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Existence Conditions for Periodic Solutions of Second-Order Neutral Delay Differential Equations

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Abstract This study focuses on the second-order neutral delay differential equation with the addition of periodic solution. The objectives of this study are to find the solution $x(t)$, of the equation for n-periodic solution where n is any integer, and the existing conditions for the solution to exist. The method adopted in this research is by simple calculation based on its periodic solution, and by using computer programming to find the solution. The results show that the problem is solvable and has many solutions. The general formula for n-periodic solution obtained with some existing conditions.

Keywords Delay differential equation; periodic solution; existence condition.

1 Introduction

Delay differential equation is a branch of differential equation and it is widely used in many sectors of science such as biological science on population dynamics and epidemiology, and model for infectious diseases [1]. There are various ways on solving this type of equation such as by the embedded singly diagonally implicit Runge-Kutta (SDIRK) method, and the two- and three-point one-step block methods [2,3]. According to Thompson [1], delay differential equations is the derivative of a function which is unknown at a certain time given in term the values of the function at preceding times, and its general form is

$$y'(t) = f\big(t, y(t), y(t-\tau_1), \dots, y(t-\tau_k)\big), \tag{1}$$

where $\tau_i = \tau_i(t, y(t))$ are time delays. There are two types of delay differential equation which are retarded and neutral.

In this paper, it focuses on the problem which involving a second-order neutral delay differential equation with the addition of the periodic solutions. Hence, the general form of second-order neutral delay differential equation is

$$x''(t) = f\big(t, x(t), x'(t)\big), \tag{2}$$

where $x''(t)$ is the second derivative and f is the function with delay time. The periodic solution of an equation is a solution that repeats the process in a similar form. The periodic solution $x(t)$ is a solution that depends on variable t periodically,

$$x(t+T) = x(t), \tag{3}$$

where variable t is an independent variable in real number, and $T \neq 0$ is the periods [4].

In the previous studies, there are many researches on second-order delay differential equation with various methods. According to Chen, Tang and Sun [5], they proved the existence of multiple delays of periodic solutions for second-order delay differential equation by applying the critical point theory and S^1 index theory with the method by Kaplan and Yorke [6]. Muminov and Murid [7] showed another way to solve this type of equation with different equation by simple calculation of algebra and application of some properties in linear algebra. The authors showed the existing conditions when solving the equation on each case of the periodic solution.

This research aims to find the solution for the specific equation which in the form of second-order neutral delay differential equation with addition of n-periodic solutions. In order to find the solution, there are several conditions applied to the problem equation as it is a need to ensure the existence of the solution.

2 Problem Statement

The problem in this study is to identify the existence conditions and solutions of second order neutral delay differential equation with periodic solutions. In this paper, the problem equation is

$$x''(t) + px''\left(t - \frac{1}{2}\right) = f(t), \tag{4}$$

where p is a nonzero real number, and $f(t)$ is continuous function with delay time on real number. This study involves 1-periodic, 2-periodic, 3-periodic, 4-periodic and n-periodic solutions.

3 Methodology

In this section, there are several steps to obtain the solution $x(t)$. First, list all the possible forms of the periodic function. The definition of periodic function is used in the form of

$$x''(t) = x''(t+k), \tag{5}$$

where k a positive integer [4]. Second, substitute the periodic function from previous step into equation (4) to form a system of equation, and determine the solvability of the system of equation by finding its determinant which is not equal to 0. Third, form an equation $x''(t)$ by using simple algebraic calculation to the system of equation and solve the system to obtain the solution $x(t)$. Next, find all the unknown variables in the solution and obtain the solution $x(t)$. Lastly, repeat the steps to 1-periodic to 4-periodic solutions and find the general formula for n-periodic solution for equation (4) with its existence conditions.

4 Results and Discussion

In this section, we solve the equation (4) by adapting the similar method used in [7] and find its existence conditions.

4.1 1-periodic solution ($\boldsymbol{k = 1}$)

From equation (4), we consider the periodicity by applying definition of periodic solution which equivalent to 1-periodic solution, $x''(t) = x''(t+1)$.

Substituting t with $\left(t+\frac{1}{2}\right)$ into equation (4), gives

$$x''\left(t+\frac{1}{2}\right)+px''(t)=f\left(t+\frac{1}{2}\right), \tag{6}$$

where this equation (6) is the equivalent to equation (4). Since $x''\left(t+\frac{1}{2}\right)=x''\left(t-\frac{1}{2}\right)$, the equation becomes

$$x''\left(t-\frac{1}{2}\right)+px''(t)=f\left(t+\frac{1}{2}\right). \tag{7}$$

Then, we obtain the system of linear equation as the following,

$$\left.\begin{aligned} x''(t)+px''\left(t-\frac{1}{2}\right)&=f(t) \\ x''\left(t-\frac{1}{2}\right)+px''(t)&=f\left(t+\frac{1}{2}\right) \end{aligned}\right\}. \tag{8}$$

By assuming the right sides of the system of equation (8) is known, we consider the system with respect to $x''\left(t-\frac{1}{2}\right), x''(t)$. It also can be written in matrix form,

$$\begin{bmatrix} p & 1 \\ 1 & p \end{bmatrix}\begin{bmatrix} x''\left(t-\frac{1}{2}\right) \\ x''(t) \end{bmatrix}=\begin{bmatrix} f(t) \\ f\left(t+\frac{1}{2}\right) \end{bmatrix}. \tag{9}$$

Then, the solvability of the system of equations (9) is

$$\begin{vmatrix} p & 1 \\ 1 & p \end{vmatrix}=p^2-1\neq 0. \tag{10}$$

Since the determinant of the system of equations is not equal to zero, then the solution $x(t)$ is solvable. Now, we solve the system of linear equations (8) and we obtain

$$x''(t)=\frac{1}{1-p^2}\left[-pf\left(t+\frac{1}{2}\right)+f(t)\right]. \tag{11}$$

Next, we integrate the equation (10) twice,

$$\int_0^t\int_0^{t_1} x''(s)\ dsdt_1=\int_0^t\int_0^{t_1}\frac{1}{1-p^2}\left[-pf\left(s+\frac{1}{2}\right)+f(s)\right]dsdt_1, \tag{12}$$

where $t<1$, and obtain the solution of $x(t)$ as

$$x(t)=x(0)+x'(0)t+F_1(t), \tag{13}$$

where

$$\begin{aligned} F_1(t)&=\frac{1}{1-p^2}\int_0^t\int_0^{t_1}\left[-pf\left(s+\frac{1}{2}\right)+f(s)\right]dsdt_1 \\ &=\frac{1}{1-p^2}\int_0^t\int_0^{t_1} f(s)+\sum_{k=1}^{1}(-1)^k p^{2-k} f\left(s+\frac{k}{2}\right)dsdt_1. \end{aligned} \tag{14}$$

The equation in (13) and (14) are the general solution of equation (4) for 1-periodic solution. From solution (13), we have two unknown variables which are $x(0)$ and $x'(0)$. Hence, we need to find the values for $x(0)$ and $x'(0)$ as it is essential to ensure the equation (4) has solution, and find its existing conditions. We find the values of $x(0)$ and $x'(0)$ by using solution (13) and (14),

$$\left.\begin{aligned} x(1)&=x(0)+x'(0)+F_1(1) \\ x'(1)&=x'(0)+{F_1}'(1) \end{aligned}\right\}. \tag{15}$$

We know that $x(0)=x(1)$ and $x'(0)=x'(1)$ since the solution is continuous and periodic, then we obtain

$$\left.\begin{aligned} x'(0)&=-F_1(1) \\ 0&={F_1}'(1) \end{aligned}\right\}. \tag{16}$$

It also can be written in matrix form,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x(0) \\ x'(0) \end{bmatrix} = \begin{bmatrix} -F_1(1) \\ F_1{}'(1) \end{bmatrix}. \tag{17}$$

From equation (17), the right side of the equation, we already assumed as known variables while the left side of the equation contains the unknown variables $x(0)$ and $x'(0)$. In order to verify the values for $x(0)$ and $x'(0)$, we find determinant of the system of equation $\left(x(0), x'(0)\right)$ from (17), and we obtain

$$D_1 = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0, \tag{18}$$

which means that the system can have many solutions or no solution. In our case, since $F_1'(1) = 0$, then the value of $x(0)$ can be any real number while $x'(0)$ is a unique value. From the equation (17), we know the existing condition for the solution $x(t)$ as in the following theorem.

Theorem 1 *The equation* (4) *has 1-periodic solution if and only if* $F_1'(1) = 0$. *Otherwise, the equation* (4) *has no solution. In this case, the equation* (4) *has infinitely many solutions of the form* $x(t) = x(0) + x'(0)t + F_1(1)$ *where* $x(0)$ *is any real number and* $x'(0) = -F_1(1)$.

4.2 2-periodic solution ($\boldsymbol{k = 2}$)

From equation (4), we consider the periodicity by applying definition of periodic solution which equivalent to 2-periodic solution, $x''(t) = x''(t+2)$. Then, we obtain the system of linear equation as the following,

$$\left.\begin{aligned} x''(t) + px''\left(t-\frac{1}{2}\right) &= f(t) \\ x''\left(t+\frac{1}{2}\right) + px''(t) &= f\left(t+\frac{1}{2}\right) \\ x''(t+1) + px''\left(t+\frac{1}{2}\right) &= f(t+1) \\ x''\left(t-\frac{1}{2}\right) + px''(t+1) &= f\left(t+\frac{3}{2}\right) \end{aligned}\right\}. \tag{19}$$

By assuming the right sides of the system of equation (19) is known, we consider the system with respect to $x''\left(t-\frac{1}{2}\right), x''(t), x''\left(t+\frac{1}{2}\right), x''(t+1)$. Then, the solvability condition of the system of equations (19) is

$$\begin{vmatrix} p & 1 & 0 & 0 \\ 0 & p & 1 & 0 \\ 0 & 0 & p & 1 \\ 1 & 0 & 0 & p \end{vmatrix} = p^4 - 1 \neq 0. \tag{20}$$

Since the determinant of the system of equations is not equal to zero, then the we can obtain the solution $x(t)$ as the system is solvable. Now, we solve the system of equations (19) and we obtain

$$x''(t) = \frac{1}{1-p^4}\left[-p^3 f\left(t+\frac{1}{2}\right) + p^2 f(t+1) - pf\left(t+\frac{3}{2}\right) + f(t)\right]. \tag{21}$$

Next, we integrate the equation (21) twice and obtain

$$x(t) = x(0) + x'(0)t + F_2(t), \tag{22}$$

where

$$\begin{aligned} F_2(t) &= \frac{1}{1-p^4}\int_0^t \int_0^{t_1} \left[-p^3 f\left(s+\frac{1}{2}\right) + p^2 f(s+1) - pf\left(s+\frac{3}{2}\right) + f(s)\right] ds\,dt_1 \\ &= \frac{1}{1-p^4}\int_0^t \int_0^{t_1} f(s) + \sum_{k=1}^{3} (-1)^k p^{4-k} f\left(s+\frac{k}{2}\right) ds\,dt_1. \end{aligned} \tag{23}$$

The equation in (22) and (23) are the general solution for equation (4) for 2-periodic solution. From solution (22), we have two unknown variables which are $x(0)$ and $x'(0)$. Hence, we find the values of $x(0)$ and $x'(0)$ by using solution (22) and (23),

$$\left.\begin{array}{l} x(2) = x(0) + 2x'(0) + F_2(2) \\ x'(2) = x'(0) + F_2{}'(2) \end{array}\right\}. \tag{24}$$

We know that $x(0) = x(2)$ and $x'(0) = x'(2)$, since the solution is continuous and periodic, then we obtain

$$\left.\begin{array}{r} 2x'(0) = -F_2(2) \\ 0 = F_2{}'(2) \end{array}\right\}. \tag{25}$$

From equation (25), the right side of the equation, we already assumed as known variables while the left side of the equation is the unknown variables $x(0)$ and $x'(0)$. In order to verify the values for $x(0)$ and $x'(0)$, we find determinant of the system of equation $\left(x(0), x'(0)\right)$ from (25), and we obtain

$$D_2 = \begin{vmatrix} 0 & 2 \\ 0 & 0 \end{vmatrix} = 0, \tag{26}$$

which means that the system can have many solutions or no solution. In our case, since $F_2'(2) = 0$, then the value of $x(0)$ can be any real number while $x'(0)$ is a unique value. From equation (25), we know the existing condition of the solution (22) and (23), as in the following theorem.

Theorem 2 *The equation* (4) *has 2-periodic solution if and only if* $F_2'(2) = 0$. *Otherwise, the equation* (4) *has no solution. In this case, the equation* (4) *has infinitely many solutions of the form* (22), *where* $x(0)$ *is any real number while* $x'(0) = -\frac{1}{2}F_2(2)$.

4.3 3-periodic solution ($\boldsymbol{k = 3}$)

From equation (4), we consider the periodicity by applying definition of periodic solution which equivalent to 3-periodic solution, $x''(t) = x''(t+3)$. Then, we obtain the system of linear equation as the following,

$$\left.\begin{array}{r} x''(t) + px''\left(t - \frac{1}{2}\right) = f(t) \\ x''\left(t + \frac{1}{2}\right) + px''(t) = f\left(t + \frac{1}{2}\right) \\ x''(t+1) + px''\left(t + \frac{1}{2}\right) = f(t+1) \\ x''\left(t + \frac{3}{2}\right) + px''(t+1) = f\left(t + \frac{3}{2}\right) \\ x''(t+2) + px''\left(t + \frac{3}{2}\right) = f(t+2) \\ x''\left(t - \frac{1}{2}\right) + px''(t+2) = f\left(t + \frac{5}{2}\right) \end{array}\right\}. \tag{27}$$

By assuming the right sides of the system of equation (27) is known, we consider the system with respect to $x''\left(t - \frac{1}{2}\right), x''(t), x''\left(t + \frac{1}{2}\right), x''(t+1), x''\left(t + \frac{3}{2}\right), x''(t+2)$. Then, the determinant of the system of equations (27) is $p^6 - 1 \neq 0$, then the solution $x(t)$ is solvable. Now, we solve the system of linear equations (27) and we obtain

$$x''(t)=\frac{1}{1-p^6}\left[-p^5f\left(t+\frac{1}{2}\right)+p^4f(t+1)-p^3f\left(t+\frac{3}{2}\right)+p^2f(t+2)-pf\left(t+\frac{5}{2}\right)\right.$$
$$\left.+f(t)\right]. \tag{28}$$

Next, we integrate the equation (28) twice and obtain the solution for 3-periodic solution as

$$x(t)=x(0)+x'(0)t+F_3(t), \tag{29}$$

where

$$F_3(t)=\frac{1}{1-p^6}\int_0^t\int_0^{t_1}\left[-p^5f\left(s+\frac{1}{2}\right)+p^4f(s+1)-p^3f\left(s+\frac{3}{2}\right)+p^2f(s+2)\right.$$
$$\left.-pf\left(s+\frac{5}{2}\right)+f(s)\right]dsdt_1$$
$$=\frac{1}{1-p^6}\int_0^t\int_0^{t_1}f(s)+\sum_{k=1}^{5}(-1)^kp^{6-k}f\left(s+\frac{k}{2}\right)dsdt_1. \tag{30}$$

From solution (29), we have two unknown variables which are $x(0)$ and $x'(0)$. Hence, we find the values of $x(0)$ and $x'(0)$ by using solution (29) and (30),

$$\left.\begin{array}{l} x(3)=x(0)+3x'(0)+F_3(3) \\ x'(3)\ =x'(0)+F_3{}'(3) \end{array}\right\}. \tag{31}$$

We know that $x(0)=x(3)$ and $x'(0)=x'(3)$, since the solution is continuous and periodic, then we obtain

$$\left.\begin{array}{r} 3x'(0)=-F_3(3) \\ 0=F_3{}'(3) \end{array}\right\}. \tag{32}$$

From equation (32), the right side of the equation, we already assumed as known variables while the left side of the equation is the unknown variables $x(0)$ and $x'(0)$. In order to verify the values for $x(0)$ and $x'(0)$, we find determinant of the system of equation $\big(x(0),x'(0)\big)$ from (32), and we obtain the determinant is equal to 0, which means that the system can have many solutions or no solution. In our case, since $F_3'(3)=0$, then the value of $x(0)$ can be any real number while $x'(0)$ is a unique value. From the equation (32), we produce the following theorem.

Theorem 3 *The equation* (4) *has 3-periodic solution if and only if* $F_3'(3)=0$*. Otherwise, the equation* (4) *has no solution. In this case, the equation* (4) *has infinitely many solutions of the form* (29)*, where* $x(0)$ *is any real number while* $x'(0)=-\frac{1}{3}F_3(3)$*.*

4.4 4-periodic solution ($\boldsymbol{k=4}$)

From equation (4), we consider the periodicity by applying definition of periodic solution which equivalent to 4-periodic solution, $x''(t)=x''(t+4)$. Then, we obtain the system of linear equation as the following,

$$\begin{gathered} x''(t)+px''\left(t-\frac{1}{2}\right)=f(t) \\ x''\left(t+\frac{1}{2}\right)+px''(t)=f\left(t+\frac{1}{2}\right) \\ x''(t+1)+px''\left(t+\frac{1}{2}\right)=f(t+1) \\ x''\left(t+\frac{3}{2}\right)+px''(t+1)=f\left(t+\frac{3}{2}\right) \\ x''(t+2)+px''\left(t+\tfrac{3}{2}\right)=f(t+2) \end{gathered} \tag{33}$$

$$x''\left(t+\frac{5}{2}\right)+px''(t+2)=f\left(t+\frac{5}{2}\right)$$
$$x''(t+3)+px''\left(t+\frac{5}{2}\right)=f(t+3)$$
$$x''\left(t-\frac{1}{2}\right)+px''(t+3)=f\left(t+\frac{7}{2}\right)$$

By assuming the right sides of the system of equation (33) is known, we consider the system with respect to $x''\left(t-\frac{1}{2}\right), x''(t), x''\left(t+\frac{1}{2}\right), x''(t+1), x''\left(t+\frac{3}{2}\right), x''(t+2), x''\left(t+\frac{5}{2}\right), x''(t+3)$. Then, the determinant of the system of equations (33) is $p^8-1\neq 0$, then the solution $x(t)$ is solvable. Now, we solve the system of linear equations (33) and obtain function $x''(t)$,

$$x''(t)=\frac{1}{1-p^8}\left[-p^7 f\left(t+\frac{1}{2}\right)+p^6 f(t+1)-p^5 f\left(t+\frac{3}{2}\right)+p^4 f(t+2)-p^3 f\left(t+\frac{5}{2}\right)\right.$$
$$\left.+p^2 f(t+3)-pf\left(t+\frac{7}{2}\right)+f(t)\right]. \tag{34}$$

Next, we integrate the equation (34) twice and obtain the solution for 4-periodic solution as

$$x(t)=x(0)+x'(0)t+F_4(t), \tag{35}$$

where

$$F_4(t)=\frac{1}{1-p^8}\int_0^t\int_0^{t_1}\left[-p^7 f\left(s+\frac{1}{2}\right)+p^6 f(s+1)-p^5 f\left(s+\frac{3}{2}\right)+p^4 f(s+2)\right.$$
$$\left.-p^3 f\left(s+\frac{5}{2}\right)+p^2 f(s+3)-pf\left(s+\frac{7}{2}\right)+f(s)\right]dsdt_1. \tag{36}$$

From solution (35), we have two unknown variables which are $x(0)$ and $x'(0)$. Hence, we find the values of $x(0)$ and $x'(0)$ by using solution (35) and (36), and since $x(0)=x(4)$ and $x'(0)=x'(4)$ as the solution is continuous and periodic, then we obtain

$$\left.\begin{aligned} 4x'(0)&=-F_4(4) \\ 0&=F_4{}'(4) \end{aligned}\right\}. \tag{37}$$

From equation (37), the right side of the equation, we already assumed as known variables while the left side of the equation is the unknown variables $x(0)$ and $x'(0)$. In order to verify the values for $x(0)$ and $x'(0)$, we need to find determinant of the system of equation $\left(x(0),x'(0)\right)$ from (37). Since the determinant is equal to 0, then the system can have many solutions or no solution. In our case, since $F_4'(4)=0$, then the value of $x(0)$ can be any real number while $x'(0)$ is a unique value. From the equation (37), we produce the following theorem.

Theorem 4 *The equation* (4) *has 4-periodic solution if and only if* $F_4'(4)=0$. *Otherwise, the equation* (4) *has no solution. In this case, the equation* (4) *has infinitely many solutions of the form* (35), *where* $x(0)$ *is any real number while* $x'(0)=-\frac{1}{4}F_4(4)$.

4.5 n-periodic solution ($k=n$)

From equation (4), we consider the periodicity by applying definition of periodic solution which equivalent to n-periodic solution, $x''(t)=x''(t+n)$. Then, we obtain the system of linear equation as the following,

$$x''(t)+px''\left(t-\frac{1}{2}\right)=f(t)$$
$$x''\left(t+\frac{1}{2}\right)+px''(t)=f\left(t+\frac{1}{2}\right)$$

$$
\begin{gathered}
x''(t+1)+px''\left(t+\frac{1}{2}\right)=f(t+1) \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
x''(t+n-1)+px''\left(t+n-\frac{3}{2}\right)=f(t+n-1) \\
x''\left(t-\frac{1}{2}\right)+px''(t+n-1)=f\left(t+n-\frac{1}{2}\right)
\end{gathered} \tag{38}
$$

By assuming the right sides of the system of equation (38) is known, we consider the system with respect to $x''\left(t-\frac{1}{2}\right), x''(t), x''\left(t+\frac{1}{2}\right), x''(t+1), x''\left(t+\frac{3}{2}\right), x''(t+2), \dots, x''(t+n-1)$. Then, we determine the solvability condition of the system of equations (38) by using determinant,

$$
\begin{vmatrix}
p & 1 & 0 & \dots & 0 & 0 \\
0 & p & 1 & \dots & 0 & 0 \\
0 & 0 & p & \ddots & 0 & 0 \\
\vdots & \vdots & \dots & \ddots & \dots & \vdots \\
0 & 0 & 0 & \dots & p & 1 \\
1 & 0 & 0 & \dots & 0 & p
\end{vmatrix} = p^{2n}-1 \neq 0. \tag{39}
$$

Since the determinant of the system of equations is not equal to zero, then the we can obtain the solution $x(t)$ as the system is solvable. Now, we solve the system of linear equations (38) and obtain

$$
\begin{aligned}
x''(t)=\frac{1}{1-p^{2n}}\Big[&-p^{2n-1}f\left(t+\frac{1}{2}\right)+p^{2n-2}f(t+1)-p^{2n-3}f\left(t+\frac{3}{2}\right)+\cdots \\
&+p^4f(t+n-2)-p^3f\left(t+n-\frac{3}{2}\right)+p^2f(t+n-1)-pf\left(t+n-\frac{1}{2}\right) \\
&+f(t)\Big].
\end{aligned} \tag{40}
$$

Next, we integrate the equation (40) twice and obtain solution for n-periodic solution as

$$
x(t)=x(0)+x'(0)t+F_n(t), \tag{41}
$$

where

$$
\begin{aligned}
F_n(t)=\frac{1}{1-p^{2n}}\int_0^t\int_0^{t_1}\Big[&-p^{2n-1}f\left(s+\frac{1}{2}\right)+p^{2n-2}f(s+1)-p^{2n-3}f\left(s+\frac{3}{2}\right)+\cdots \\
&+p^4f(s+n-2)-p^3f\left(s+n-\frac{3}{2}\right)+p^2f(s+n-1)-pf\left(s+n-\frac{1}{2}\right) \\
&+f(s)\Big]dsdt_1 \\
=\frac{1}{1-p^{2n}}\int_0^t\int_0^{t_1}&f(s)+\sum_{k=1}^{2n-1}(-1)^kp^{2n-k}f\left(s+\frac{k}{2}\right)dsdt_1.
\end{aligned} \tag{42}
$$

From solution (41), we have two unknown variables which are $x(0)$ and $x'(0)$. Hence, we need to find the values for $x(0)$ and $x'(0)$ as it is essential to ensure the equation (4) has solution, and find its existing conditions. We find the values of $x(0)$ and $x'(0)$ by using solution (41) and (42),

$$
\left.\begin{aligned}
x(n)&=x(0)+nx'(0)+F_n(n) \\
x'(n)&=x'(0)+F_n{}'(n)
\end{aligned}\right\}. \tag{43}
$$

Since the solution is continuous and periodic, we know that $x(0)=x(n)$ and $x'(0)=x'(n)$ which obtains

$$
\left.\begin{aligned}
nx'(0)&=-F_n(n) \\
0&=F_n{}'(n)
\end{aligned}\right\}. \tag{44}
$$

In order to verify the values for $x(0)$ and $x'(0)$, we find determinant of the system of equation $\big(x(0), x'(0)\big)$ from (44), and we obtain

$$D_n = \begin{vmatrix} 0 & n \\ 0 & 0 \end{vmatrix} = 0, \tag{45}$$

which means that the system can have many solutions or no solution. In our case, since $F_n'(n) = 0$, then the value of $x(0)$ can be any real number while $x'(0)$ is a unique value. From the equation (44), we produce the following theorem.

Theorem 5 *The equation* (4) *has n-periodic solution if and only if* $F_n'(n) = 0$. *Otherwise, the equation* (4) *has no solution. In this case, the equation* (4) *has infinitely many solutions of the form* (41), *where* $x(0)$ *is any real number while* $x'(0) = -\frac{1}{n}F_n(n)$.

4.6 Examples

The following are the examples for 2-periodic, and 3-periodic solution problems.

Example 1 Assume $p = 3, f(t) = \cos \pi t$ and apply the 2-periodic solution. After substituting the values of p and function $f(t)$ into equation (4), the equation has the form

$$x''(t) + 3x''\left(t - \frac{1}{2}\right) = \cos \pi t. \tag{46}$$

Therefore, the solution $x(t)$ for $t \in [0,2)$ when substitute $x'(0)$ and $F_2(t)$ into solution (22) is

$$x(t) = \mathrm{x}(0) + 0.0955\mathrm{t} - \frac{3\pi t - 3\sin \pi t + \cos \pi t - 1}{10\pi^2}, \tag{47}$$

where $x(0)$ is any real number. Since $x(0)$ can be any real number, we can plot the graph of $x(t)$ with variety values of $x(0)$. In this example, we show the graph for $x(0) = 0$ and $x(0) = 0.1$.

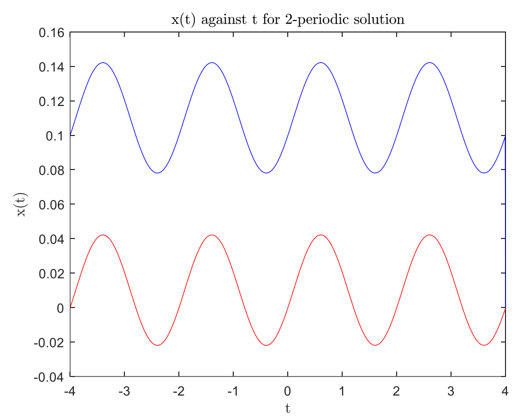

Figure 1: The graph of $x(t)$ for 2-periodic solution of Example 1 when $x(0) = 0$ (red) and $x(0) = 0.1$ (blue)

Example 2 Assume $p = 3, f(t) = \cos \frac{2\pi}{3} t$ and apply the 3-periodic solution. After substituting the values of p and function $f(t)$ into equation (4), the equation has the form

$$x''(t) + 3x''\left(t - \frac{1}{2}\right) = \cos \frac{2\pi}{3} t. \tag{48}$$

The solution, $x(t)$ for $t \in [0,3)$ when substitute $x'(0)$ and $F_3(t)$ into solution (30) is

$$x(t) = \mathrm{x}(0) + 0.0304t + \frac{1}{52\pi^2}\left[\frac{27\sqrt{3}\tan\left(\frac{\pi t}{3}\right) - 45}{\tan^2\left(\frac{\pi t}{3}\right) + 1} - 9\sqrt{3}t + 45\right], \qquad (49)$$

where $x(0)$ is any real number. Since $x(0)$ can be any real number, we can plot the graph of $x(t)$ with variety values of $x(0)$. In this example, we show the graph for $x(0) = 0$ and $x(0) = 0.1$.

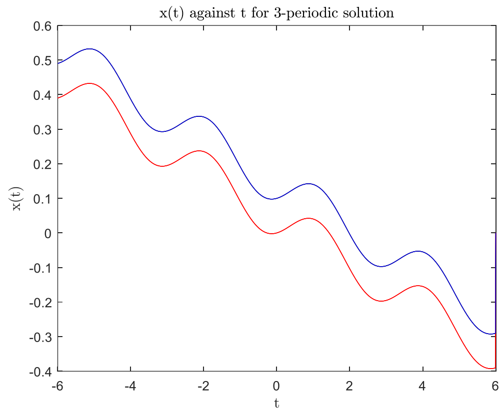

Figure 2: The graph of $x(t)$ for 3-periodic solution of Example 2 when $x(0) = 0$ (red) and $x(0) = 0.1$ (blue)

5 Conclusion

In summary, we obtain the final result of the research. Based on the patterns obtained in 1-periodic, 2-periodic, 3-periodic and 4-periodic solutions, we find the general formula for n-periodic solution of equation (4) with the similar existing conditions. The results show that the solution has infinitely many solutions since value of $x(0)$ can be any real number while the value of $x'(0)$ is a unique real number for each case. We also obtain the necessary condition and sufficient condition for the solution exists. Generally, the equation (4) is solvable in any periodic if and only if it satisfies the existing conditions.

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