Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia
Vol. 4, 2021, page 58-71

Prey Predator Model with Prey Refuge Stage Structure on Predator

[1] Mohamad Muzairy Mahmud and [2] Faridah Mustapha

[1,2]Department of Mathematical Sciences
Faculty of Science, Universiti Teknologi Malaysia,
81310 Johor Bahru, Johor, Malaysia.

e-mail: [1]muzairy@graduate.utm.my , [2]faridahmustapha@utm.my

Abstract This paper investigates a model of a prey predator with a Michaelis Menten-Holling type functional response and a constant proportion of prey refuge and predator stage structure. The predator is divided into two parts which are juvenile predator and mature predator. As a defensive property against predation, prey has a refuge capability. All of the possible equilibrium points of the model have been identified. They are investigated in terms of their stability. The model under consideration is made up of three nonlinear ordinary differential equations that describe the interaction between prey, juvenile predator and adult predators. The two numerical simulations using MAPLE software are shown for each cases where the equilibrium points is stable. The results show the condition for each equilibrium points is stable.

Keywords Global stability; Equilibrium points; Local stability

1 Introduction

According to Sundari *et al.* [1], in recent years, numerous researchers have generously investigated the link between prey and predator species. Their interaction has been characterised theoretically as differential equations, and the qualitative analysis of ODE has been used to examine its dynamical properties. Sundari *et al.* [1] went on to say that the modelling endeavour began separately in the 1920s by two mathematicians, Vito Volterra and Lotka. In actuality, the prey-predator interaction is controlled by a number of factors, including the biological system's carrying capacity, competition for food among the same species, predator reaction to prey, and harvesting individual or both species.

Aziz *et al*. [2] said that standard Lotka-Volterra systems are also known as predator-prey systems, which are based on the assumption that per capita rates of predation rely simply on the prey populations in the first instance are generalised into a broad range of current predator-prey theories. Then, Aziz *et al*. [2] said that several biologists have challenged the traditional prey-dependent predator-prey models, because functional and digital responses on the typical ecological timescale are sought according to the density of both prey and predator, especially when food is searched, share or competed with predators.

Therefore, Azar *et al*. [3] said that for a multiple predator-type Lotka-Volterra, where the predator is harvested, the stability qualities are investigated. Two separate harvest tactics, a steady quota and a continuous harvest effort are being utilised. They also said that the temporal behaviour, which involves periodical and chaotic oscillation, of population dynamics at varied intensities of the harvest is explored. Therefore Azar *et al*. said a constant harvest quota for predators is proved to destabilise a stable system, if a constant harvest effort is used. Azar *et al*. show that an increase in harvest quota leads to a stationary increasing abundance of the predator for the parameter area investigated.

Bischi *et al*. think unregulated fishing is characterised by a classic predicament of prisoners, commonly described as the 'tragedy of the common [4]. As a result, individuals optimise short-term profit rather than pursue long-term goals via overuse and financial inefficiency such as reduced resource levels and long-term profitability. Indeed, the sustainability of fishing is controlled by the increase in natural supplies as well as by ecological interactions between species' patterns of balance.They concluded that introducing harvesting activities to a dynamic system as complicated,as it can create possibilities that are not easily understood and administered. In addition, overuse of some populations of fish can effect the entire ecosystem and eventually lead to the depletion of some species, hence lowering returns to the risk of unanticipated resource extinction. Therefore, Bischi *et al*. in [4], said that central institutions normally implement forms of regulation either by placing constraints on harvesting such as continual efforts, individual fishery quotas, taxes, or by restricting fish types to be harvested or the areas where exploitation is permitted.

Ko *et al*. in [5] said that they explore a predator-prey model with a functional response of type II of Holling that integrates a prey shelter with a homogenous neumann border. Then they show that non-constant positive, static solutions exist and are available, depending on the constant m in (0, 1], which prescribes the protection of (1−m)u to prey u against predation. Furthermore, Ko *et al*. in [5] study the asymptotic conduct of spatially inhomogeneous solutions and periodic solutions on the ground. Franco *et al*. in [6] said that predator is a living thing that consumes another living thing. The prey is the predator's organism. Lion and zebra, bear and fish, fox and rabbit are some instances of the predator and prey [6]. Therefore, Franco *et al*. [6] said that almost usually, the phrases "predator" and "prey" indicate only animals which feed on animals, but also plants, such as bear and berries, rabbit and lettuce, grasshopper and leaf.

2 Mathematical Formulations

In this part , we look at a ratio-dependent Michaelis Menten – Holling type prey predator model. A predator stage structure has been introduced, along with a prey refuge. Where R, A, K_1, B are positive constants .R is the intrinsic growth rate of the prey, K is the carrying capacity of the environment , A *is* capture rate of the predator , K_1 is the predators benefit from feeding . B is the conversion coefficiency of the predator ,D is the transition rate from juvenile predator to mature predator , d_1 is the mortality rate of juvenile predator , d_2= mortality rate of mature predator.

The form of the model :

$$\frac{du}{dt} = \alpha Ru \left(1 - \frac{u}{k}\right) - \frac{Au(1-\alpha)v_2}{u(1-\alpha) + K_1 v_2}$$

.

$$\frac{dv_1}{dt} = -Dv_1 - d_1 v_1 + \frac{BAu(1-\alpha)v_2}{u(1-\alpha) + K_1 v_2} \qquad (1)$$

$$\frac{dv_2}{dt} = Dv_1 - d_2 v_2$$

$$\text{with } u(0) \geq 0, v_1(0) \geq 0, v_2(0) \geq 0 \qquad (2)$$

At any time t, where $u(t)$ is the Population densities of the prey , $v_1(t)$ is the Juvenile predator , $v_2(t)$ = Adult predator .

3 Solutions of The Problem

System (1) can be written in the dimensionless form by introducing U = population of prey, V_1 = population of juvenile predator and V_2= population of adult predator . The system (1) becomes :

$$\frac{dU}{dT} = U(1-U) - \frac{bU(1-\alpha)V_2}{U(1-\alpha) + V_2} = f_1(U, V_1, V_2)$$

$$\frac{dV_1}{dT} = -D_1 V_1 - a_1 V_1 + \frac{bcU(1-\alpha)V_2}{U(1-\alpha) + V_2} = f_2(U, V_1, V_2) \qquad (3)$$

$$\frac{dV_2}{dT} = D_1V_1 - a_2V_2 = f_3(U, V_1, V_2)$$

For the conditions,

$$U(0) = U_0 > 0, V_1(0) = V_{10} > 0, V_2(0) = V_{20} > 0 \tag{4}$$

Where,

$$b = \frac{A}{K_1R}, \qquad c = BK, D_1 = \frac{D}{R}, a_1 = \frac{d_1}{R}, a_2 = \frac{d_2}{R}$$

3.1 Equilbrium points

The equilbrium points are obtained by solving simultaneously.

$$\frac{dU}{dT} = 0, \qquad \frac{dV_1}{dT} = 0, \qquad \frac{dV_2}{dT} = 0$$

Three equilbrium points are obtained which :

a. The trivial equilbrium points $P_0(0,0,0)$. All species are 0 .

b. Only prey exist, $P_1(1,0,0)$.

c. The co-existence equilibrium point $P_2(U^*, V_1^*, V_2^*)$

Where,

$$V_1^* = \frac{a_2}{D_1}V_2^* \;;$$

$$U^* = 1 - \frac{(1-\alpha)}{D_1 c}\,[D_1 bc \;-\; (D_1 + a_1)a_2]\ ;$$

$$V_2^* = \ (1-\alpha)\left[\frac{D_1 bc}{(D_1 + a_1)a_2} - 1\right] U\ *;$$

3.2 Local stability analysis

To obtain the the local dynamical behavior of the model near equilibrium points, we need to solve the equations by using the Jacobian Matrix of system (3). Then , we will use Routh Hurwitz criterion system to determine the stability of equilbrium parts.

The Jacobian matrix of system (3) is given by

$$J = \begin{bmatrix} \frac{-(1-\alpha)bV_2}{U(1-\alpha)+V_2} + \frac{(1-\alpha)^2 bV_2U}{U(1-\alpha)+V_2} - 2U + 1 & 0 & \frac{(1-\alpha)bUV_2}{(U(1-\alpha)+V_2)^2} - \frac{(1-\alpha)bU}{U(1-\alpha)+V_2} \\ \frac{(1-\alpha)bcV_2}{U(1-\alpha)+V_2} - \frac{(1-\alpha)^2 bcV_2U}{(U(1-\alpha)+V_2)^2} & -a_1 - D_1 & \frac{(1-\alpha)bcU}{U(1-\alpha)+V_2} - \frac{(1-\alpha)bcV_2U}{(U(1-\alpha)+V_2)^2} \\ 0 & D_1 & -a_2 \end{bmatrix}$$

At the equilibrium point $P_0(0{,}0{,}0)$, the jacobian matrix of system (3) is

$$\mathrm{J}(P_0) \ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -(D_1 + a_1) & 0 \\ 0 & D_1 & -a_2 \end{bmatrix}$$

Then by solving the determinant we will get ,

$$\lambda_1 = 1, \lambda_2 = -(D_1 + a_1)\,, \lambda_3 = \ -a_2$$

We can see that $\lambda_1 = 1>0$ while the others two eigen values are negative . Thus trivial equilibrium point P_0 is a saddle point.

At equilibrium point $P_1(1,0,0)$, the Jacobian matrix of system (3) is

$$\mathrm{J}(P_1) = \begin{bmatrix} 1 & 0 & -b \\ 0 & -(D_1 + a_1) & bc \\ 0 & D_1 & -a_2 \end{bmatrix}$$

Then by solving the determinant we will get ,

$$\lambda_1 = -1 \, , \lambda_2 = -(D_1 + a_1 + a_2) \, , \lambda_3 = bcD_1 - (D_1 + a_1)a_2.$$

We can see that λ_1 and λ_2 are negative. Equilbrium point P_1 is stable if λ_3 is negative. λ_3 is negative if

$$bcD_1 - (D_1 + a_1)a_2 < 0$$

Thus the equilibrium point $P_1(1,0,0)$ of system (3) is locally asymptotically stable if

$$bc < \frac{(D_1 + a_1)a_2}{D_1} \, ,$$

Parameter *bc* is the benefit to the predator when eat prey.

The Jacobian matrix of system (3) at equilibrium point $P_2(U^*, {V_1}^*, {V_2}^*)$ is

$$J(P_2) = \begin{bmatrix} 1 - 2U* - \dfrac{bV_2^{*2}(1-\alpha)^1}{[U*(1-\alpha) + V_2*^2]} & 0 & \dfrac{-bU*^2(1-\alpha)^2}{[U*(1-M) + V_2*]^2} \\ \dfrac{bcU*^2(1-\alpha)^2 D_1}{[U*(1-\alpha) + V_2*]^2} & -(D_1 + a_1) & \dfrac{bcU*^2(1-\alpha)^2}{[U*(1-\alpha) + V_2*]^2} \\ 0 & D_1 & -a_2 \end{bmatrix}$$

and then,

$$J(P_2) = \begin{bmatrix} \mathrm{H}_{11} & \mathrm{H}_{12} & \mathrm{H}_{13} \\ \mathrm{H}_{21} & \mathrm{H}_{22} & \mathrm{H}_{23} \\ \mathrm{H}_{31} & \mathrm{H}_{32} & \mathrm{H}_{33} \end{bmatrix}$$

where

$$\mathrm{H}_1 = -(\mathrm{H}_{11} + \mathrm{H}_{22} + \mathrm{H}_{33})$$

$$\mathrm{H}_2 = \mathrm{H}_{11}\mathrm{H}_{22} + \mathrm{H}_{11}\mathrm{H}_{33} + \mathrm{H}_{22}\mathrm{H}_{33} - \mathrm{H}_{23}\mathrm{H}_{32}$$

$$\mathrm{H}_3 = \mathrm{H}_{11}(\mathrm{H}_{23}\mathrm{H}_{32} - \mathrm{H}_{22}\mathrm{H}_{33}) - \mathrm{H}_{13}\mathrm{H}_{21}\mathrm{H}_{32}$$

The characterisitic equation is :

$$\lambda^3 + \mathrm{H}_1\lambda^2 + \mathrm{H}_2\lambda + \mathrm{H}_3 = 0$$

Then we using Routh Hurwitz Criterion ,

$$P_3(s) = \lambda^3 + \mathrm{H}_1\lambda^2 + \mathrm{H}_2\lambda + \mathrm{H}_3 = 0$$

$$\begin{bmatrix} \mathrm{H}_3 & \mathrm{H}_1 \\ \mathrm{H}_2 & 1 \\ Z_{3,1} & 0 \\ Z_{4,1} & 0 \end{bmatrix}$$

$$Z_{3,1} = \frac{\mathrm{H_1(H_2) - \ (1)H_3}}{H_2}, \text{let } Z_{3,1} \text{ be } b_1$$

$$Z_{4,1} = \frac{b_1\,(1) - \ H_2\,(0)}{b_1} = a_0$$

Prove as assumption ,

$$\mathrm{H_3} > \ 0$$

$$\mathrm{H_2} > \ 0$$

$$1 > \ 0$$

Then,

$$\frac{\mathrm{H_1(H_2) - \ (1)H_3}}{H_2} > 0$$

$$If\ H_2 > \ 0, \text{then}$$

$$\mathrm{H_1(H_2) - \ H_3} > 0$$

$$\mathrm{H_1(H_2)} > \mathrm{H_3}$$

Note :

No restriction on a_1 ,

If $H_2 > 0$, $H_1 > \frac{H_3}{H_2}$, thus H_1 will automatically positive.

Thus , $\mathrm{H_1} > 0$

By using Routh Hurwitz criterion system (3) is locally asymptotically stable if

i. $H_1 > 0$
ii. $H_3 > 0$
iii. $H_1H_2 - H_3 > 0$

4 Results

Graphs are then displayed using MAPLE software in this project based on two numerical simulations. Simulation 1 is case for $(U, V_1, V_2) = (1,0,0)$, where only prey exist while simulation 2 is case for $(U, V_1, V_2) = (0.90, 0.34, 0.11)$, where co-existence equilibrium point exists.

These are the parameter values used in Numerical Simulation 1 and Numerical Simulation 2 .

Table 1 : Parameter values used in Numerical Simulation 1 and Numerical Simulation 2.

Simulation	Simulation 1 $P_1(1,0,0)$	Simulation 2 $P_2(U^*, V_1{}^*, V_2{}^*)$
The benefit of juvenile predator when consume prey	b = 1.5	b = 1
The benefit of mature predator when consume prey	c = 0.7	c = 3.0
Transition rate from juvenile predator to mature predator	D_1 = 1.5	D_1 = 0.5
Capture rate for juvenile predator	a_1= 1.0	a_1 = 0.3
Capture rate for mature predator	a_2 = 1.5	a_2 = 1.5
Proportion of prey to refuge	α= 0.7	α= 0.5

Eigen values	$\lambda_1 = -1, \lambda_2 = -3.35092, \lambda_3 = -0.64907$	$\lambda_1 = -1, \lambda_2 = -1.03839,$ $\lambda_3 = -0.76861$

Numerical simulation 1

Case for $(\boldsymbol{U}, \boldsymbol{V_1}, \boldsymbol{V_2}) = (\mathbf{1}, \mathbf{0}, \mathbf{0})$ where only prey exist.

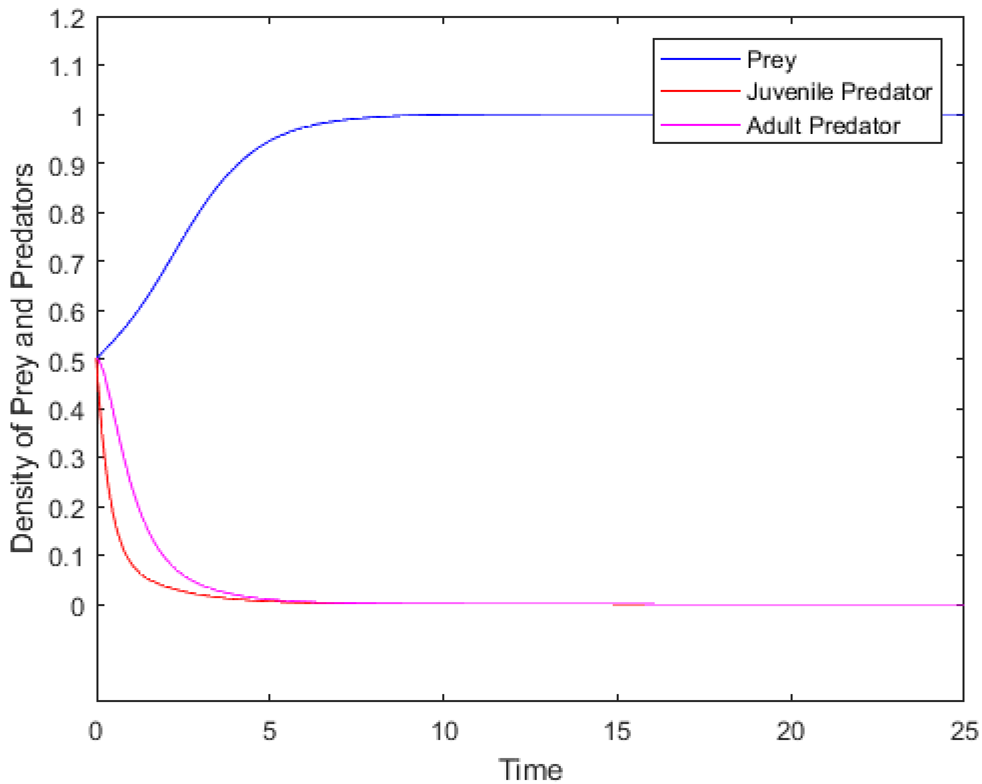

Figure 1 : Density of Prey and Predators with time for numerical simulation 1.

Numerical simulation 2

Case for $(U, V_1, V_2) = (0.90, 0.34, 0.11)$, where co-existence equilibrium point exists.

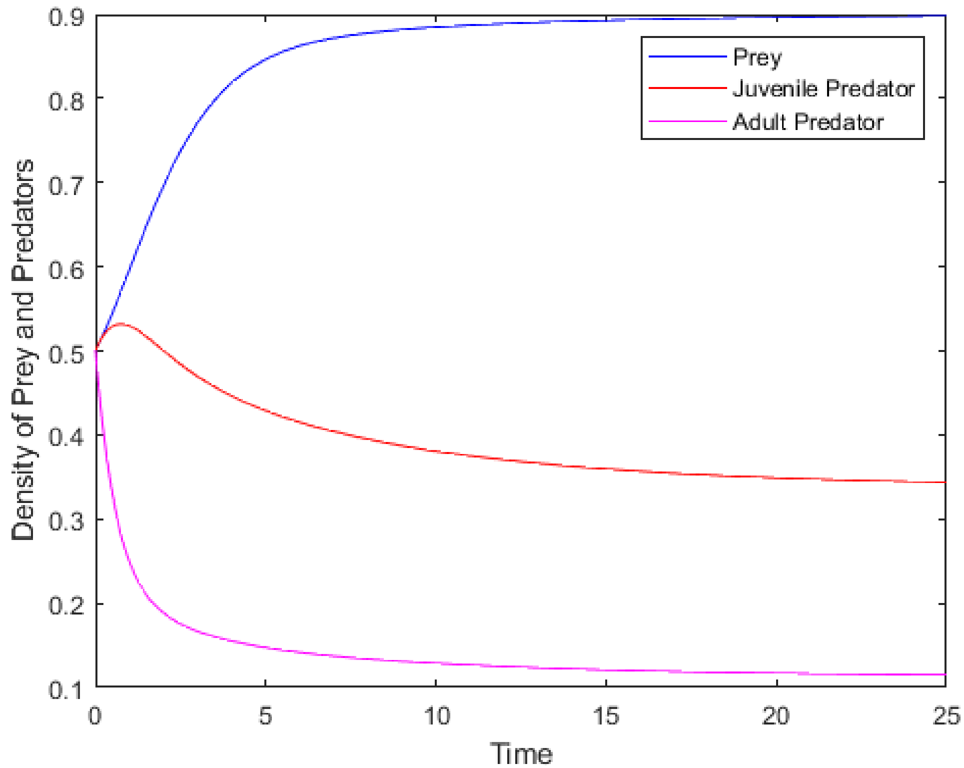

Figure 2 : Density of Prey and Predators with time for numerical simulation 2.

Figure 1 represent the graphs for results in numerical simulation 1 while figure 2 are represent the graphs for results in numerical simulation 2 . Based on figure 1, we can see that the population of mature and the juvenile predator go to zero, while the density of the prey goes to the carrying capacity 1. Based on figure 2, we can see that the population of mature and the juvenile predator go to 0.34,0.11 respectively, while the density of the prey goes to the 0.9. Thus this show that the co-existence equilibrium points is stable where the population of prey, population of juvenile predator and population of mature predator are stable and exists.

Figure 1 is the graph for the density of prey and predators with time for numerical simulation 1. Based on figure 1, we can observe that the number of prey was dropping at first, but it began to increase once the predator disappeared. After the predator has left, the prey's life returns to normal. Next, from figure 1 we can see that the population of juvenile predators is declining with time. Then, all at once, the number of juvenile predators go to zero. Lastly from the figure 1 , we can observe that the population of mature predators is steadily diminishing over time. Then the mature predator population go to zero.

Figure 2 shows the graph for the density of prey and predators with time for numerical simulation 2. Based on figure 2, we can see that the number of prey is diminishing with time. Following the absence of the predator, the development of prey increases. After the predator has left, the prey's life returns to normal. Then, in figure 2 ,we also can see that the population of juvenile predator increasing at first . Then it is decrease steeply over time and remain constant when the time is increasing. Therefore , in figure 2, we can observe that the population of mature predator is dwindling and approaching zero. The growth of mature predators is therefore steady and does not alter.

References

[1] Sundari, N. M. S., & Valliathal, M. (2018, December). Analysing a prey n predatormodel for stability with prey refuge-stage structure on predator. In J ournal of Physics: Conference Series (Vol. 1139, No. 1, p. 012016). IOP Publishing.

[2] Aziz-Alaoui, M. A., & Okiye, M. D. (2003). Boundedness and global stability for a predator-prey model with modified Leslie-Gower and Holling-type II schemes. Aplied Mathematics Letters, 16(7), 1069-1075.

[3] Azar, C., Holmberg, J., Lindgern, K. (1995): Stability Analysis of Harvesting in a

Predator-Prey, Model. Journal of Theoretical Biology 174, 13-19

[4] Bischi, G. I., Lamantia, F., & Radi, D. (2013). A predator-prey fishery model with endogenous switching of harvesting strategy. Applied Mathematics and Computation, 219(20), 10123–10142. https://doi.org/10.1016/j.amc.2013.02.061

[5] Ko, W., & Ryu, K. (2006). Qualitative analysis of a predator–prey model with Holling type II functional response incorporating a prey refuge. Journal of Differential Equations, 231(2), 534-550.

[6] Franco, G. (2019). A predator is an organism that eats another organism. Retrieved from https://necsi.edu/predator-prey-relationships