$\begin{bmatrix} x_{11}{}' \\ x_{21}{}' \\ x_{31}{}' \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{R(s)} & 0 \\ -\frac{1}{R(s)} & 0 & \frac{1}{T(s)} \\ 0 & -\frac{1}{T(s)} & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{21} \\ x_{31} \end{bmatrix}. \tag{4.6}$$

Sum of the squares in any column of $\mathcal{U}(s)$ is 1, therefore u and v can be written in the form of

$$u = \frac{x_{11} + ix_{21}}{1 - x_{31}} = \frac{1 + x_{31}}{x_{11} - ix_{21}}$$

$$-\frac{1}{v}=\frac{x_{11}-ix_{21}}{1-x_{31}}=\frac{1+x_{31}}{x_{11}+ix_{21}}.$$

Then x_{11}, x_{21} and x_{31} can be expressed in terms of u and v, that is

$$x_{11}=\frac{uv-1}{u+v}, \tag{4.7}$$

$$x_{21}=\frac{i(uv+1)}{u+v} \tag{4.8}$$

and

$$x_{31}=\frac{u-v}{u+v}. \tag{4.9}$$

Differentiate x_{11}, x_{21} and x_{31} to yield

$$x_{11}'=\frac{(v^2+uv+1)u'+v'}{(u+v)^2}, \tag{4.10}$$

$$x_{21}'=i\frac{(u^2+uv-1)v'+u'}{(u+v)^2} \tag{4.11}$$

and

$$x_{31}'=\frac{2uu'-2vv'}{(u+v)^2}. \tag{4.12}$$

Substitute (4.7) ,(4.8),(4.9),(4.10),(4.11) and (4.12) into the system (4.6) to get

$$\frac{(v^2+uv+1)u'+v'}{(u+v)^2}=\frac{i(uv+1)}{R(u+v)}, \tag{4.13}$$

$$\frac{(u^2+uv-1)v'+u'}{(u+v)^2}=\frac{i(uv-1)}{R(u+v)}+\frac{uv+1}{T(u+v)}$$

and

$$\frac{2uu'-2vv'}{(u+v)^2}=-\frac{u-v}{T(u+v)}. \tag{4.14}$$

Rearrange (4.13) and (4.14)

$$(v^2+uv+1)u'=\frac{i(uv+1)(u+v)^2}{R}-v', \tag{4.15}$$

$$v'=\frac{u^2-v^2+uu'}{2v(1+T)}. \tag{4.16}$$

Finally, the result is obtained from (4.15) and (4.16), that is

$$(v^2 + uv + 1)u' = \frac{i(uv + 1)(u + v)^2}{R} - \frac{u^2 - v^2 + uu'}{2v(1 + T)}$$
$$u' = -\frac{1}{2}\left(\frac{1}{R(s)} - \frac{i}{T(s)}\right) - \frac{1}{2}\left(\frac{1}{R(s)} + \frac{i}{T(s)}\right)u^2. \quad (4.17)$$

From the Frenet frame, complex Riccati equation (4.17) can be obtained. If $R(s)$ and $T(s)$ are given, then by using complex Riccati equation, the Frenet frame can be determined. Conversely, it is possible to determine $R(s)$ and $T(s)$ using $\mathbf{x}_1(s), \mathbf{x}_2(s)$ and $\mathbf{x}_3(s)$.

5 Conclusion

Solutions for certain forms of complex Riccati equation have been discussed in this study. Under a certain transformation, complex Riccati equation is equivalent to a general complex linear second order ODE. Geometric application of Riccati equation, that is solving Frenet frame problem, particularly in differential geometry also demonstrated in this study, where from the radius of curvature and torsion along with vector equation, the Riccati equation can be determined.

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