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A Modified Numerical Approximation for Thermal Post-Buckling Behaviour of Isotropic Circular Plate

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Abstract This study investigates a suitable mathematical formulation to determine the thermal post buckling behaviour of isotropic circular plates under clamped boundary conditions. The governing equations of strain energy and work done are non-dimensionalised using von Karman's approximation. By integration techniques, the linear buckling load and the radial edge load are obtained. The ratio between these two loads will give us the value of post buckling load for different b_0/t ranging from 0.0 to 1.0. The numerical results are compared with previous literature for 3 terms, and it is in good agreement within engineering accuracy.

Keywords post buckling load; isotropic circular plates; clamped boundary conditions; linear buckling load; integration techniques.

1 Introduction

Buckling is a sudden change in the shape of a structure when subjected to compressive load [1]. The structure will undergo a large deformation and will lose its ability to carry load at the point of critical load value. Usually it is caused by pressure, concentrated force, and temperature gradient. Post-buckling is the continuation after the buckling stage. The load value may not change or may start to decrease after it reaches the critical value whereas at the same time, deformation will continue to increase. Thermal buckling occurs when there is a temperature rise in a heated structural member, causes the compressive radial thermal to emerge and thus contributed to the failure of a structural member. Thermal buckling plays an important part in many fields such as aerospace, automotive, naval, and engineering structures. In automotive industry, thermal buckling could happen in the automotive clutch and brake discs where heat is generated from friction and is distributed non-uniformly which leads to non-uniform temperature gradient across the disc. Thermal stress can occur which causes the failure of the disc [2].

Li et.al. [3] studied the vibration of thermally post-buckled isotropic and orthotropic circular plates. A structure which is thermally buckled can carry additional load in the post-buckled system [3]. Unfortunately, there exists differences in the vibrational characteristics between a buckled plate and an unbuckled plate. In this study, the author used von Karman plate theory and Hamilton principle to study the axisymmetric vibrations of a thermally loaded circular plate. Then, shooting method is applied to the nonlinear differential equations of thermal post-buckling and the linear equations for free vibrations which are solved simultaneously. As a result, the author compared the thermal load parameter, λ/λ_{cr} for both simply supported and clamped isotropic circular plates with previous literature and the results are in good agreement with each other.

Ramaraju and Gundabathula [4] reinvestigated the approach for thermal post-buckling of circular plates by using Berger's approximation. According to the author, it is hard to calculate the radial tension of circular plates due to explicit coupling in the expression for radial and circumferential strains because of radial displacement. Therefore, to calculate the radial tension, some assumptions must be made. The author used Berger's approximation and it is treated as a constant to find the functional form of radial displacement. The result is then compared with the formulation using finite element method. As a result, both methods match very well, with its difference percentage ranging from -1.30% to 0.22% [4].

Varma and Rao [5] conducted a research of using novel formulation to study the thermal post-buckling behaviour of uniform thin circular plate with a more realistic edge rotational restraint. An elastic rotational spring is introduced to stimulate elastic rotational restraint. The author stated that due to explicit coupling between in-plane strains and radial displacement, it is hard to formulate circular plates compared to the common plates. An improved Berger's approximation, where the second strain invariant is neglected, is used to avoid the difficulty when calculating radial tensile load. It is concluded that the results match well with those obtained by using FEM analysis [5].

Noh and Waleed [6] did a buckling analysis of an isotropic circular plates with piezoceramic annular plate attached to it. The author used finite difference method to solve the governing equations. From the results, it is observed that the radial and hoop stresses depend on the radial throughout the annular region but constant throughout the circular region. The results were then compared with FEM results, and it is found to be in good agreement. It can be concluded that the critical buckling voltage increases as the annular thickness increases.

In this study, mathematical modeling involved reduction of nonlinear governing equations of strain energy and work done to non-dimensional governing equations. These non-dimensional governing equations are further transformed into ordinary differential equations by substitution method. In this formulation, parameter values b_0/t taken from 0 to 1 in equal intervals. The differential equation thus obtained represents the total potential energy of the system. The differential equation for potential energy is further integrated to evaluate the linear buckling load. An algebraic function is identified as suitable admissible functions for the lateral displacement '*w*' satisfying clamped boundary conditions. Further, the radial edge tensile load of the isotropic circular plate is evaluated using integration method with the help of MAPLE software. The ratio of radial edge load to linear buckling load determines the thermal post-buckling load carrying capacity (γ) of the circular plate. The numerical results obtained from the present study are compared with the results from [4].

2 Formulation

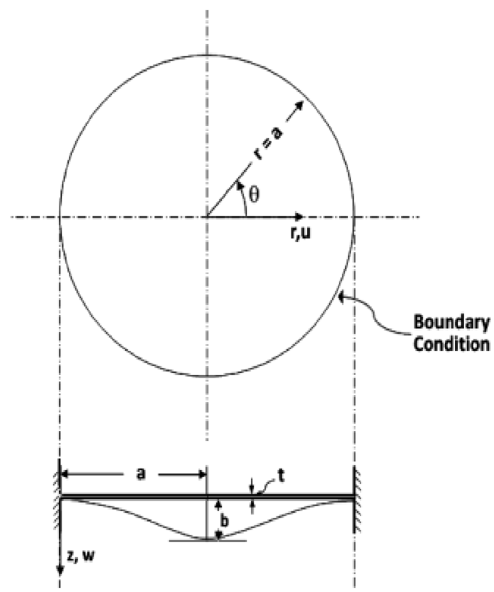

Figure 1: Circular plate showing coordinate system and lateral deflection pattern for clamped boundary condition.

A mathematical formulation for the thermal post buckling of isotropic circular plates by evaluating linear buckling load parameters due to large lateral displacements are presented in this section. From Figure 2, consider a circular plate with radius '*a'* and uniform thickness subjected to a radial uniform compressive load 'N_r', the von Karman strain-displacement relations can be written as

$$\varepsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \tag{1}$$

$$\varepsilon_\theta = \frac{u}{r} \tag{2}$$

$$\chi_r = -\frac{d^2w}{dr^2} \tag{3}$$

$$\chi_\theta = -\frac{1}{r}\left(\frac{dw}{dr}\right) \tag{4}$$

The strain energy *U* with isotropic material properties is given by,

$$U = \int_0^{2\pi}\int_{r_1}^{r_2}\frac{1}{2}\left[C_1\varepsilon_r^2 + C_2\varepsilon_\theta^2 + C_{12}\varepsilon_r\varepsilon_\beta + D_1\chi_r^2 + D_2\chi_\theta^2 + D_{12}\chi_r\chi_\theta\right] r dr d\theta \tag{5}$$

where $C_1 = \frac{E_r h}{1-\nu^2}$, $C_2 = \frac{E_\theta h}{1-\nu^2}$, $C_{12} = \nu_r C_2 = \nu_\theta C_1$, $D_1 = \frac{E_r h^3}{12(1-\nu^2)}$, $D_2 = \frac{E_\theta h^3}{12(1-\nu^2)}$ and $D_{12} = \nu_r D_2 = \nu_\theta D_1$.

Using appropriate substitutions, equation (5) can be written as

$$U = \frac{1}{2}\int_0^{2\pi}\int_{r_1}^{r_2} \frac{E_r h}{1-v^2}\left(\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2\right)^2 + \frac{E_\theta h}{1-v^2}\left(\frac{u}{r}\right)^2 + v_r . \frac{E_\theta h}{1-v^2}\left(\frac{du}{dr}+\frac{1}{2}\left(\frac{dw}{dr}\right)^2\right)^2\left(\frac{u}{r}\right)$$
$$+ \frac{E_r h^3}{12(1-v^2)}\left(-\frac{dw}{dr}\right)^2 + \frac{E_\theta h^3}{12(1-v^2)}\left(-\frac{1}{r}\frac{dw}{dr}\right)^2 + v_r \frac{E_\theta h^3}{12(1-v^2)}\left(-\frac{dw}{dr}\right)^2$$
$$* \left(-\frac{1}{r}\frac{dw}{dr}\right) r\, dr d\theta \tag{6}$$

After substitution of the values and elimination of *h*, equation (6) can be reduced as shown in equation (7),

$$U = \frac{1}{2}\int_0^1 \left(\frac{d^2w}{dr^2}\right)^2 + \frac{1}{r^2}\left(\frac{dw}{dr}\right)^2 + 2v\frac{1}{r^2}\left(\frac{d^2}{dr^2}\right)\left(\frac{dw}{dr}\right) dr \tag{7}$$

where the isotropic parameter $\beta = 1(E_\theta = E_r)$.
The work done, *W* is given by,

$$W = \frac{1}{2}\int_0^{2\pi}\int_{r_1}^{r_2} 2\bar{N}_r \left(\frac{dw}{dr}\right)^2 r dr d\theta \tag{8}$$

Equation (8) is then reduced using substitution and becomes,

$$W = \frac{\lambda\beta}{2}\int_0^1 \frac{d^2w}{dr^2} dr \tag{9}$$

The total potential energy of isotropic circular plate can be stated as,

$$\Pi = U - W \tag{10}$$

The total potential energy Π is minimized with respect to the undetermined coefficients of the assumed admissible function using Rayleigh-Ritz method [7]. The value of Poisson's ratio ν is taken as 0.3 with only clamped boundary condition is considered in this present study. The following admissible function F is taken for the lateral displacement '*w*' and is used to calculate thermal post-buckling behavior of circular plates,

$$F = \sum_{i=0}^{n} w_0 \left[\left(\frac{r}{a}\right)^{2i} + \alpha_{2i+1} \left(\frac{r}{a}\right)^{2i+2} + \alpha_{2i+2} \left(\frac{r}{a}\right)^{2i+4} \right]$$

where the function F above satisfy the boundary condition for clamped plates as given as below,

At $r = 0, w' = 0$
At $r = a, w = 0, w' = 0$

The values of α_{2i+1} *and* α_{2i+2} are obtained from the boundary conditions of the plate. Following Yamaki [8] study, the value of α_1 and α_2 is taken by -2 and 1 which satisfies the boundary condition. The admissible function is calculated up until the value of $n = 3$ *whi*ch represents the term of the equation.

An equivalent uniform compressive radial edge load N_r will be evolved in the plate if it is heated to a temperature ΔT from the stress-free plate [9]. The plate would buckle if the temperature reached critical temperature ΔT_r, due to the critical uniform radial edge compressive load $N_{r_{cr}}$ is developed. When the temperature ΔT is increased further, the lateral displacement β will took place and when the lateral displacement gets higher, an additional uniform radial edge tensile load N_{r_T} is developed; allowing the plate to withstand more N_r than $N_{r_{cr}}$. Hence, the total equivalent compressive uniform compressive radial edge load $N_{r_{NL}}$ *is* developed and is represented as,

$$\overline{\mathrm{N}}_{\mathrm{r_{NL}}} = \overline{\mathrm{N}}_{\mathrm{r_{Cr}}} + \overline{\mathrm{N}}_{\mathrm{r_T}} \tag{11}$$

in non-dimensionalised form, where each term in equation (11) is non-dimensionalised as

$$\overline{N_r} = \frac{N_r a^2}{D} \tag{12}$$

and D is the plate flexural rigidity.
The radial edge tensile load can be calculated using von Karman's non-linear strain displacement relation as shown in equation (1), (2), (3) and (4). From equation (1) and (2), we can write the expression of N_r as

$$N_r = \frac{\beta \mathrm{t}}{1 - v^2} \left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 + v\frac{u}{r} \right] \tag{13}$$

According to Berger's approximation, the second invariant of the strains are neglected or $\varepsilon_r \ll \varepsilon_\theta$, then, N_r can be written as

$$N_r = \frac{\beta \mathrm{t}}{1 - v^2} \left[\frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \right] \tag{14}$$

The numerical results are obtained for clamped edges boundary condition, $u = 0$ and $\frac{dw}{dr} = 0$. The uniform radial edge load developed due to large deflections in circular plates in non-dimensional form is given as,

$$N_{r_T} = \frac{12}{\beta(1 - v^2)} \int_0^1 \left(\frac{dw}{dr}\right)^2 dr \tag{15}$$

The thermal post-buckling load carrying capacity γ is given by,

$$\gamma = \frac{N_{r_{NL}}}{N_{r_{cr}}} = 1 + c\left(\frac{b_0}{t}\right)^2 \tag{16}$$

3 Result and Discussion

This study investigates the thermal post buckling behaviour of isotropic circular plates under clamped boundary conditions. The radial tension for circular plate is hard to evaluate due to the coupling of between two strain components. Hence, an admissible function for the lateral displacement is assumed as below.

$$F = \sum_{i=0}^{n} \mathrm{w}_0 \left[\left(\frac{\mathrm{r}}{\mathrm{a}}\right)^{2\mathrm{i}} + \alpha_{2\mathrm{i}+1}\left(\frac{\mathrm{r}}{\mathrm{a}}\right)^{2\mathrm{i}+2} + \alpha_{2\mathrm{i}+2}\left(\frac{\mathrm{r}}{\mathrm{a}}\right)^{2\mathrm{i}+4}\right]$$

The values of the value of α_1 and α_2 is taken by -2 and 1 from [8] study which satisfy the boundary condition of clamped plates. These values will be substituted into the admissible function to calculate the thermal post buckling of clamped circular plates. The numerical results are obtained by using the Maple software. The result is evaluated using three terms ($n = 1, 2, 3$), where the convergence of each term is checked. The post buckling load is compared with results from [4] that uses finite element method (FEM). The comparison is presented in Table 1.

Table 1: The values of post-buckling load carrying capacity of clamped isotropic circular plates for 1, 2 and 3 terms with the error percentage between present and FEM study.

b_0/t	Post buckling load (γ)				Error percentage with previous literature		
	Present			FEM [4]			
	$n = 1$	$n = 2$	$n = 3$		$n = 1$	$n = 2$	$n = 3$
0	1.0000	1.0000	1.0000	1.0000	0.17%	0.00%	0.00%
0.1	1.0046	1.0050	1.0083	1.0050	0.35%	0.00%	0.33%
0.2	1.0184	1.0201	1.0333	1.0201	0.62%	0.00%	1.29%

0.3	1.0415	1.0452	1.0749	1.0452	0.92%	0.00%	2.84%
0.4	1.0737	1.0804	1.1331	1.0804	1.27%	0.00%	4.88%
0.5	1.1152	1.1256	1.2080	1.1256	1.64%	0.00%	7.32%
0.6	1.1659	1.1809	1.2995	1.1809	2.02%	0.00%	10.04%
0.7	1.2258	1.2462	1.4077	1.2462	2.40%	0.00%	12.96%
0.8	1.2949	1.3216	1.5324	1.3216	2.02%	0.00%	15.95%
0.9	1.3733	1.4071	1.6739	1.4070	2.40%	0.01%	18.97%
1	1.4608	1.5025	1.8320	1.5025	2.78%	0.00%	21.93%
λ_{cr}	16.0000	14.7017	14.6820	14.6896			
C	0.4608	0.5025	0.8320	0.5286			

The value of c for one, two and three terms are 0.4608, 0.5025, and 0.8320, respectively. The value for λ_{cr} is obtained based on [4] where the values were acquired from the admissible function F that is used in present study. In this study, the calculation of λ_t is emphasised, whereas the value of λ_{cr} is taken from the admissible function.

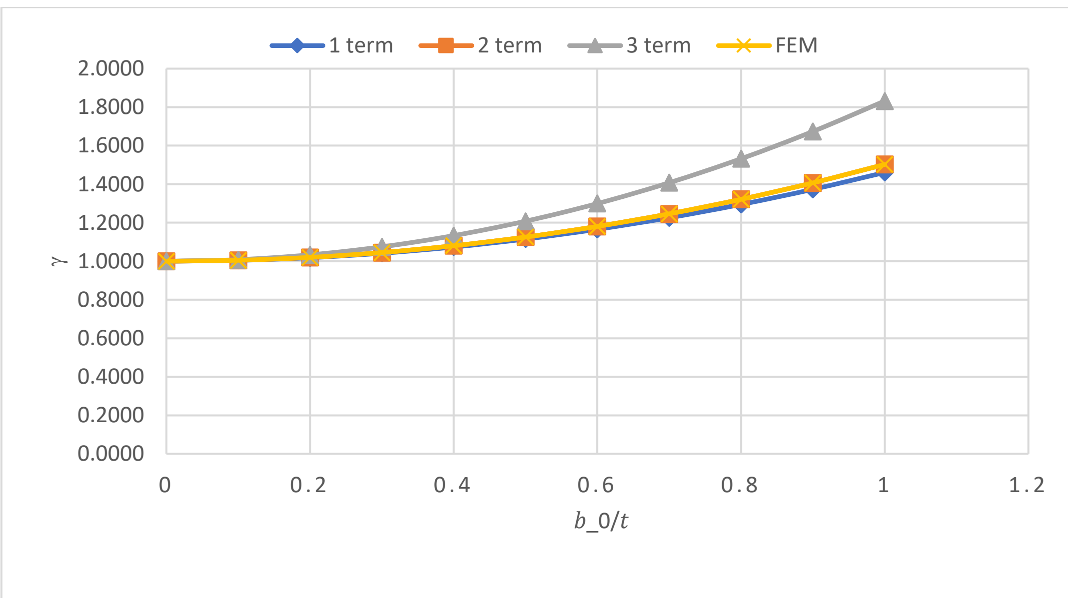

Figure 2: Comparison of present numerical results and FEM results [2]

The result in present study shows the same trend for all the terms, the γ increases as b_0/t increases. However, the error percentage differs for all the three terms. It can be seen from Figure 2 that the values for one and two terms are similar with the FEM results. The three terms result showed excellent agreement from b_0/t 0 until 0.3, however, it started to experience obvious change when for the value 0.4 and above. This can be seen from the percentage error calculated in Table 1. This is because, for three term, the admissible function results in a higher order polynomial function and according to Ilanko et. al [10], higher order polynomial in an admissible function causes numerical instabilities and ill-conditioning. Therefore, the minimum number of polynomial functions with the lowest order possible should be obtained in our admissible function to keep the solution simple and free of numerical problems.

4 Conclusion

It is clear from the results that the present method yields accurate result with FEM results with the right number of terms. The simple formulation evaluated for this study to calculate the post-buckling of isotropic circular plates is proven to show similar result with previous literature that uses finite element method. However, the explicit coupling between circumferential and radial strains making it difficult to evaluate, for cases of circular plates. Thus, some approximations and assumptions were made for easy derivation. To study the convergence of our result, the admissible function is evaluated for three terms. Based on our results and discussion, it is found that our method yields accurate result similar with the results from FEM for one and two term whereas for three term, a higher percentage error can be seen. This is due to higher order of polynomials function obtained in the admissible function for three term, thus making it not suitable.

As we know, the thermal post-buckling of circular plates can be calculated using various methods such as finite element method, shooting method, iterative method, and many more. The present method reduces the complexity of the formulation and gives accurate results for suitable terms. The benefits of using this method are to make the calculation of finding linear buckling load of isotropic circular plates easier, thus it is less time-consuming compared to other methods. Easy mathematical operations and methods such as differentiation, integration, and substitution are used in this method, and this is very useful for the future engineers and designers.

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