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Solution of Time Independent Emden-Fowler Type Equation By Homotopy Perturbation Method

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Abstract In this study, the nonlinear time-dependent singular initial value problem is considered. The homotopy perturbation method (HPM) is applied to obtain the solution of the nonhomogeneous time-dependent Emden – Fowler type equation. A new algorithm based on HPM to overcome the difficulty of the singular nonlinear problem is developed. Then, the algorithm is utilized and discussed in detail to obtain the approximate analytical solution of the time-dependent Emden – Fowler type equation. The analytical results of the problem are obtained by the help of MAPLE software. The analysis shown that the HPM is an effective, easy, and accurate method to solve non-linear and singular problems.

Keywords Homotopy Perturbation Method; Time-dependent Emden–Fowler equation; Approximate solution.

1 Introduction

A semi-analytical method known as the Homotopy Perturbation Method (HPM) is utilized to solve a nonlinear differential equation. The present approach can be used to solve a nonlinear ordinary differential equation, a nonlinear partial differential equation, and a coupled nonlinear system of partial differential equations. In this paper, the HPM is used to obtain the approximate analytical solutions of the time-dependent Emden-Fowler type equations. A stable new algorithm based on HPM to overcome a set of the singular point at $x = 0$ is implemented. The analysis is accompanied by any linear and nonlinear time-dependent singular initial value questions. The findings show that HPM is very powerful and efficient method to solve the mentioned equations.

2 Literature Review

2.1 Nonlinear Singular Initial Value Problems

For solving linear and nonlinear differential as well as integral equations, the HPM was first suggested by He (1999) and He (2000). The method, which is a coupling of the conventional perturbation method and homotopy in topology, deforms continuously to a simple problem that can be solved easily. The HPM was applied to solve the integro-differential equation of Volterra (El-Shahed, 2005); nonlinear oscillators (He, 2004); bifurcation of nonlinear problems (He, 2005a); bifurcation of delay differential equations (He, 2005b); nonlinear wave equations (He, 2005b); boundary value problems (He, 2006); quadratic Riccati differential equation of fractional order (Golbabai and Sayevand, 2010; Odibat and Momani, 2008); singular differential equations (Chowdhury and Hashim, 2007; Yıldırım and Ozi, 2007) and other fields (Abbasbandy, 2006; Ariel, 2010; He, 2003; Siddiqui et al., 2006). Many improvements have been made on HPM in recent years to overcome various forms of differential equations (Ghorbani and Saberi-Nadjafi, 2008; Lu, 2009; Odibat, 2007; Siddiqui et al., 2009).

Wazwaz (2002) implemented a new differential operator to solve singular Lane-Emden equations by using a convenient modification of the Adomian decomposition process. Hosseini and Nasabzadeh (2007) solved a unique singular ODE and new operator has been expanded. In Hosseini and Jafari (2009), Adomian decomposition method has been used for nearly all forms of singular differential equations.

To apply a new reliable modification according to the given operator on the HPM has been shown in Hosseini and Jafari (2009). The latest improvement reveals that the sequence solution converges quickly as compared to the standard HPM. The solutions of HPM valid for only a small-time span as comparing to the solution of MHPM. MHPM is used to extend the validity domains by recursively applying the HPM over successive time intervals. The solution of HPM for linear BVPs of heat problems indicate a great result that quickly converges to the exact solution. The results obtained show that this recently improved method introduces a powerful improvement in solving single nonlinear problems. HPM may be concluded to be a very powerful and efficient tool for solving a wide range of initial and boundary value problems.

2.2 Nonlinear Emden-Fowler Type Equation

The Emden-Fowler type of equation is a singular initial value problem related to a second-order ordinary differential equation (ODE). This equation is widely used in mathematical physics and astrophysics to model many phenomena, such as thermal explosions, stellar structure, gas spherical cloud thermal activity, isothermal gas spheres, and thermionic currents, the attraction of its molecules and subject to the classical laws of thermodynamics. The Emden-Fowler type equations have significant applications in many fields of the scientific and in technical world where a variety of forms of these functions have been investigated by researchers. Many methods including numerical and perturbation methods have been used to solve the Emden-Fowler type equations. The approximate solutions to the Emden-Fowler type equations have been presented by Shawagfeh (1993) and Wazwaz (2005) by using the Adomian decomposition method (ADM). Wazwaz (2005) applied ADM to solve the time-dependent Emden–Fowler type of equations.

Nouh (2004) accelerated the convergence of power series solution of Lane-Emden type equations by using Euler-Abel transformation and Padé approximation. Liao (2003) solved Lane-Emden type equations by applying the HAM. Further, Bataineh (2009), Noorani (2009), and Hashim (2007) applied HAM to solve the Emden-Fowler type of equations and the time-dependent Emden-Fowler type of equations. In Dehghan and Shakeri (2008); Wawaz (2009); Shang et. al. (2009) and Yildirim and Öziş (2009), the variational iteration method (VIM) was used to solve the Emden-Fowler type of equations. On the other hand, Parand et. al. (2010) utilized the Hermite function collocation (HFC) method to solve these equations.

To solve nonlinear and singular time-dependent Emden-Fowler type equations with the Neumann and Dirichlet boundary conditions, a new modification of the HPM is suggested. First, the single problem is transformed into an equivalent integral equation and then apply the HPM to obtain an approximate series solution. When computing the successive solution elements, this new modified HPM can be used without unknown constants, and therefore stop solving a series of transcendental equations to evaluate the unknown constants. In addition, the proposed method is sufficiently efficient to solve the difficulty of the singular point at *x*=0.

To solve the Emden-Fowler-types equations, this current study is aim to develop a modified HPM. Such nonlinear problems pose difficulties in finding their solutions in the presence of singularity at *x*=0. The recommended solution, which is discussed in this study, is based on the HPM. However, before defining the recursive scheme for the solution of the problems, all type of boundary conditions can be used in the proposed scheme to obtain an integral equation. For solving the nonlinear singular time-dependent Emden-Fowler type equations in this study the initial boundary value problem is translated into an equivalent integral in the proposed process. Then the HPM will be applied to get an approximate solution to the problem. This technique is reliable enough to overcome the difficulty of the singular point at *x*=0.

3 Methodology

3.1 Homotopy Perturbation Method

The heat equation can be used to model several problems in mathematical physics and astrophysics concerning the diffusion of heat perpendicular to the surface of parallel planes. The equation is given as

$$y_{xx} + \frac{r}{x} y_x + af(x,t)g(y) + h(x,t) = y_t, \quad 0 < x \leq L, \; 0 < t < T, \; r > 0 \tag{1}$$

with boundary conditions

$$y(0,t) = \alpha, \qquad y'(0,t) = 0, \tag{2}$$

where α is a constant and $f(x,t)g(y) + h(x,t)$ is the nonlinear heat source, $y(x,t)$ is the temperature, and t is the dimensionless time variable. For steady state case consider

$$r = 2 \quad \text{and} \quad h(x,t) = 0.$$

Then the equations (1) and (2) becomes,

$$y_{xx} + \frac{2}{x} y_x + af(x,t)g(y) = 0, \tag{3}$$
$$y(0,t) = \alpha, \quad y_x(0,t) = 0,$$

This equation (3) is known as Emden-Fowler equation, where $f(x)$ and $g(y)$ are two different x and y functions.

When $f(x) = 1$ and $a = 1$, the equation (3) becomes,

$$y_{xx} + \frac{2}{x} y_x + g(y) = 0 \tag{4}$$
$$y(0,t) = 1, \quad y_x(0,t) = 0,$$

This known as Lane -Emden equation. This Equation (4) was used to model a variety of phenomena in mathematical physics and astrophysics by reducing to the Lane–Emden equation with a given $f(y)$.

3.2 Basic Idea of Homotopy Pertubation Method

The homotopy perturbation method is a combination of the classical perturbation technique and homotopy technique, which has eliminated the limitations of the traditional perturbation methods. This technique can have full advantage of the traditional perturbation techniques. To illustrate the basic idea of the homotopy perturbation method for solving nonlinear differential equations, the following nonlinear differential equation is considered:

$$A(u)-f\ (r)=0, r \in \ \Omega \tag{5}$$

Also considering the boundary conditions of:

$$B\left(u, \frac{\partial u}{\partial n}\right)=0, r \in \Gamma \tag{6}$$

where, A is a general differential operator

B is a boundary operator

$f(r)$ as analytical function

Γ is the boundary of the domain Ω

The operator A can be, generally divided into two parts of L and N, where L is the linear part, while N is the nonlinear one. To achieve this study goal, the following nonlinear differential equation is consider:

$$L(u) - N(u) - f(r) = 0 \tag{7}$$

By the homotopy technique, we construct a homotopy $y(r,p)$: $\Omega \times [0,1] \rightarrow \Re$, which satisfies,

$$H(v,p) = (1 - p)[L(v) - L(u_0)] + p\,[L(v) + N(v) - f(r)] = 0, \tag{8}$$

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\,[N(v) - f(r)] = 0, \tag{9}$$

where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation of (5) which satisfies the boundary conditions. It follows from (8) and (9) that we will have,

$$H(v,0) = L(v) - L(u_0), \quad H(v,1) = A(v) - f(r) \tag{10}$$

Thus, the changing process of p from zero to unity is just that of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are said to be homotopic in topology.

According to the HAM, firstly, the embedding parameter p can be used as a small parameter, and assume that the solution of Eq. (8) and Eq. (9) can be expressed as a power series in p, that is,

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{11}$$

Setting $p = 1$ results in the approximate solution of (5) ;

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \ldots \tag{12}$$

The series Equation (12) is convergent for most cases; however, the convergent rate depends upon the nonlinear operator $A(v)$,

- The second derivative of $N(v)$ with respect to V must be small because the parameter may be relatively large; that is, $p \to 1$.

- The norm of $\frac{L^{-1}\partial N}{\partial V}$ must be smaller than one so that the series converges.

4 Results and Discussion

4.1 Alternative approach of HPM

Using the time-dependent Emden–Fowler equation as a model problem, we will present a stable new algorithm for dealing with time-dependent singular initial value problems (IVPs) in a practical and efficient method. The HPM will be used in a straightforward fashion, but with a new differential operator L option. While it is well recognised that HPM starts by separating the problem's linear and nonlinear components, this method does not always provide sufficient results in singular IVPs. However, a minor adjustment is needed to avoid the singularity condition at $x = 0$. Defining the operator L in terms of the second order derivatives, $y_{xx} + ry_x/x$, found in the problem is an alternative solution.

From Eq. (1), we can create a homotopy that satisfies the following relation.

$$y_{xx} + \frac{r}{x} y_x - y_{0_{xx}} - \frac{r}{x} y_{0_x} + p\,(y_{0_{xx}} + \frac{r}{x} y_{0_x} + af(x,t)g(y) + h(x,t) - y_t = 0 \tag{13}$$

where $p \in [0,1]$ is an embedding parameter and y_0 is an initial approximation which satisfies the boundary conditions. Let us consider the solution form of Equation (13) as

$$y(x) = u_0(x,t) + pu_1(x,t) + p^2 u_2(x,t) + \cdots \qquad (14)$$

and the initial approximation

$$y_0 = \alpha + \int_0^x x^{-r} \int_0^x x^r h(x,t)dx\, dx \qquad (15)$$

Now, by substituting (14) into (13) and substituting (14) into (2) and equating the coefficient terms by power of *p*, we can obtain;

$$u_{0_{xx}} + \frac{r}{x}u_{0_x} - y_{0_{xx}} - \frac{r}{x}y_{0_x} = 0, \quad u_0(0,t) = \alpha, \quad u_{0_x}(0,t) = 0 \qquad (16)$$

$$u_{1_{xx}} + \frac{r}{x}u_{1_x} + y_{0_{xx}} + \frac{r}{x}y_{0_x} + af(x,t)g(u_0) + h(x,t) - u_{0_t} = 0,$$
$$u_1(0,t) = 0, u_{1_x}(0,t) = 0, \qquad (17)$$

$$u_{2_{xx}} + \frac{r}{x}u_{2_x} + af(x,t)g(u_1) - u_{1_t} = 0, u_2(0,t) = 0, u_{2_x}(0,t) = 0, \qquad (18)$$

$$u_{3_{xx}} + \frac{r}{x}u_{3_x} + af(x,t)g(u_2) - u_{2_t} = 0, u_3(0,t) = 0, u_{3_x}(0,t) = 0 \qquad (19)$$

Using the MAPLE package, we can now conveniently solve the above equations for u_0, u_1, u_2, u_3, and so on. Finally, if a four-term approximation is sufficient, the solution can be written as below.

$$y \simeq u_0 + u_1 + u_2 + u_3 + \cdots \qquad (20)$$

According to HPM, the approximate solution of Equation (20) can be expressed as a series of the power of *p*.

4.2 Applications of alternative approach of HPM (MAPLE)

The following is a linear nonhomogeneous equation that needs to be solved.

$$y_{xx} + \frac{2}{x}y_x - (5 + 4x^2)y = y_t + (6 - 5x^2 - 4x^4) \qquad (21)$$

subject to the boundary conditions,

$$y(0,t) = e^t, \qquad y_x(0,t) = 0. \tag{22}$$

Having obtained the Homotopy Pertubation equation called zero order deformation,

$$HPME1 = (1-p)(\frac{d^2U}{dx^2} + \frac{2}{x}\frac{dU}{dx} - \frac{d^2}{dx^2}\left(e^t + x^2 - \frac{1}{4}x^4 - \frac{2}{21}x^6\right) - \frac{2}{x}\frac{d}{dx}(e^t + x^2 - \frac{1}{4}x^4 - \frac{2}{21}x^6)) + p(\frac{d^2U}{dx^2} + \frac{2}{x}\frac{dU}{dx} - U(5+4x^2) - (6-5x^2-4x^4) - \frac{dU}{dt}) \tag{23}$$

By assuming the initial approximation, we obtain,

$$p^0: \frac{\partial^2}{\partial x^2}u_0(x,t) + \frac{2\frac{\partial}{\partial x}u_0(x,t)}{x} - 2 + 3x^2 + \frac{20}{7}x^4 - \frac{2\left(2x - x^3 - \frac{4}{7}x^5\right)}{x} = 0$$

$$p^1: \frac{\partial^2}{\partial x^2}u_1(x,t) + \frac{2\frac{\partial}{\partial x}u_1(x,t)}{x} - 4 + 2x^2 + \frac{8}{7}x^4 - \frac{2\left(2x - x^3 - \frac{4}{7}x^5\right)}{x} - u_0(x,t)(4x^2+5) - \frac{\partial}{\partial t}u_0(x,t) = 0$$

$$p^2: \frac{\partial^2}{\partial x^2}u_2(x,t) + \frac{2\frac{\partial}{\partial x}u_2(x,t)}{x} - u_1(x,t)(4x^2+5) - \frac{\partial}{\partial t}u_1(x,t) = 0$$

$$p^3: \frac{\partial^2}{\partial x^2}u_3(x,t) + \frac{2\frac{\partial}{\partial x}u_3(x,t)}{x} - u_2(x,t)(4x^2+5) - \frac{\partial}{\partial t}u_2(x,t) = 0$$

$$p^4: \frac{\partial^2}{\partial x^2}u_4(x,t) + \frac{2\frac{\partial}{\partial x}u_4(x,t)}{x} - u_3(x,t)(4x^2+5) - \frac{\partial}{\partial t}u_3(x,t) = 0 \tag{24}$$

and boundary condition equations based on coefficient are

Boundary Conditions 1 $p^0 = u_0(c,t) = e^t, D_1(u_0)(c,t) = 0$

Boundary Conditions 2 $p^1 = u_1(c,t) = e^t, D_1(u_1)(c,t) = 0$

Boundary Conditions 3 $p^2 = u_2(c,t) = e^t, D_1(u_2)(c,t) = 0$

Boundary Conditions 4 $p^3 = u_3(c,t) = e^t, D_1(u_3)(c,t) = 0$

Boundary Conditions 5 $p^4 = u_4(c,t) = e^t, D_1(u_4)(c,t) = 0$ (25)

Actually, $u(0,t)$ is sub with $u(c,t)$ because if using $x=0$ the pdsolve will generate error of 'numeric division zero.'

By Equations (24) and boundary equations (25), it can be solved with *pdsolve* in MAPLE, and we obtain the following solutions for u_0, u_1, u_2, u_3 and u_4.

Then, we combine all u_0, u_1, u_2, u_3 and u_4 as follow

$$y \cong u_0 + u_1 + u_2 + u_3 + u_4$$

and will get,

$$u(x,t) = e^t + x^2 - \frac{114}{24845813}x^{14} - \frac{40}{14745843}x^{16} - \frac{58}{19262251}x^{12} - \frac{19}{24338822}x^{18} - \frac{1}{163960010}x^{22} - \frac{13}{118280278}x^{20} + x^2e^t + \frac{1}{2}x^4e^t + \frac{1}{6}x^6e^t + \frac{1}{24}x^8e^t + \frac{151}{128700}e^tx^{12} + \frac{47}{128700}x^{10}e^t + \frac{1}{238680}x^{16}e^t + \frac{1785}{16929134}x^{14}e^t$$

or

$$y(x,t) \cong x^2 + e^t\left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots\right) \tag{26}$$

Since the number of denominators is much larger than the numerator, we assume that the equation is equal to zero. According to HPM, the approximate solution of equation can be expressed as a series of the power of *p*. In the limit of infinitely many terms, this will result in the closed-form solution.

Thus, finally, the approximate solution in a series form is,

$$y(x,t) = x^2 + e^{x^2+t} \tag{27}$$

5 Conclusion

Overall, the present study is about solving time dependent singular IVPs, we provide an efficient technique based on the HPM. The Adomian's Decomposition Method is used to compare with the resulting solutions. The example shows that the solution of the current method is the same as those obtained by Adomian's decomposition, proving the validity and accuracy of the procedure. We also notice the efficiency of the method, which gives quite pleasing results in terms of power series. The HPM offers many advantages and characteristics out over Adomian's decomposition method. The primary advantage of this method is that it overcomes the difficulties related to obtaining Adomian polynomials, and the calculations in HPM are simple and clear. Because of its dependability and reductions in computation size, the HPM has recently become widely used in many fields of research and engineering to solve these types of problems. The homotopy-perturbation approach is defined to be a useful tool for both linear and nonlinear time-dependent singular IVPs.

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