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On Algebraic Properties of Two–Dimensional *b*–Bistochastic Genetic Algebra

[1]Norhuda Afini Mohd Rosli and [2]Ahmad Fadillah Embong

[1,2]Department of Mathematical Sciences
Faculty of Science, Universiti Teknologi Malaysia,
81310 Johor Bahru, Johor, Malaysia.

e-mail: [1]hudaafini@gmail.com, [2]ahmadfadillah@utm.my

Abstract Genetic algebra is known as non-associative in general. In this paper, we consider genetic algebras induced by *b*-bistochastic Quadratic Stochastic Operators (QSOs) which are called *b*-bistochastic genetic algebras, limited on two-dimensional space. We study the condition of associativity on two-dimensional *b*-bistochastic genetic algebra. Moreover, full description of derivation of two-dimensional *b*-bistochastic genetic algebra are presented. It is well-known that commutative and associative algebras have only trivial derivations, thus, the existence of non-trivial derivations on such algebras is given.

Keywords *b*-Bistochastic; Genetic Algebra; Quadratic Stochastic Operator; Associative Algebra; Derivative Algebra

1 Introduction

Quadratic Stochastic Operators (QSOs) are an operator that early discovered from the problems of population genetics which can be traced back to Bernstein's work [1]. The QSOs were used to describe the distribution evolution of individual in a population like in populations genetic [2,3]. Besides, QSOs played an important role of analysis for the study of dynamics properties and models in different fields such as biology, physics, economics and mathematics.

Besides Bernstein's work, the applications of QSOs on population genetics were given by Lyubich [3]. The species time evolution can be identified by the following situation. Let $I = \{1,2,\dots,n\}$ be the n type of species in a population and we denote $x^{(0)} = \left(x_1^{(0)},\dots,x_n^{(0)}\right)$ as the probability distribution of the species in an early state of that population. The probability of an individual in the i^{th} species and j^{th} species to cross-fertilize and produce an individual from the k^{th} species denoted as $P_{ij,k}$. Given $x^{(0)} = \left(x_1^{(0)},\dots,x_n^{(0)}\right)$, we can find the probability distribution of the first generation, $x^{(1)} = \left(x_1^{(1)},\dots,x_n^{(1)}\right)$ by applying QSO as a total probability, i.e.

$$x_k^{(1)} = \sum_{i,j=1}^{n} P_{ij,k} x_i^{(0)} x_j^{(0)}, \qquad k \in \{1,2,\dots,n\}.$$

The operator above is denoted by the symbol V. This means that, starting from the initial arbitrary state of the probability distribution $x^{(0)}$ in a population, then it continues to evolve to the probability distribution of the first generation, $x^{(1)} = V\left(x^{(0)}\right)$, the second generation $x^{(2)} = V\left(x^{(1)}\right) = V\left(V\left(x^{(0)}\right)\right) = V^{(2)}\left(x^{(0)}\right)$, and so on. Thus, the probabilities distribution of the population can be described as follows:

$$x^{(0)}, \quad x^{(1)} = V\left(x^{(0)}\right), \quad x^{(2)} = V^{(2)}\left(x^{(0)}\right), \quad x^{(3)} = V^{(3)}\left(x^{(0)}\right), \cdots$$

In other words, each QSOs describe the evolution of generations in terms of probabilities distribution.

Furthermore, according to Ganikhodzhaev, Mukhamedov, Pirnapasov and Qaralleh [4], each QSO defines an algebraic structure on the vector space $\mathbb{R}^n$ containing the simplex. Such an algebra is called genetic algebra. Lyubich [5] states that it is known that QSO generated genetic algebra is commutative and non-associative in general. Note that for any algebra, the space of all derivations is a Lie algebra with the commutator multiplication. Particularly, the theory of non-associative algebras, in genetic algebra, the Lie algebra of derivations of a given algebra is one of the important tools for studying its structure.

Based on the previous studies, it is apparent that investigation of QSOs in general setting is challenging (unlike in case of linear operators), therefore the researchers are likely to introduce classes of QSOs such as Volterra-QSOs, *b*-bistochastic QSOs, doubly QSOs and separable QSOs. Genetic Volterra algebras were introduced and some of their algebraic properties were studied [4]. Recently, connections between the evolution algebras and the associated dynamical system have been made for the case of Volterra QSOs [6]. Motivated from those ideas, we are going to consider genetic algebras generated by *b*-bistochastic QSOs which is simply called *b*-bistochastic genetic algebras.

The properties of *b*-bistochastic QSO were studied in [7]. However, genetic algebras associated to these operators were not completely studied yet. In this work, we are limited ourselves to study on two-dimensional *b*-bistochastic genetic algebra. Specifically, we are going to describe the condition for associativity of such algebra. In general, genetic algebras generated by QSOs are commutative but non-associative. Based on the well-known Kadisons Theorem, it states that all derivations of associative and commutative algebras are trivial. Therefore, we are going to describe the derivation of *b*-bistochastic genetic algebra.

2 Preliminaries

This section introduces terms, definitions and theorems related to QSO and *b*-bistochastic operators.

2.1 Quadratic Stochastic Operator

Let V be a mapping on the $(n-1)$-dimensional simplex

$$S^{n-1} = \left\{\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n | x_i \geq 0, \sum_{i=1}^{n} x_i = 1\right\} \tag{1}$$

maps into itself, $V: S^{n-1} \longrightarrow S^{n-1}$. V has such a form

$$V: x_k' = \sum_{i,j=1}^{n} P_{ij,k} x_i x_j, \quad k = 1,2, \dots, n \tag{2}$$

where $P_{ij,k}$ are coefficient of heredity and

$$P_{ij,k} \geq 0, \qquad P_{ij,k} = P_{ji,k}, \qquad \sum_{k=1}^{n} P_{ij,k} = 1 \;, \qquad i,j,k = 1,2,\dots,n \tag{3}$$

Then, V is called Quadratic Stochastic Operator (QSO).

Other than that, QSO also being studied in terms of order. Hardy [8] introduced an order called x majorized by y, $(x \prec y)$. The order was used by to initiate a definition of bistochastic QSO in terms of classical majorization [9]. QSO is called bistochastic (also called doubly stochastic) if $V(x) \prec x$ for all x taken from the $(n-1)$-dimensional simplex. The necessary and sufficient conditions were given for the bistochasticity QSO in [10] and [11]. Moreover, besides these orders, there is another order that has been performed to introduce new operator, *b*-bistochastic operator.

2.2 *b*-Bistochastic Operator

A new order called *b*-order were introduced in [7]. They were motivated to use majorization that was introduced in Parker and Ram [12] to define a bistochasticity QSO with respect to the *b*-order and call it *b*-bistochastic QSO. They described several properties of the *b*-bistochastic QSO.

Definition 1 [7] Let V be the QSO given by equation (2) and S^{n-1} is a simplex given by equation (1). V is called *b*-bistochastic operator if

$$V(\mathbf{x}) \leq^{b} \mathbf{x} \quad \forall \mathbf{x} \in S^{n-1}$$

From equation (1), if $n = 2$, then the simplex reduces to

$$S^1 = \{\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 0,\ x_1 + x_2 = 1\}, \tag{4}$$

which is called 1-dimensional simplex. Furthermore, the QSOs V given by equation (2) on this simplex can be written as $V(\mathbf{x}) = (V(\mathbf{x})_1, V(\mathbf{x})_2)$ where,

$$V(\mathbf{x})_1 = P_{11,1}x_1^2 + 2P_{12,1}x_1x_2 + P_{22,1}x_2^2$$
$$V(\mathbf{x})_2 = P_{11,2}x_1^2 + 2P_{12,2}x_1x_2 + P_{22,2}x_2^2$$

Theorem 1 [7] Let V be a *b*-bistochastic QSO defined on S^{n-1}, then the following statements hold:

i. $P_{ij,k} = 0$ for all $i,j \in \{k+1,\dots,n\}$ where $k \in \{1,\dots,n-1\}$

ii. $P_{ij,k} \leq \frac{1}{2}$ for all $j \geq l+1,\ l \in \{1,\dots,n-1\}$

With this theorem, the condition for associativity of *b*-bistochastic genetic algebra can easily determine. In this research, we only involve for two-dimensional *b*-bistochastic genetic algebra. Hence, we have the properties

$$P_{12,1} \leq \frac{1}{2} \;,\;\; P_{22,1} = 0$$

2.2 Genetic Algebra

In mathematical genetics, genetic algebras are (possibly non-associative) used to model inheritance in genetic. Many authors have tried to investigate and study the difficult problem of classification of these algebras [13]. In mathematics, the algebras that occur in genetic (via gametic or zygotic) are very interesting structures. Reed [14] states that these algebras are not necessarily Lie or Jordan or any alternative algebra although they are generally commutative but non-associative.

2.2.1 Definitions on Some Properties of Genetic Algebra

Definition 2 The genetic algebra A is associative if satisfy the following condition:

$$\big((X \circ Y) \circ Z\big)_k = \big(X \circ (Y \circ Z)\big)_k \quad , \qquad k = 1, \dots, m$$

Definition 3 A derivation of the genetic algebra A is defined as a linear operator $D : A \to A$ satisfying

$$D(u, v) = D(u)v + uD(v)$$

for all $u, v \in E$, where E is algebra.

3 Associativity of Two-Dimensional *b*-Bistochastic Genetic Algebra

In this section, we are going to determine the condition for *b*-bistochastic genetic algebra to be associative on two-dimensional space, $\mathbb{R}^2$. Throughout this paper, we will let $P_{11,1} = a$ and $P_{12,1} = b$ for one-dimensional simplex. Due to conditions in equation (3), $P_{11,2} = 1 - a$ and $P_{12,2} = 1 - b$.

Based on equation (2), equation (3) and properties above, we can expand $(\mathbf{x} \circ \mathbf{y})$ given $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$ as:

$$\begin{aligned} \mathbf{x} \circ \mathbf{y} &= \left(\sum_{i,j=1}^{2} P_{ij,1} x_i y_j \ , \sum_{i,j=1}^{2} P_{ij,2} x_i y_j \right) \\ &= \big(P_{11,1} x_1 y_1 + P_{12,1}(x_1 y_2 + x_2 y_1)\ , P_{11,2} x_1 y_1 + P_{12,2}(x_1 y_2 + x_2 y_1) + x_2 y_2\big) \\ &= (a x_1 y_1 + b(x_1 y_2 + x_2 y_1), (1-a) x_1 y_1 + (1-b)(x_1 y_2 + x_2 y_1) + x_2 y_2) \end{aligned} \tag{5}$$

Corollary 1 From Theorem 1, we have the following table, where $e_1 = (1,0)$ and $e_2 = (0,1)$.

Table 1: Multiplication table ($\mathbb{R}^2$)

$\circ$	e_1	e_2
e_1	$(a\ , 1-a)$	$(b\ , 1-b)$
e_2	$(b\ , 1-b)$	$(0\ , 1)$

Proposition 1 Let $(A,\circ)$ be a genetic algebra and e_i be its standard basis. Then,

$$(\mathbf{x} \circ \mathbf{y}) = x_1 y_1 (e_1 \circ e_1) + x_1 y_2 (e_1 \circ e_2) + x_2 y_1 (e_2 \circ e_1) + x_2 y_2 (e_2 \circ e_2)$$

Proof First, let us compute the multiplication of **x** and **y** with respect to the operation, $\circ$, where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. We can see that

$$\begin{aligned} (\mathbf{x} \circ \mathbf{y}) &= \left(\sum_{i,j=1}^{2} P_{ij,1} x_i y_j \ , \sum_{i,j=1}^{2} P_{ij,2} x_i y_j \right) \\ &= \big(P_{11,1} x_1 y_1 + P_{12,1} x_1 y_2 + P_{21,1} x_2 y_1 + P_{22,1} x_2 y_2\ , P_{11,2} x_1 y_1 + P_{12,2} x_1 y_2 + P_{21,2} x_2 y_1 \\ &\qquad + P_{22,2} x_2 y_2\big) \\ &= x_1 y_1 \big(P_{11,1}\ , P_{11,2}\big) + x_1 y_2 \big(P_{12,1}\ , P_{12,2}\big) + x_2 y_1 \big(P_{21,1}\ , P_{21,2}\big) + x_2 y_2 \big(P_{22,1}\ , P_{22,2}\big) \\ &= x_1 y_1 (e_1 \circ e_1) + x_1 y_2 (e_1 \circ e_2) + x_2 y_1 (e_2 \circ e_1) + x_2 y_2 (e_2 \circ e_2) \end{aligned}$$

So, this proves the proposition.

Lemma 1 Let $(A,\circ)$ be a genetic algebra on $\mathbb{R}^2$ and e_i be its standard basis. A is associative if and only if $(e_i \circ e_j) \circ e_k = e_i \circ (e_j \circ e_k)$ for all $i, j, k \in I_2$.

Proof ($\Rightarrow$) If $(A,\circ)$ is associative, then $(e_i \circ e_j) \circ e_k = e_i \circ (e_j \circ e_k)$ for all $i, j, k \in I_2$.

($\Leftarrow$) Assume $(e_i \circ e_j) \circ e_k = e_i \circ (e_j \circ e_k)$ for all $i, j, k \in I_2$. Let $\mathbf{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$ and $\mathbf{z} = (z_1, z_2)$ where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^2$. From $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$, we have $\boldsymbol{x} = x_1e_1 + x_2e_2$, $\boldsymbol{y} = y_1e_1 + y_2e_2$ and $\boldsymbol{z} = z_1e_1 + z_2e_2$. Now, from Proposition 1, we get

$$\begin{aligned}(\mathbf{x} \circ \boldsymbol{y}) \circ \mathbf{z} &= \big((x_1e_1 + x_2e_2) \circ (y_1e_1 + y_2e_2)\big) \circ (z_1e_1 + z_2e_2) \\ &= \big(x_1y_1(e_1 \circ e_1) + x_1y_2(e_1 \circ e_2) + x_2y_1(e_2 \circ e_1) + x_2y_2(e_2 \circ e_2)\big) \\ &\qquad \circ (z_1e_1 + z_2e_2) \\ &= x_1y_1z_1(e_1 \circ e_1) \circ e_1 + x_1y_2z_1(e_1 \circ e_2) \circ e_1 + x_2y_1z_1(e_2 \circ e_1) \circ e_1 \\ &\qquad + x_2y_2z_1(e_2 \circ e_2) \circ e_1 + x_1y_1z_2(e_1 \circ e_1) \circ e_2 + x_1y_2z_2(e_1 \circ e_2) \circ e_2 \\ &\qquad + x_2y_1z_2(e_2 \circ e_1) \circ e_2 + x_2y_2z_2(e_2 \circ e_2) \circ e_2\end{aligned}$$

Due to early assumption, we can rewrite the last expression in the following form

$$\begin{aligned}(\mathbf{x} \circ \mathbf{y}) \circ \mathbf{z} &= x_1y_1z_1e_1 \circ (e_1 \circ e_1) + x_1y_2z_1e_1 \circ (e_2 \circ e_1) + x_2y_1z_1e_2 \circ (e_1 \circ e_1) \\ &\qquad + x_2y_2z_1e_2 \circ (e_2 \circ e_1) + x_1y_1z_2e_1 \circ (e_1 \circ e_2) + x_1y_2z_2e_1 \circ (e_2 \circ e_2) \\ &\qquad + x_2y_1z_2e_2 \circ (e_1 \circ e_2) + x_2y_2z_2e_2 \circ (e_2 \circ e_2) \\ &= x_1e_1 \circ y_1z_1(e_1 \circ e_1) + x_1e_1 \circ y_2z_1(e_2 \circ e_1) + x_2e_2 \circ y_1z_1(e_1 \circ e_1) + x_2e_2 \\ &\qquad \circ y_2z_1(e_2 \circ e_1) + x_1e_1 \circ y_1z_2(e_1 \circ e_2) + x_1e_1 \circ y_2z_2(e_2 \circ e_2) + x_2e_2 \\ &\qquad \circ y_1z_2(e_1 \circ e_2) + x_2e_2 \circ y_2z_2(e_2 \circ e_2) \\ &= (x_1e_1 + x_2e_2) \\ &\qquad \circ \big(y_1z_1(e_1 \circ e_1) + y_2z_1(e_2 \circ e_1) + y_1z_2(e_1 \circ e_2) + y_2z_2(e_2 \circ e_2)\big) \\ &= (x_1e_1 + x_2e_2) \circ \big((y_1e_1 + y_2e_2) \circ (z_1e_1 + z_2e_2)\big) \\ &= \mathbf{x} \circ (\mathbf{y} \circ \mathbf{z})\end{aligned}$$

We show that $(\mathbf{x} \circ \mathbf{y}) \circ \mathbf{z} = \mathbf{x} \circ (\mathbf{y} \circ \mathbf{z})$. Hence, A is associative. This completes the proof.

Now the next theorem will describe the condition for associativity of two-dimensional b-bistochastic genetic algebra.

Theorem 2 Let V be a b-bistochastic genetic algebra on $\mathbb{R}^2$. Then, V is associative if and only if $b = 0$.

Proof Let V be a b-bistochastic genetic algebra on $\mathbb{R}^2$

($\Rightarrow$) Assume V is associative. By Lemma 1, $(e_i \circ e_j) \circ e_k = e_i \circ (e_j \circ e_k)$ for all $i, j, k \in I_2$. Therefore, according to Table 1, we can have

$$(e_1 \circ e_1) \circ e_2 = e_1 \circ (e_1 \circ e_2) \Rightarrow (ab, 1 - ab) = (ab + b(1 - b), (1 - a)b + (1 - b)^2)$$

Solving the equality above, we get the following equation

$$b(1 - b) = 0$$

where $b = 0$ and $b = 1$. Since $b \leq \frac{1}{2}$, so we have $b = 0$. This is satisfied for all $(e_i \circ e_j) \circ e_k = e_i \circ (e_j \circ e_k)\ \forall\, i, j, k \in I_2$.

($\Leftarrow$) Assume $b = 0$ and let $\mathbf{x} = (x_1, x_2)$, $\boldsymbol{y} = (y_1, y_2)$ and $\boldsymbol{z} = (z_1, z_2)$ where $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.

$$\begin{aligned}\mathbf{x} \circ \mathbf{y} &= (ax_1y_1\,,\ (1 - a)x_1y_1 + x_1y_2 + x_2y_1 + x_2y_2) \\ &= \big(ax_1y_1\,,\ (1 - a)x_1y_1 + x_1y_2 + x_2(y_1 + y_2)\big)\end{aligned}$$

$$\begin{aligned}\mathbf{y} \circ \mathbf{z} &= (ay_1z_1\,,\ (1 - a)y_1z_1 + y_1z_2 + y_2z_1 + y_2z_2) \\ &= \big(ay_1z_1\,,\ (1 - a)y_1z_1 + y_1z_2 + y_2(z_1 + z_2)\big)\end{aligned}$$

$$
\begin{aligned}
(\mathbf{x} \circ \mathbf{y}) \circ \mathbf{z} = \Big(&a(ax_1y_1)z_1\,,\ (1-a)(ax_1y_1)z_1 + (ax_1y_1)z_2 \\
&+ \big[(1-a)x_1y_1 + x_{1y_2} + x_2y_1 + x_{2y_2}\big](z_1+z_2)\Big) \\
= {}& (a^2x_1y_1z_1\,,\ (1-a)ax_1y_1z_1 + ax_1y_1z_2 + (1-a)x_1y_1z_1 + x_1y_2z_1 + x_2y_1z_1 \\
&+ x_2y_2z_1 + (1-a)x_1y_1z_2 + x_1y_2z_2 + x_2y_1z_2 + x_2y_2z_2) \\
= {}& (a^2x_1y_1z_1\,,\ (1-a)ax_1y_1z_1 + x_1y_1z_2 + (1-a)x_1y_1z_1 + x_1y_2z_1 + ax_2y_1z_1 \\
&+ (1-a)x_2y_1z_1 + x_2y_2z_1 + x_1y_2z_2 + x_2y_1z_2 + x_2y_2z_2) \\
= {}& \big(ax_1(ay_1z_1)\,,\ (1-a)x_1(ay_1z_1) \\
&+ (x_1+x_2)[(1-a)y_1z_1 + y_1z_2 + y_2z_1 + y_2z_2] + x\_2(ay_1z_1)\big) \\
= {}& \mathbf{x} \circ (\mathbf{y} \circ \mathbf{z})
\end{aligned}
$$

This shows that V is associative. Therefore, this proves the theorem.

4 Derivation of Two-Dimensional *b*-Bistochastic Genetic Algebra

This section will describe the derivation of two-dimensional *b*-bistochastic genetic algebra. Let $(A,\circ)$ be a genetic algebra on $\mathbb{R}^2$. A linear operator mapping $D: A \to A$ given by

$$D(e_i) = \sum_{j=1}^{n} d_{ij} e_j \tag{6}$$

is called derivation if it satisfies

$$D(\mathbf{x} \circ \mathbf{y}) = D(\mathbf{x}) \circ \mathbf{y} + \mathbf{x} \circ D(\mathbf{y}), \forall\, \mathbf{x}, \mathbf{y} \in A \tag{7}$$

Moreover, for any $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ in *b*-bistochastic genetic algebra one has

$$
\begin{aligned}
\mathbf{x} \circ (\mathbf{y} + \mathbf{z}) &= \sum_{i,j=1}^{n} P_{ij,k} x_i (y_j + z_j) \\
&= \sum_{i,j=1}^{n} P_{ij,k} (x_i y_j + x_i z_j) \\
&= \sum_{i,j=1}^{n} P_{ij,k} x_i y_j + \sum_{i,j=1}^{n} P_{ij,k} x_i z_j \\
&= \mathbf{x} \circ \mathbf{y} + \mathbf{x} \circ \mathbf{z}
\end{aligned} \tag{8}
$$

Therefore, this algebra satisfies distributive law.

Throughout this section, we will only consider our *b*-bistochastic on two-dimensional genetic algebra that is $n = 2$.

Lemma 2 The linear operator D given by (6) is derivative if and only if $D(e_i \circ e_j) = D(e_i) \circ e_j + e_i \circ D(e_j)$ for all $i, j = 1,2,\dots,n$

Proof (⇒) If D is derivative, then obviously $D(e_i \circ e_j) = D(e_i) \circ e_j + e_i \circ D(e_j)$.

(⇐) Assume $D(e_i \circ e_j) = D(e_i) \circ e_j + e_i \circ D(e_j)$ for all $i, j = 1,2$. Let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$. By Proposition 1, we have

$$(\mathbf{x} \circ \mathbf{y}) = x_1y_1(e_1 \circ e_1) + x_1y_2(e_1 \circ e_2) + x_2y_1(e_2 \circ e_1) + x_2y_2(e_2 \circ e_2)$$

Then, the derivation of $(\mathbf{x} \circ \mathbf{y})$ is

$$
\begin{aligned}
D(\mathbf{x} \circ \mathbf{y}) &= D\big(x_1y_1(e_1 \circ e_1) + x_1y_2(e_1 \circ e_2) + x_2y_1(e_2 \circ e_1) + x_2y_2(e_2 \circ e_2)\big) \\
&= x_1y_1D(e_1 \circ e_1) + x_1y_2D(e_1 \circ e_2) + x_2y_1D(e_2 \circ e_1) + x_2y_2D(e_2 \circ e_2)
\end{aligned}
$$

$$
\begin{aligned}
&= x_1y_1[D(e_1)\circ e_1 + e_1\circ D(e_1)] + x_1y_2[D(e_1)\circ e_2 + e_1\circ D(e_2)] \\
&\qquad + x_2y_1[D(e_2)\circ e_1 + e_2\circ D(e_1)] + x_2y_2[D(e_2)\circ\ e_2 + e_2\circ D(e_2)] \\
&= x_1y_1[D(e_1)\circ e_1] + x_1y_2[D(e_1)\circ e_2] + x_2y_1[D(e_2)\circ e_1] + x_2y_2[D(e_2)\circ e_2] \\
&\qquad + x_1y_1[e_1\circ D(e_1)] + x_1y_2[e_1\circ D(e_2)] + x_2y_1[e_2\circ D(e_1)] \\
&\qquad + x_2y_2[e_2\circ D(e_2)] \\
&= x_1[y_1(D(e_1)\circ e_1) + y_2(D(e_1)\circ e_2)] + x_2[y_1(D(e_2)\circ e_1) + y_2(D(e_2)\circ e_2)] \\
&\qquad + y_1\big[x_1\big(e_1\circ D(e_1)\big) + x_2\big(e_2\circ D(e_1)\big)\big] \\
&\qquad + y_2\big[x_1\big(e_1\circ D(e_2)\big) + x_2\big(e_2\circ D(e_2)\big)\big] \\
&= x_1D(e_1)\circ y_1e_1 + x_1D(e_1)\circ y_2e_2 + x_2D(e_2)\circ y_1e_1 + x_2D(e_2)\circ y_2e_2 + x_1e_1 \\
&\qquad \circ y_1D(e_1) + x_2e_2\circ y_1D(e_1) + x_1e_1\circ y_2D(e_2) + x_2e_2\circ\ y_2D(e_2) \\
&= D(x_1e_1)\circ y_1e_1 + D(x_1e_1)\circ y_2e_2 + D(x_2e_2)\circ y_1e_1 + D(x_2e_2)\circ y_2e_2 + x_1e_1 \\
&\qquad \circ D(y_1e_1) + x_2e_2\circ D(y_1e_1) + x_1e_1\circ D(y_2e_2)\ + x_2e_2\circ D(y_2e_2) \\
&= D(x_1e_1 + x_2e_2)\circ(y_1e_1 + y_2e_2) + (x_1e_1 + x_2e_2)\circ D(y_1e_1 + y_2e_2) \\
&= D(x_1, x_2)\circ(y_1, y_2) + (x_1, x_2)\circ\ D(y_1, y_2) \\
&= D(\mathbf{x})\circ\mathbf{y} + \mathbf{x}\circ D(\boldsymbol{z})
\end{aligned}
$$

This finishes the proof.

Theorem 3 Let A be a two-dimensional b-bistochastic genetic algebra. Then the following statements holds:

(i) If $b < \frac{1}{2}$ and $b \neq \frac{a}{2}$, then all derivatives are trivial.

(ii) If $b = \frac{a}{2}$ for any $a \in [0,1)$, then the derivative has the following form:

$$D(\mathbf{x}) = x_1t(e_2 - e_1) \text{ for any } t \in \mathbb{R}$$

(iii) If $b = \frac{1}{2}$ and $a = 1$, then the derivative has the following form:

$$D(\mathbf{x}) = \mathrm{e}_1(sx_2 - tx_1) + e_2(tx_1 - sx_2) \text{ for any } s, t \in \mathbb{R}$$

Proof Let D be a derivative for two-dimensional b-bistochastic genetic algebra. By Lemma 2, D is derivative if and only if $D\big(e_i\circ e_j\big) = D(e_i)\circ e_j + e_i\circ D\big(e_j\big)$ for all $i, j = 1,2$. Therefore, we obtain a system of equations as follow:

$$ad_{11} + 2bd_{12} - (1-a)d_{21} = 0 \quad (9)$$

$$2(1-a)d_{11} + [2(1-b) - a]d_{12} - (1-a)d_{22} = 0 \quad (10)$$

$$(1-b-a)d_{21} - bd_{22} = 0 \quad (11)$$

$$(1-b)d_{11} + (1-b)d_{12} + (1-a)d_{21} = 0 \quad (12)$$

$$(2b-1)d_{21} = 0 \quad (13)$$

$$2(1-b)d_{21} + d_{22} = 0 \quad (14)$$

Recall from Theorem 1, where $b \leq \frac{1}{2}$. So, we divide equation (13) into two cases:

Case I: Let $b < \frac{1}{2}$, then $d_{21} = 0$. Substitute $d_{21} = 0$ into equation (14) implies $d_{22} = 0$. Then, substitute $d_{21} = 0$ and $d_{22} = 0$ into the system of equations above and get:

$$ad_{11} + 2bd_{12} = 0 \quad (15)$$

$$(2-a)(d_{11} + d_{12}) = 0 \quad (16)$$

$$(1-b)(d_{11} + d_{12}) = 0 \quad (17)$$

Since $a \neq 2$, then $d_{12} = -d_{11}$ due to (16). So (15) will be $(a-2b)d_{11} = 0$. Therefore, we divide (15) into two parts:

i. Let $b \neq \frac{a}{2}$, then $d_{11} = d_{12} = d_{21} = d_{22} = 0$. The derivative D is trivial, hence proves (i).

ii. Let $b = \frac{a}{2}$, then d_{11} become a free variable. Let $d_{12} = t \in \mathbb{R}$. From previous, we have $d_{21} = d_{22} = 0$ and $d_{12} = d_{11}$. Therefore, from the definition of derivation D,

$$\begin{aligned} D(e_1) &= d_{11}e_1 + d_{12}e_2 \\ &= -d_{12}e_1 + d_{12}e_2 \\ &= -te_1 + te_2 \\ &= t(e_2 - e_1) \\ D(e_2) &= d_{21}e_1 + d_{22}e_2 = 0 \end{aligned}$$

Then let a variable $\mathbf{x} = (x_1 x_2) = x_1e_1 + x_2e_2$ be any variable. Hence,

$$\begin{aligned} D(\mathbf{x}) &= D(x_1e_1 + x_2e_2) \\ &= x_1D(e_1) + x_2D(e_2) \\ &= x_1D(e_1) \\ &= x_1t(e_2 - e_1) \end{aligned}$$

This proves (ii).

Case II: Let $b = \frac{1}{2}$. the system of equations will reduce to:

$$ad_{11} + bd_{12} - (1-a)d_{21} = 0 \quad (18)$$
$$(1-a)(2d_{11} + d_{12} - d_{22}) = 0 \quad (19)$$
$$\frac{1}{2}(d_{21} - d_{22}) - ad_{21} = 0 \quad (20)$$
$$\frac{1}{2}(d_{11} + d_{12}) + (1-a)d_{21} = 0 \quad (21)$$
$$d_{21} + d_{22} = 0 \quad (22)$$

Due to equation (22), $d_{21} = -d_{22}$. The system of equations become:

$$ad_{11} + bd_{12} - (1-a)d_{21} = 0 \quad (23)$$
$$(1-a)(2d_{11} + d_{12} + d_{21}) = 0 \quad (24)$$
$$(1-a)d_{21} = 0 \quad (25)$$
$$\frac{1}{2}(d_{11} + d_{12}) + (1-a)d_{21} = 0 \quad (26)$$

Refer to equation (25), we can divide into 2 parts:

i. Let $a \neq 1$, thus $d_{21} = 0$. From equation (22) one has $d_{11} = -d_{12}$, together with equation (23) produce $d_{11} = d_{12} = 0$. Therefore, $d_{11} = d_{12} = d_{22} = 0$, the derivation D is trivial.

ii. Let $a = 1$. Due to (23), $d_{11} = -d_{12}$. Let $d_{12} = t = -d_{11}$ and $d_{21} = s = -d_{22}$ such that $t, s \in \mathbb{R}$. From the definition of derivation D,

$$D(e_1) = -d_{12}e_1 + d_{12}e_2 = t(e_2 - e_1)$$
$$D(e_2) = d_{21}e_1 - d_{21}e_2 = s(e_1 - e_2)$$

Then,

$$\begin{aligned} D(\mathbf{x}) &= D(x_1e_1 + x_2e_2) \\ &= x_1D(e_1) + x_2D(e_2) \\ &= x_1t(e_2 - e_1) + x_2s(e_1 - e_2) \\ &= e_1(x_2s - x_1t) + e_2(x_1t - x_2s) \end{aligned}$$

which proves (iii). Thus, this completes the proof.

5 Conclusion

This paper study the associativity and derivation of two-dimensional *b*-bistochastic genetic algebra. The condition for two-dimensional *b*-bistochastic genetic algebra to be associative are determined. Furthermore, the trivial and non-trivial derivations of two-dimensional *b*-bistochastic genetic algebra are also fully described.

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