Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia
Vol. 4, 2021, page 135-144

The Forced Vibration of a Rectangular Membrane with Moving Boundaries

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Abstract – In this work the forced vibration of rectangular membranes with moving boundaries has been carried out. This research is including the nonhomogeneous problems which is a time dependent boundary value problem. The method of eigenfunctions is used in this study to solve the problem. The aim of this study is to solve the problem of forced vibration of rectangular membranes with moving boundaries by using the method of eigenfunctions and then to plot a graph for solution in two dimensions. A computer algorithm has been written to plot the graph of solution. The graph of solution that was developed during the research will help researchers to understand the effect of t as time step, a as length of rectangular membranes, b as width of rectangular membranes and c as wave speed on the graph of the rectangular membranes when the value of t, a, b and c is increased.

Keywords – Forced Vibration; Rectangular Membrane; Wave Equation; Method of Eigenfunctions; Time Dependent; Nonhomogeneous Problems

1 Introduction

Nowadays, the wave equations have been very popular among mathematicians and also physicians. It can be seen in many situations such as electromagnetic waves and sea waves. The vibration membrane also was related to the wave equation. Myinth-U and Debnath [1] have studied that for deriving the vibrating membrane, there are certain assumptions that should be made which is the membrane is flexible and elastic, there is no elongation of a single segment of the membrane, the tension in the membrane is large, the minimal diameter of the membrane is large, the slope of the displayed membrane at any point is small than the unity and there is only pure transverse vibration.

From previous studies, there are numerous studies that involve the wave equation and also the vibration membrane. Amjad and Khan [2] have studied about the forced vibration analysis of rectangular membranes with clamped edges. During their study, they have analysed the fundamental frequency and its variation based on their dimensions of the membrane, density and tension in the membrane. They also analysed the effect of aspect ratio on the natural frequencies of the membrane.

According to Haertel and Rodriguez [3], their research focus on how to solve the vibrating membrane problem based on the basic Newton's principles and also some simulations. Their aims are to show that the same results can be achieved with much less mathematical effort, based only on Newton's principles and linear elastic forces. There are also another interesting articles that related to the wave equations and also vibrating membrane. The interesting article is about the Fourier solution of the wave equation for a star-like-shaped vibrating membrane by Caratelli, Natalini and Ricci [4]. In their article, their aims are to show how they modify some classical formulas and also they have does some derivation of the methods to compute the coefficients of Fourier type expansions representing solutions of the classical wave equation in complex domains.

In this research, we will show the method to solve the problems of forced vibration of rectangular membranes which is a non-homogeneous problem. The problems were considered as the time-dependent boundary value problems. The method that was used to solve the problem later is known as the method of eigenfunctions. This method is closely related to the method of separation of variables which is aimed for finding a particular solution of a differential equation.

2 Problem Statement

In this paper, we will discuss on the problems of a forced vibration of a rectangular membrane with moving boundaries. The problem here is to determine the displacement function u which satisfies

$$u_{tt} - c^2\nabla^2 u = F(x,y,t), \quad 0 < x < a, \ 0 < y < b, \tag{2.1}$$

$$u(x,y,0) = f(x,y), \quad 0 \le x \le a, \ 0 \le y \le b, \tag{2.2}$$

$$u_t(x,y,0) = g(x,y), \quad 0 \le x \le a, \ 0 \le y \le b, \tag{2.3}$$

$$u(0,y,t) = p_1(y,t), \quad 0 \le y \le b, \ t \ge 0, \tag{2.4}$$

$$u(a,y,t) = p_2(y,t), \quad 0 \le y \le b, \ t \ge 0, \tag{2.5}$$

$$u(x,0,t) = q_1(x,t), \quad 0 \le x \le a, \ t \ge 0, \tag{2.6}$$

$$u(x,b,t) = q_2(x,t), \quad 0 \le x \le a, \ t \ge 0. \tag{2.7}$$

For this problems, a solution will be in the form $u(x,y,t) = U(x,y,t) + v(x,y,t)$ which is equation (2.8), where v is the new dependent variable to be determined. Before finding v, we must first determine U. Then after we get U, we need to find the wave equation for v by substituting equation (2.1) – (2.7) into $u(x,y,t) = U(x,y,t) + v(x,y,t)$. Then, the new problem is to find the function $v(x,y,t)$ which satisfies

$$v_{tt} - c^2\left(v_{xx} + v_{yy}\right) = \tilde{F}(x,y,t),$$

$$v(x,y,0) = \tilde{f}(x,y), \qquad v_t(x,y,0) = \tilde{g}(x,y),$$

$$v(0,y,t) = 0, \qquad v(a,y,t) = 0,$$

$$v(x,0,t) = 0, \qquad v(x,b,t) = 0.$$

This is an initial boundary value-problem with homogeneous boundary conditions, which can be solved by using the method of eigenfunctions.

4 Methodology

In this research, the method of eigenfunctions is applied to solve the problem of forced vibration of rectangular membranes with moving boundaries. In Sub-section 4.1, the method of eigenfunctions in two dimension is shown step by step. In Sub-section 4.2, after the new problem for v was determined, then the method of eigenfunctions can be applied.

4.1 Method of Eigenfunctions in Two Dimension

Consider the nonhomogeneous initial boundary-value problem

$$L[u] = \rho u_{tt} - G \quad \text{in } D \tag{4.1}$$

with prescribed homogeneous boundary conditions on the boundary *B* of *D*, and the initial conditions

$$u(x_1, x_2, \dots, x_n, 0) = f(x_1, x_2, \dots, x_n), \tag{4.2}$$

$$u_t(x_1, x_2, \dots, x_n) = g(x_1, x_2, \dots, x_n). \tag{4.3}$$

Here $\rho \equiv \rho(x_1, x_2, \dots, x_n)$ is a real-valued positive continuous function and $G \equiv G(x_1, x_2, \dots, x_n)$ is a real-valued continuous function. We assume that the only solution of the associated homogeneous problem

$$L[u] = \rho u_{tt} \tag{4.4}$$

with the prescribed boundary conditions is the trivial solution. Then, if there exists a solution of the given problem in Equation (4.1) – (4.3), it can be represented by a series of eigenfunctions of the associated eigenvalue problem

$$L[\varphi] + \lambda\rho\varphi = 0 \tag{4.5}$$

with φ satisfying the boundary conditions given for u.

As a specific example, we shall determine the solution of the problem of forced vibration of a rectangular membrane of length a and width b. The problem is

$$u_{tt} - c^2\nabla^2 u = F(x, y, t) \qquad \text{in} \qquad D \tag{4.6}$$

$$u(x, y, 0) = f(x, y), \qquad 0 \le x \le a, \quad 0 \le y \le b, \tag{4.7}$$

$$u_t(x, y, 0) = g(x, y), \qquad 0 \le x \le a, \quad 0 \le y \le b, \tag{4.8}$$

$$u(0, y, t) = 0, \qquad u(a, y, t) = 0, \tag{4.9}$$

$$u(x, 0, t) = 0, \qquad u(x, b, t) = 0. \tag{4.10}$$

The associated eigenvalue problem is

$$\nabla^2\varphi + \lambda\varphi = 0 \quad \text{in} \quad D,$$

$$\varphi = 0 \quad \text{on the boundary} \quad B \text{ of } D.$$

We have just shown that the separated equations for the wave equation are

$$T'' + \lambda c^2 T = 0, \tag{4.11}$$

$$\nabla^2 U + \lambda U = 0, \tag{4.12}$$

where, in this case, $\nabla^2 U = U_{xx} + U_{yy}$. Let $\lambda = \alpha^2$. Then the solution of Equation (3.10) is

$$T(t) = A(cos\ (\alpha ct)) + B(sin\ (\alpha ct))$$

Now we look for a nontrivial solution of Equation (3.11) in the form

$$U(x, y) = X(x)Y(y).$$

Substituting this into Equation (3.11) yields

$$X'' - \mu X = 0, \qquad Y'' + (\lambda + \mu)Y = 0.$$

If we let $\mu = -\beta^2$, then the solutions of these equations take the form

$$X(x) = C\ (\cos(\beta x)) + D(\sin(\beta x)).$$

$$Y(y) = E\ (cos\ \ (\gamma y)) + F\ (\sin\ \ (\gamma y)),$$

where

$$\gamma^2 = (\lambda + \mu) = \alpha^2 - \beta^2.$$

The homogeneous boundary conditions in x require that $C = 0$ and

$$D\ (\sin(\beta a)) = 0$$

which implies that $\beta = \left(\frac{m\pi}{a}\right) with\ D \neq 0$. Similarly, the homogeneous boundary conditions in y require that $E = 0$ and

$$F\ (\sin(\gamma b)) = 0$$

which implies that $\gamma = \left(\frac{n\pi}{b}\right)$ with $F \neq 0$. Noting that m and n are independent integers, we obtain the displacement function in the form

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{mn} \cos \alpha_{mn}\ ct + b_{mn} sin\ \alpha_{mn} ct) \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$$

(4.13)

where $\alpha_{mn} = \left(\frac{m^2\pi^2}{a^2}\right) + \left(\frac{n^2\pi^2}{b^2}\right)$, α_{mn} and b_{mn} are constants.

The eigenvalues for this problem are given by

$$\alpha_{mn} = \left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right), \qquad m, n = 1,2,3, \ldots$$

and the corresponding eigenfunctions are

$$\varphi_{mn}(x, y) = \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right).$$

Thus, we assume the solution

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(t)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

and the forcing function

$$F(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F_{mn}(t)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

Here $F_{mn}(t)$ are given by

$$F_{mn}(t) = \frac{4}{ab}\int_0^a\int_0^b F(x,y,t)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy.$$

Note that u automatically satisfies the homogeneous boundary conditions. Now inserting $u(x,y,t)$ and $F(x,y,t)$ in Equation (4.6), we obtain

$$\ddot{u}_{mn} + c^2\alpha_{mn}^2 = F_{mn},$$

where $\alpha_{mn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$. We have assumed that u is twice continuously differentiable with respect to t. Thus, the solution of the preceeding ordinary differential equation takes the form

$$u_{mn}(t) = A_{mn}\cos(\alpha_{mn}ct) + B_{mn}\sin(\alpha_{mn}ct) + \frac{1}{\alpha_{mn}c}\int_0^t F_{mn}(\tau)\sin[\alpha_{mn}c(t-\tau)]\,d\tau.$$

The first initial condition gives

$$u(x,y,0) = f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

Assuming that $f(x,y)$ is continuous in x and y, the coefficient A_{mn} of the double Fourier series is given by

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy.$$

Similarly, from the remaining initial condition, we have

$$u_t(x,y,0) = g(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}(\alpha_{mn}c)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right),$$

and hence, for continuous $g(x,y)$,

$$B_{mn} = \frac{4}{ab\alpha_{mn}c}\int_0^a\int_0^b g(x,y)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dxdy.$$

The solution of the given initial boundary-value problem is therefore given by

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(t)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right).$$

4.2 Time-Dependent Boundary-Value Problems

We consider the forced vibration of a rectangular membrane with moving boundaries. The problem here is to determine the displacement function u which satisfies

$$u_{tt} - c^2\nabla^2 u = F(x,y,t), \quad 0 < x < a, \quad 0 < y < b, \tag{4.11}$$

$$u(x,y,0) = f(x,y), \quad 0 \le x \le a, \quad 0 \le y \le b, \tag{4.12}$$

$$u_t(x,y,0) = g(x,y), \qquad 0 \le x \le a, \quad 0 \le y \le b, \tag{4.13}$$

$$u(0,y,t) = p_1(y,t), \qquad 0 \le y \le b, \quad t \ge 0, \tag{4.14}$$

$$u(a,y,t) = p_2(y,t), \qquad 0 \le y \le b, \quad t \ge 0, \tag{4.15}$$

$$u(x,0,t) = q_1(x,t), \qquad 0 \le x \le a, \quad t \ge 0, \tag{4.16}$$

$$u(x,b,t) = q_2(x,t), \qquad 0 \le x \le a, \quad t \ge 0. \tag{4.17}$$

For such problems, we seek a solution

$$u(x,y,t) = U(x,y,t) + v(x,y,t), \tag{4.18}$$

where v is the new dependent variable to be determined. Before finding v, we must first determine U. If we substitute equation (4.18) into equations (4.11) – (4.17), we respectively obtain

$$v_{tt} - c^2(v_{xx} + v_{yy}) = F - U_{tt} + c^2(U_{xx} + U_{yy}) = \tilde{F}(x,y,t)$$

and

$$v(x,y,0) = f(x,y) - U(x,y,0) = \tilde{f}(x,y),$$

$$v_t(x,y,0) = g(x,y) - U_t(x,y,0) = \tilde{g}(x,y),$$

$$v(0,y,t) = p_1(y,t) - U(0,y,t) = \widetilde{p_1}(y,t),$$

$$v(a,y,t) = p_2(y,t) - U(a,y,t) = \widetilde{p_2}(y,t)$$

$$v(x,0,t) = q_1(x,t) - U(x,0,t) = \widetilde{q_1}(x,t),$$

$$v(x,b,t) = q_2(x,t) - U(x,b,t) = \widetilde{q_2}(x,t).$$

In order to make the conditions on v homogeneous, we set

$$\widetilde{p_1} = \widetilde{p_2} = \widetilde{q_1} = \widetilde{q_2} = 0,$$

so that

$$U(0,y,t) = p_1(y,t), \quad U(a,y,t) = p_2(y,t), \tag{4.19}$$

$$U(x,0,t) = q_1(x,t), \quad U(x,b,t) = q_2(x,t). \tag{4.20}$$

In order that the boundary conditions be compatible, we assume that the prescribed functions take the forms

$$p_1(y,t) = \varphi(y)p_1^*(y,t), \quad p_2(y,t) = \varphi(y)p_2^*(y,t),$$

$$q_1(x,t) = \mu(x)q_1^*(x,t), \quad q_2(x,t) = \mu(x)q_2^*(x,t),$$

where the function φ must vanish at the end points $y = 0, y = b$ and the function μ must vanish at $x = 0, x = b$. Thus, $U(x,y,t)$ which satisfies equations (4.19) – (4.20) takes the form

$$U(x,y,t) = \varphi(y)\left[p_1^* + \frac{x}{a}(p_2^* + p_1^*)\right] + \mu(x)\left[q_1^* + \frac{y}{b}(q_2^* + q_1^*)\right].$$

The problem then is to find the function $v(x,y,t)$ which satisfies

$$v_{tt} - c^2\left(v_{xx} + v_{yy}\right) = \tilde{F}(x, y, t),$$

$$v(x, y, 0) = \tilde{f}(x, y), \qquad v_t(x, y, 0) = \tilde{g}(x, y),$$

$$v(0, y, t) = 0, \qquad v(a, y, t) = 0,$$

$$v(x, 0, t) = 0, \qquad v(x, b, t) = 0.$$

This is an initial boundary-value problem with homogeneous boundary condition, which has already been solved on Sub section 4.1.1.

5 Example and Discussion of Problem of Forced Vibration

$$u_{tt} - c^2\left(u_{xx} + u_{yy}\right) = 0,$$

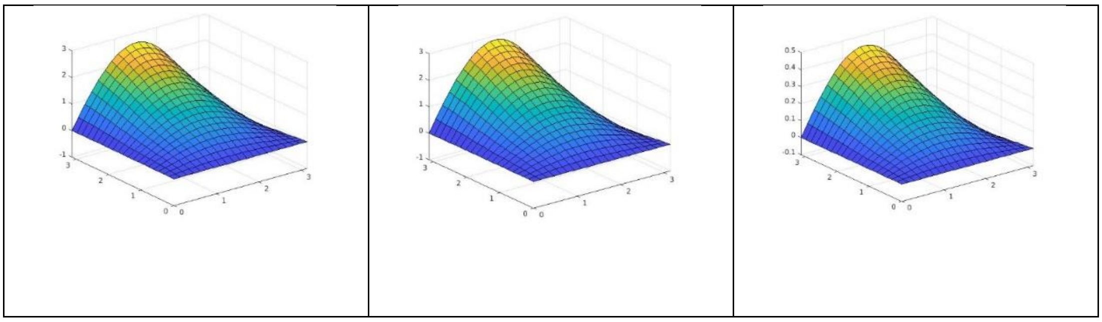
$$u(x, y, 0) = 0, \quad u_t(x, y, 0) = \frac{y}{b}\sin\left(\frac{\pi x}{a}\right),$$

$$u(0, y, t) = 0, \qquad u(a, y, t) = 0,$$

$$u(x, 0, t) = 0, \qquad u(x, b, t) = \sin\left(\frac{\pi x}{a}\right)\sin(t).$$

Solution: $u(x, y, t) = \frac{y}{b}\sin\left(\frac{\pi x}{a}\right)\sin(t) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} v_{mn}(t)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$,

where $v_{mn}(t) = \frac{2(-1)^n}{\alpha_{mn} c a^3 (1 - a^2 c^2)}(a^2 - c^2\pi^2)(\sin(\alpha_{mn} ct) - ac\sin(t))$.



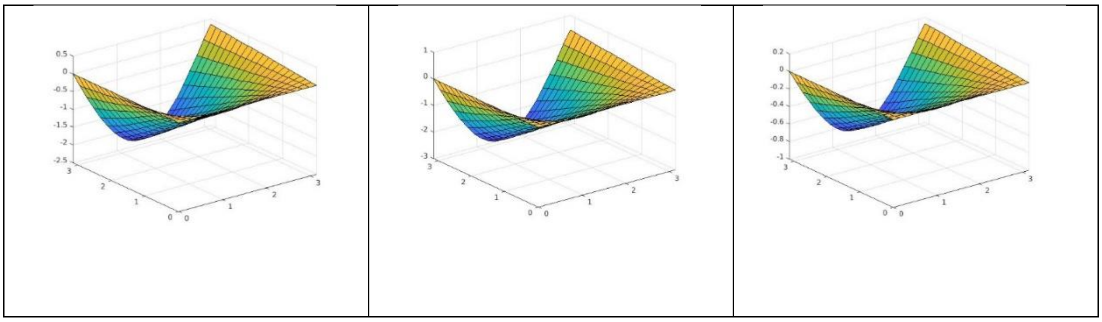


Figure 5.1: The graph solution for Example 1 with the same value of $a = 3, b = 1, c = 3$ and the value of t is changed with $t = 1,2,3, \dots ,6$.

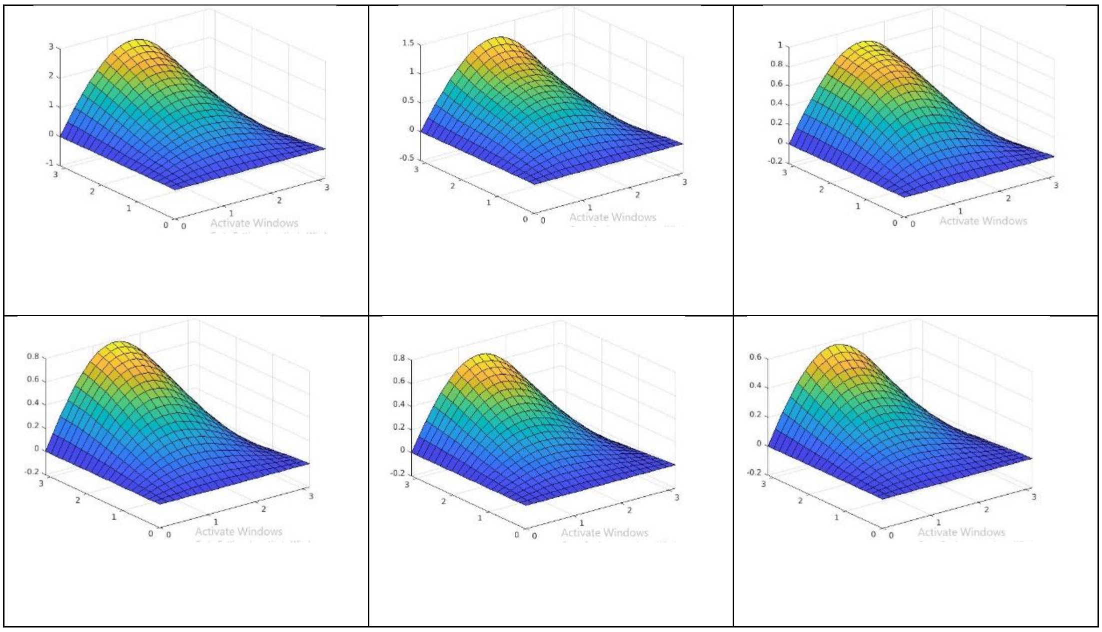


Figure 5.2: The graph of solution for Example 1 as $a = 3$, and the value of b is changed which is $b = 1,2, \dots ,6$ and $c = 3, t = 1$.

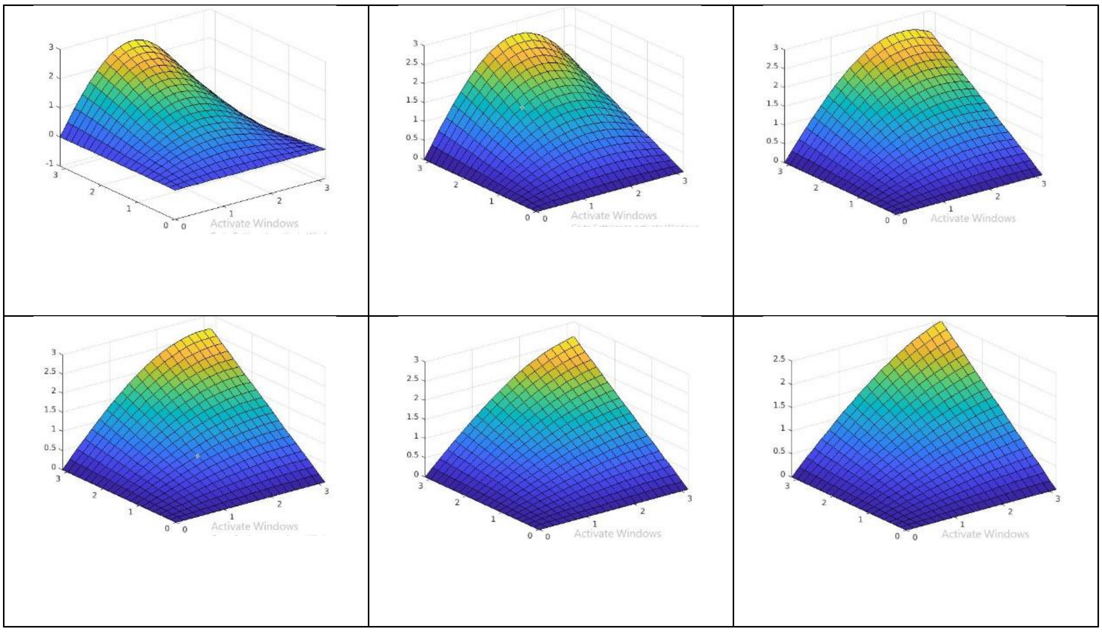


Figure 5.3: The graph of solution for Example 1 as the value of a is changed which is $a = 3, 4, 5,6,7,8$ and $b = 1, c = 3, t = 1$.

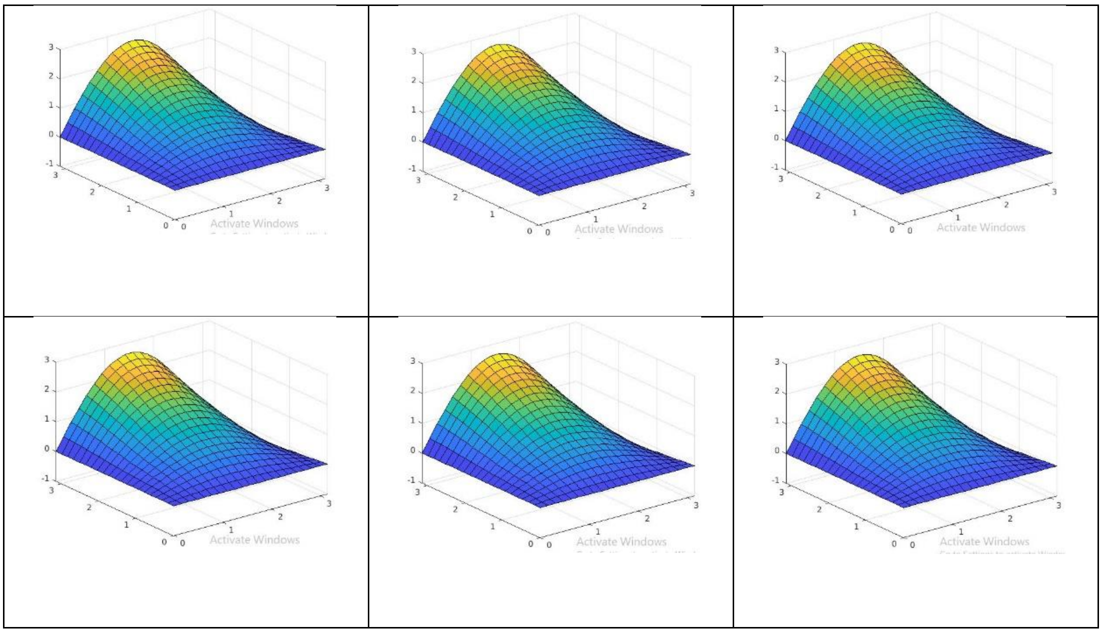


Figure 5.4: The graph of solution for Example 1 as $a = 3, b = 1$, and the value of c is changed which the value of $c = 3,4,5,6,7,8$ and $t = 1$.

5.1 Discussion

From Figure 5.1, Figure 5,2 and Figure 5.3, it was observed that when the value of t, a and b is increased, the graph also starts to show the disturbances. But then the value of c is increased, the graph does not show any obvious disturbances. According to Amjad and Khan (2018), they have mentioned in their studies that when one of the values of a or b is increases, then the fundamental frequency will become decreases and then become constant.

6 Conclusion

As a conclusion, it obvious that when the value of t, a and b is increases, the graph also starts to show the disturbances as well, while when the value of c is increases, the graph does not show any disturbances. This show that t, a and b have a great impact on the graph disturbances.

References

[1] Myinth-U, T. and Debnath, L. (2007). Linear Partial Differential Equations. New York, United States.

[2] Amjad, S.N. and Khan, N.S. (2018). Forced vibration analysis of rectangular membranes with clamped edges. *2018 3rd International Conference on Emerging Trends in Engineering, Sciences and Technology (ICEEST),* (pp.1-20).

[3] Haertel, H. and Rodriguez, E.M. (2000). The vibrating-membrane problem-based on basic principles and simulation.1-5

[4] Caratelli, D., Natalini, P., and Ricci, P.E. (2009). Fourier solution of the wave equation for a star-like-shaped vibrating membrane. *Computer and Mathematics with Applications*, 176-184.