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Mathematical Modelling of COVID-19 in Malaysia using Riccati Equation

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Abstract This research focuses on the study of mathematical modelling of COVID-19 in Malaysia using Riccati equation. The graphs of actual active cases and predict active cases in Malaysia from 17 March 2020 until 21 April 2021 are presented. The logistic growth models are constructed using Riccati equation to predict the pattern of active cases. MATLAB software is used to solve the equation numerically and plot the data and results. By numerical formulation, the result is studied to understand how well the model works. The piecewise model from combination of three models is also presented.

Keywords COVID-19; mathematical modelling; Riccati equation

1 Introduction

SARS-CoV-2 is a virus that causes COVID-19, a serious and life-threatening disease. On 11 March 2020, the epidemic was designated a global pandemic and by 12 June 2021, there had been a total of 176 102 484 positive cases with 3 802 121 deaths. A total of 652 204 cases and 3 844 related deaths were reported across the Malaysia as of 12 June 2021 [1]. Presently, because of the high rate of infection propagation and spread, COVID-19 is of great concern to researchers, governments, and all individuals. The social and economic ravages of COVID-19 worldwide have immediately motivated the use of mathematical models to understand the epidemic's path. Mathematical models in epidemiology have long developed quantitative information and offered useful guidance for disease management and policy development [2].

These mathematical models, however, have some limitations, as the main challenges are the translation of observations. In formulations in mathematics, if this conversion is inefficient, then deficiencies will be given by mathematical models with some predictions. There may then be more than one mathematical model of a particular real-world problem, each model with its advantages and disadvantages [3].

Many academics have produced a variety of predicting methods for COVID-19 trend predictions in some severe countries and around the world, discussing mathematical models, infectious disease models, and artificial intelligence models. The prediction of time series of epidemic development has been done using models based on mathematical statistics, machine learning, and deep learning [4].

The population of growth rate can make more specific form of it to describe two different kinds of growth models which are exponential and logistic. However, if there are few people and many resources, exponential growth will happen for a while. But when the number of individuals gets large enough, resources start to get used up, slowing the growth rate. In this study, the number of new cases who infected with COVID-19 were large and not suitable to use exponential growth. At the same time, the number of infected cases were not always increase since there will be the time where the number of infected decrease either because immunity or protective measures inhibiting the case growth.

The research carried out specifically studied the mathematical modelling of COVID-19 in Malaysia using Riccati equation which include logistic growth with parameters. In previous study, Pelinovsky et al. [5] has determined that logistic growth model is commonly used to interpret the COVID-19 epidemic data. This study focuses on mathematical model that predict the pattern of COVID-19 number of active cases (in treatment) in Malaysia. Besides that, this study investigates whether the use of Riccati equations could accurately fit the actual cases and determine a better piecewise model from combination of two or more models.

2 Literature Review

2.1 Coronavirus Disease 2019

The world is currently dealing with continuing outbreaks of coronavirus disease, namely, Coronavirus Disease 2019 (COVID-19) caused by the novel coronavirus, SARS-COV2, is extremely virulent virus that caused COVID-19 to be a fatal disease and directed at the human respiratory system [6]. By March 2020, the World Health Organization (WHO) proclaimed the situation to be a pandemic, the first of its kind in our century. To date, several countries and regions have been locked down and have applied strict social distance measures to avoid the spread of the virus. From here, the strategic and health care management viewpoint, the distribution pattern of the disease and the estimation of its spread over time are of great importance for saving lives and mitigating the social and economic effects of the disease [3].

2.2 Mathematical Models of COVID-19

Abusam et al. [7] described that mathematical modelling is the only way to predict the occurrence of epidemics that have no known cure, such as COVID-19 and agreeing on the efficacy and accessibility of the intervention steps to be imposed. It can be used to help health agencies in determining what control steps to be taken and how to distribute limited health resources accordingly.

According to Mishra et al. [3], mathematical models have predicted a variety of circumstances that are likely to arise in the near future, such as the second wave spread across Europe. There is no question that mathematical models, while not all reliable, are capable of helping people to see what could happen in the near future.

2.2.1 Logistic Growth Model

In a previous study, Wu et al. [8] calibrated four models to the reported number of infected cases for the whole of China, 29 provinces in China, and 33 countries and regions.
The generalized growth model (GGM):

$$\frac{dC(t)}{dt} = rC^p(t),$$

where $C(t)$ represents the cumulative number of confirmed cases at the time t, $p \in [0,1]$ is an exponent that allows the model to capture different growth profiles. In the case of exponential growth, the solution is $C(t) = C_0 e^{rt}$, where r is the growth rate and C_0 is the initial number of confirmed cases at the time when the count starts.

However, the outbreak is slowing down and approaching its limit at the end of the decaying transmission rate. As a consequence, the growth trend departs from the sub-exponential path as the cumulative number cases are reaching the point of inflection and daily incidence curve is approaching its limit. Then, logistic type model could have improved efficiency. The exponential and classical logistic models are, in reality, first- and second-order approximations to the growth phase of an epidemic curve formed by the typical Kermack–McKendrick SIR model, respectively. There are three types of logistic models used to describe the outbreak beyond the early stage of growth:

- Classical Logistic growing model:

$$\frac{dC(t)}{dt} = rC(t)\left(1 - \frac{C(t)}{K}\right).$$

- Generalized Logistic Model (GLM):

$$\frac{dC(t)}{dt} = rC^p\left(1 - \frac{C(t)}{K}\right),$$

  where one additional parameter $p \in [0,1]$ is introduced on top of the classical logistic model to capture different growth profiles.

- Generalized Richards model (GRM):

$$\frac{dC(t)}{dt} = r[C(t)]^p(1 - (\frac{C(t)}{K})^{\alpha},$$

  where the exponent α is introduced to measure the deviation from the symmetric S-shaped dynamics of the simple logistic curve.

All three models include two parameters, that is the generalised rate of growth r sets the typical time scale of the epidemic growth process and the final capacity K, which is the asymptotic total number of infections throughout the epidemic.

The results of the modelling clearly indicate longer after-peak trajectories in Western countries, as opposed to most provinces in China, where the after-peak trajectory is marked by much faster decay.

2.3 Riccati equation

Svirin [9] described that one of the most interesting first-order nonlinear differential equations is the Riccati equation. The non-linear equation Riccati equation can always be reduced to a second order linear ordinary differential equation (ODE). It is written in the form

$$y' = q_0(x) + q_1(x)y + q_2(x)y^2, \tag{2.1}$$

then, wherever q_2 is non-zero and differentiable, $v = yq_2$ satisfies a Riccati equation of the form

$$v' = v^2 + R(x)v + S(x),$$

where $S = q_2 q_0$ and $R = q_1 + \frac{q_2'}{q_2}$, because

$$v' = (yq_2)' = y'q_1 + yq_2'$$
$$= (q_0 + q_1 y + q_2 y^2)q_2 + v\frac{q_2{}'}{q_2} = q_0 q_2 + \left(q_1 + \frac{q_2'}{q_2}\right)v + v^2.$$

Substituting $v = -\frac{u''}{u}$, it follows that u satisfies the linear second order ODE

$$u'' - R(x)u' + S(x)u = 0,$$

since

$$v' = -\left(\frac{u'}{u}\right)' = -\left(\frac{u''}{u}\right) + \left(\frac{u'}{u}\right)^2 = -\left(\frac{u''}{u}\right) + v^2,$$

so that

$$\frac{u''}{u} = v^2 - v' = -S - Rv = -S + R\frac{u'}{u}.$$

Hence

$$u'' - Ru' + Su = 0.$$

A solution of this equation (2.1) will lead to a solution $y = -\frac{u'}{q_2 u}$ of the original Riccati equation [10].

3 Mathematical Modelling of COVID-19 cases in Malaysia using Riccati Equation

3.1 Model Formulation

The mathematical model formulation of cases in Malaysia was separated into three phases which are Phase 1 until Phase 7, Phase 7 until Phase 10 and Phase 10 until Phase 17.
Define the following variables and parameters:

t = time (day),
$b(t)$ = number of new cases at time t,
$r(t)$ = number of patients treated (active) at time t,
$p(t)$ = number of patients cured at time t,
$d(t)$ = number of patients died at time t.

The plot of the four variables based on the data given from phase 1 until phase 17 are shown in Figure 3.1

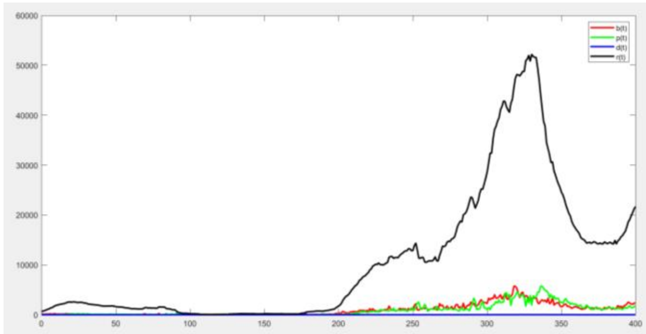


Figure 3.1: Data plots (17 March 2020 until 21 April 2021)

Generally, from the balance law,

rate of change = (rate in) – (rate out).

Thus,

rate of change of patients under treatment

= (rate of new cases) – (rate of cured) – (rate of deaths)

$$\frac{dr}{dt} = \frac{\text{number of new cases}}{\text{number of days}} - \frac{\text{number of cured}}{\text{number of days}} - \frac{\text{number of deaths}}{\text{number of days}}$$

The graph of $b(t)$ from the plot shown in Figure 3.1 is almost horizontal. Thus,

$$\text{Average rate of new cases} = \frac{\text{Total number of new cases}}{\text{number of days}} = \frac{1}{95}\sum_{i=0}^{94} b(i) = 89.8421.$$

Therefore,

$$\frac{dr}{dt} = c - (\text{rate of cured}) - (\text{rate of deaths}),$$

where $c = 89.8421$. But the rates of cured and deaths are proportional to the number patients under treatment. Hence

$$\frac{dr}{dt} = c - \alpha r(t) - kr(t),$$

where α and k are the cured rate and death rate coefficients respectively. Furthermore, observe that

$$k = \text{rate of deaths} = \frac{\text{number of deaths}}{\text{number of days}} = \frac{\frac{\text{number of deaths}}{\text{number of cured}}\ (\text{number of cured})}{\text{number of days}}$$

$$= \frac{\frac{\text{number of deaths/day}}{\text{number of cured/day}}\ (\text{fraction of } r(t))}{\text{number of days}}$$

$$= \beta\frac{d(t)}{p(t)} r(t),$$

where β is some constant. Thus k depends on $r(t)$. Summarizing,

$$\frac{dr}{dt} = c - \alpha r(t) - \beta \frac{d(t)}{p(t)} \big(r(t)\big)^2, \tag{3.1}$$

which is a well-known Riccati equation.
Generally, the solution of the Riccati equation (as discussed in Section 2)

$$y' = q_0(t) + q_1(t)y + q_2(t)y^2$$

is

$$y = -\frac{u'(t)}{q_2(t)u(t)},$$

where $u(t)$ solves the ordinary differential equation (ODE)

$$u''(t) - R(t)u'(t) + S(t)u(t) = 0,$$

with

$$S(t) = q_2(t)q_0(t) \text{ and } R(t) = q_1(t) + \frac{q_2'(t)}{q_2(t)}. \tag{3.2}$$

The solution of equation (3.1) using the equation (3.2) is

$$r(t) = \frac{u'(t)}{\beta(d(t)/p(t))u(t)}, \tag{3.3}$$

where $u(t)$ solves the ordinary differential equation

$$u''(t) - R(t)u'(t) + S(t)u(t) = 0, \tag{3.4}$$

with

$$R(t) = -\alpha + q'(t)/q(t), \qquad S(t) = cq(t), \qquad q(t) = -\beta\, d(t)/p(t). \tag{3.5}$$

Observe that $q(t)$ depends on the ratio $d(t)/p(t)$ which is the ratio of death cases over cured cases at time t.

Since the ratio $d(t)/p(t)$ shows a decreasing trend, the ratio $d(t)/p(t)$ is approximated by a curve

$$f(t) = \frac{A}{1+Bt}. \tag{3.6}$$

By substituting $d(t)/p(t)$ at selected $t = 0$ and $t = 19$, the solutions are $A = \frac{2}{7}$ and $B = 0.3$. Then, substituting into (3.6) gives

$$f(t) = \frac{2}{7(1 + 0.3t)}.$$

Figure 3.2 below shows the plot of $f(t)$ together with the plot of ratio $d(t)/p(t)$.

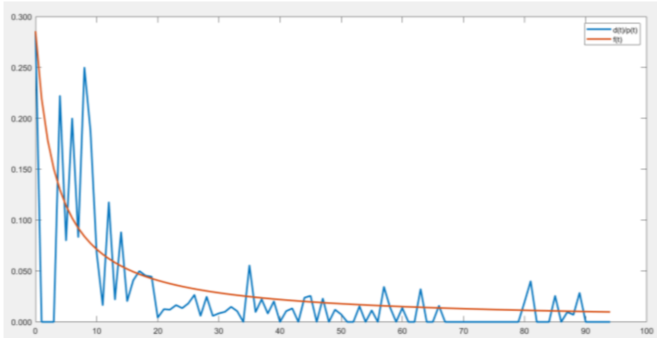


Figure 3.2: The curve $f(t)$ and the ratio $d(t)/p(t)$

With the approximation, the equations in (3.5) become

$$q(t) = -\frac{2\beta}{7(1+0.3t)}, \quad R(t) = -\alpha - \frac{0.3}{1+0.3t}, \quad S(t) = -\frac{2\beta c}{7(1+0.3t)}, \tag{3.7}$$

where $c = 89.8421$. Substituting these equation (3.7) into equation (3.4) yields

$$u''(t) + \left(\alpha + \frac{0.3}{1+0.3t}\right)u'(t) - \frac{179.6842\beta}{7(1+0.3t)}u(t) = 0. \tag{3.8}$$

This shows that $t = -\frac{1}{0.3} < 0$ is a regular singular point of ODE in equation (3.8). Rewrite the equation (3.8) as

$$(1 + 0.3t)u''(t) + (0.3 + (1 + 0.3t)\alpha)u'(t) - \frac{179.6842\beta u(t)}{7} = 0. \tag{3.9}$$

Because $t = 0$ is an ordinary point of equation (3.8), the solution $u(t)$ has a Taylor series expansion about $t = 0$. For simplicity, assume

$$u(t) = u_0 + u_1 t + u_2 t^2. \tag{3.10}$$

Hence the function $r(t)$ in equation (3.3) has the form of a rational function

$$r(t) = \frac{7(1+0.3t)}{2\beta}\frac{(u_1+2u_2 t)}{(u_0+u_1 t+u_2 t^2)}, \tag{3.11}$$

which is equivalent to

$$r(t) = \frac{7u_1(1+0.3t)}{2\beta u_0}\frac{(1+2u_2/u_1 t)}{(1+u_1 t/u_0+u_2 t^2/u_0)}. \tag{3.12}$$

Setting $t = 0$ yields

$$r(0) = \frac{7u_1}{2\beta u_0} = 622.$$

After some relabelling, (3.12) reduces to

$$r(t) = 621(1 + 0.3t)\frac{(1+B_1 t)}{(1+b_1 t+b_2 t^2)}, \tag{3.13}$$

where the constants B_1, B_2, b_2 are to be determined. These three unknowns can be estimated from the given data for $r(t)$ at selected $t = 19, 30, 46$. By substituting $r(t)$ at selected $t = 19, 30, 46$ into equation (3.13) with the help of MATLAB, the solutions are

$$B_1 = -0.00259293, \qquad b_1 = -0.00851151, \qquad b_2 = 0.00190567.$$

Substituting these values into equation (3.13),

$$r(t) = 621(1 + 0.3t)\frac{(1-0.00259293t)}{(1-0.00851151t+0.00190567t^2)}. \tag{3.14}$$

The following Figure 3.3 below shows the plot of $r(t)$ based on equation (3.14) together with the plot of $r(t)$ based on the actual data.

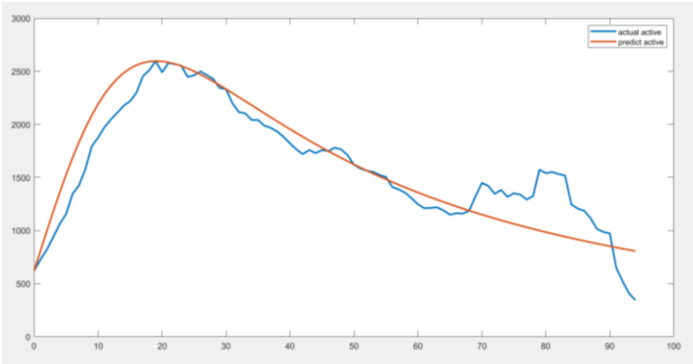
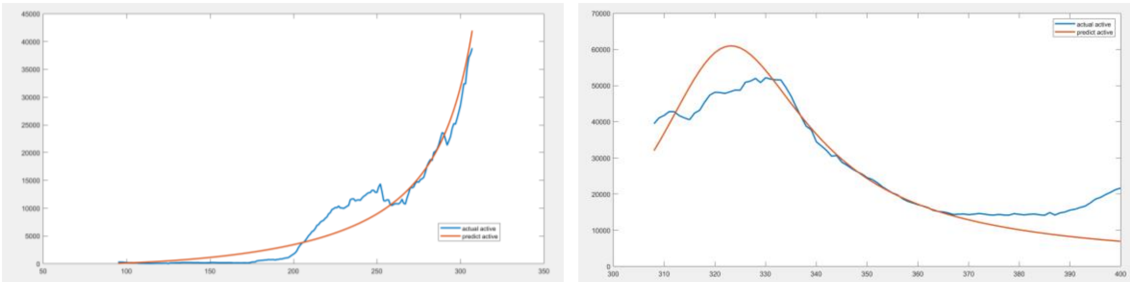


Figure 3.3: Plot of $r(t)$ based on mathematical model and actual data for Phase 1 until Phase 7

By using the same method from equation (3.6), the $r(t)$ based on mathematical model and actual data for Phase 7 until Phase 10 and Phase 10 until Phase 17 were obtained as illustrated in Figure 3.4.



a) Phase 7 until Phase 10 b) Phase 10 until Phase 17

Figure 3.4: Plots of $r(t)$ based on mathematical model and actual data

The following Figure 3.5 shows the plot of the combined three models from Phase 1 until Phase 7, Phase 7 until Phase 10 and Phase 10 until Phase 17.

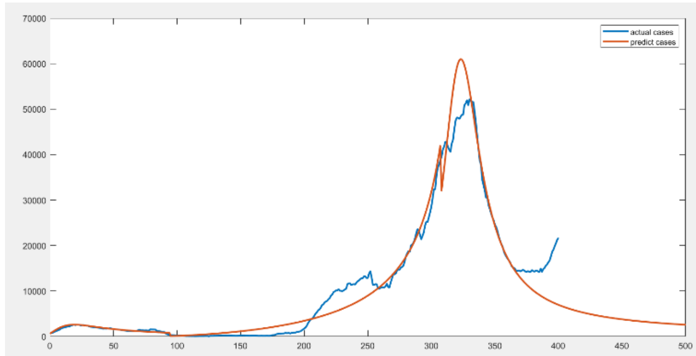


Figure 3.5: Plot of the combined three models

3.2 Discussion

According to the experimental results, despite its simplicity and crudeness, the logistic model well predicts the number of COVID-19 active cases over time. Since the daily number of active cases that reported did not uniformly increase or decrease, it is quite difficult to construct only one model that represents predict pattern of active cases in Malaysia. Hence, there are three models that constructed in this study represents the approximated $r(t)$. Figure 3.5 illustrates the capabilities of the mathematical model using Riccati equation for predict the pattern of active cases from 18 March 2020 until 21 April 2021.

On Phase 1 until Phase 7 which the early stage of outbreak, Malaysian government has implemented movement control order for two weeks for preventing the virus from spreading. After two weeks, the authorities extended the lockdown to all states in an effort to stop the virus from spreading further. At the early stage of COVID-19, the predict pattern of active cases were increase and slowly decrease starting from $t = 20$ which approximate the actual data. Figure 3.3 shown that the actual cases was increase starting at $t = 68$ and continually decrease at $t = 89$.

On Phase 7 until Phase 10 was the fast growth phase approaching the peak of incidence curve as illustrated in Figure 3.4 (a). The Malaysian government implemented the MCO 2.0 in Phase 10 due to the increasing cases. It is observed that from $t = 95$ until $t = 173$, there was a steady difference in the number of actual active cases. The active cases were gradually increased from $t = 174$ until $t = 252$ but slowly decrease after that while for predict cases were strictly increase from $t = 95$ until $t = 307$. The actual cases were slowly decrease at $t = 253$ with the steady difference until $t = 267$ but continue to increase at the early December ($t = 268$).

On Phase 10 and Phase 11, the actual and predict cases were approaching the highest number of active cases in Malaysia throughout this study as illustrated in Figure 3.4 (b). According to Figure 3.5, starting from Phase 12, there was slow growth phase which approaching the end of the outbreak for predict cases while for actual active cases, the number of cases were increase at $t = 388$.

4 Conclusion

The effectiveness of models that use the Riccati equation to describe the dynamics of the ongoing COVID-19 was evaluated in this study. From the application to the outbreak in Malaysia, the mathematical model using Riccati equation can approach the predict pattern of active cases.

While more data is needed to create more comprehensive predictions, these models may be able to anticipate future confirmed cases assuming the virus's propagation does not change in ways that are unexpected. This virus, as we all know, is relatively new and has the potential to spread rapidly. Some characteristics may have an impact on all of our predictions, but the proposed model is effective to the best of our knowledge at the time we were writing this paper.

5 References

[1] Worldometer (2020). *Malaysia Coronavirus: 3,116 Cases and 50 Deaths - Worldometer*. Retrieved from https://www.worldometers.info/coronavirus/country/malaysia/.

[2] Krishna, M. V. (2020). Mathematical modelling on diffusion and control of COVID–19. *Infectious Disease Modelling*. 5, pp.588-597.

[3] Mishra, J. (2020). A study on the spread of COVID 19 outbreak by using mathematical modeling. *Results in Physics*, 19, p.103605.

[4] Wang, P., Zheng, X., Li, J. and Zhu, B. (2020). Prediction of epidemic trends in COVID-19 with logistic model and machine learning technics. *Chaos, Solitons & Fractals*, 139, p.110058.

[5] Pelinovsky, E., Kurkin, A., Kurkina, O., Kokoulina, M. and Epifanova, A. (2020). Logistic equation and COVID-19. *Chaos, Solitons & Fractals*, 140, p.110241.

[6] Okuonghae, D. and Omame, A. (2020). Analysis of a mathematical model for COVID-19 population dynamics in Lagos, Nigeria. *Chaos, Solitons & Fractals*, 139, p.110032.

[7] Abusam, A., Abusam, R. and Al-Anzi, B. (2020). Adequacy of Logistic models for describing the dynamics of COVID-19 pandemic. *Infectious Disease Modelling*, 5, pp.536–542.

[8] Wu, K., Darcet, D., Wang, Q. and Sornette, D. (2020). Generalized logistic growth modeling of the COVID-19 outbreak: comparing the dynamics in the 29 provinces in China and in the rest of the world. *Nonlinear Dynamics*, 101(3), pp.1561–1581.

[9] *Svirin, A. (2020). Riccati equation*, Math24. Retrieved from https://www.math24.net/riccati-equation/.

[10] Wikipedia. (2020). *Riccati equation*. Retrieved from https://en.wikipedia.org/wiki/Riccati_equation.