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Implementation of Interval-Valued Fuzzy Sets in Evaluating Students Answerscripts

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Abstract This paper study the approach of using similarity measure between interval-valued fuzzy sets for evaluating students' answerscripts. The marks awarded to the answers in the students' answerscripts are represented by the interval-valued fuzzy sets, where each element in the universe of discourse belonging to an interval-valued fuzzy sets is represented by an interval between zero and one. An index of optimism λ determined by the evaluator is used to indicate the degree of optimism of the evaluator, where $\lambda \in [0, 1]$. This method has achieved the purpose of an educational institutions which is to provide the students a more flexible and intelligent evaluation system.

Keywords Similarity function; students' answerscripts; interval-valued fuzzy grade sheet; interval-valued fuzzy sets; interval-valued fuzzy evaluation method.

1 Introduction

Evaluating the quality of students' answerscripts is a crucial step for grading their performances in study. For years, researchers in several fields attempted to present the most appropriate formula for each evaluation. In 1995, Biswas [1] pointed that a chief goal of educational institutions is to provide students evaluation reports regarding their examination as efficient as possible and with the smallest percentages of unavoidable error possible.

The evaluation of students' answerscripts done manually by common academicians in majority of universities is non-transparent and may lead to dissatisfaction among students. Poor evaluation done by evaluators may affect students' mental health, results, scholarship and their career. Various method has been proposed in evaluating students' answerscripts using fuzzy sets. In 1995, Biswas [1] presented the method of using fuzzy evaluation method (fem) and a generalized fuzzy evaluation method (gfem). His method has major drawbacks because a matching function is used to measure the degrees of similarity between the standard fuzzy sets and the fuzzy marks of the questions which take a large amount of time to perform the matching operations and the two different fuzzy marks may be translated into the same awarded grade which is unfair to students' answerscripts evaluation.

Then, Chen and Wang [2] presented a method of evaluating students' answerscripts based on the interval-valued fuzzy sets using an interval-valued fuzzy grade sheet. In their paper, a method of representing the marks awarded to the answers by an interval-valued fuzzy sets were proposed. The degree of similarity between the standard interval-valued fuzzy sets and an interval-valued fuzzy mark of each question is calculated by a similarity function. This method is widely accepted as it enables evaluators to evaluate students' answerscripts in a more flexible and more intelligent manner without any software involved.

Hence, this study intended to investigate the method of evaluating students' answerscripts based on the interval-valued fuzzy sets introduced by Chen and Wang. Additionally, this method allows fairer evaluation of students' answerscripts since they use the index of optimism λ determined by the evaluator to indicate evaluator's degree of optimism during the evaluation process, where $\lambda \in [0, 1]$. If the evaluator is pessimistic, then the index of optimism will be $0 \leq \lambda < 0.5$. If the evaluator's behavior is normal, then $\lambda = 0.5$ and if the evaluator is optimistic, the index of optimism is $0.5 < \lambda \leq 1.0$.

2 Mathematical Background

Brief introduction of interval-valued fuzzy sets is discussed in Section 2.1, while Section 2.2 presented the similarity measure between interval-valued fuzzy sets.

2.1 Interval-Valued Fuzzy Sets

Interval-valued fuzzy sets were proposed as a natural extension of fuzzy sets. The difficulty arises from the uncertainty associated with allocating an exact numerical membership value for each element within the considered fuzzy sets. In the year of 1975, Sambuc [3] presented interval-valued fuzzy sets which can be used when there is a problem in determining the exact membership values of the given elements.

In interval-valued fuzzy sets, intervals may be used as membership values in such way that the exact numerical membership degree is a value inside the considered interval. During the same year, Jahn [4] wrote about the notion of interval-valued fuzzy set. In 1976, Grattan-Guinness [5] established a definition of an interval-valued membership function.

Definition 2.1 Interval-valued Fuzzy Sets [3]
An interval-valued fuzzy set in the universe of discourse X is given by an expression A,

$$A = \{\langle x, M_A(x) \rangle \mid x \in X\},$$

where the function $M_A: X \to D[0,1]$ defines the degree of membership of an element x to A.

Interval-valued fuzzy set theory is an extension of fuzzy set theory in which to each element in the universe of a closed subinterval of the unit interval is assigned which approximates the unknown membership degree [6]. In this study, the approach for students' answerscripts evaluation based on interval-valued fuzzy sets. The marks awarded to the answers in the students' answerscripts are represented by interval-valued fuzzy sets. The arithmetic operations involving closed intervals and the properties they satisfy are also studied in this section.

Definition 2.2 Arithmetic Operations on Intervals [7]
Let $*$ denote any of the arithmetic operations (addition $\oplus$ and multiplication $\otimes$) on closed intervals. Then, the general property of all arithmetic operation on closed intervals can be simplified as:

(1)

$$[p,q] * [x,y] = \{f * g \mid p \leq f \leq q, x \leq g \leq y\}.$$

The equation (1) for the arithmetic operations of addition and multiplication can be defined as follows:

(i) $[p,q] \oplus [x,y] = [\,p + x, q + y]$,

(ii) $[p,q] \otimes [x,y] = [\min(px,py,qx,qy), \max(px,py,qx,qy)]$.

2.2 Similarity Measures Between Interval-Valued Fuzzy Sets

Similarity measure is a measure that depicts the difference among interval-valued fuzzy sets. In other words, for this study, the larger the value of the degree of similarity obtained, the higher the similarity between the two interval-valued sets. The objective of this section is to present the method of obtaining the similarity function, T that is used to measure the degree of similarity between interval-valued fuzzy sets.

The method for measuring the distance between two real intervals was introduced by Zwick *et al.* [8]. Let P and Q be two intervals in $[\beta_1, \beta_2]$, where $P = [p_1, p_2]$ and $Q = [q_1, q_1]$. Then, the distance, $D(P,Q)$ between two intervals P and Q can be calculated as follows:

$$D(P,Q) = \frac{|p_1 - q_1| + |p_2 - q_2|}{2(\beta_2 - \beta_1)}. \tag{2}$$

Hence, the degree of similarity, $S(P,Q)$ between the intervals P and Q can be obtained by using equation (2) [9], as follows:

$$S(P,Q) = 1 - D(P,Q). \tag{3}$$

Then, let P and Q be two intervals in $[0,1]$, where $P = [p_1, p_2]$ and $Q = [q_1, q_1]$. Based on equation (2) and equation (3), the degree of similarity $S(P,Q)$ between intervals P and Q may be calculated as follows:

$$S(P,Q) = \begin{cases} 1, \ \ if\ q_1 \leq p_1 \leq p_2 \leq q_2 \\ 1 - \dfrac{|p_1 - q_1| + |p_2 - q_2|}{2}, \ \ otherwise \end{cases} \tag{4}$$

where $S(P,Q) \in [0,1]$. From equation (4), it is obvious that when P and Q are identical intervals, then the $D(P,Q) = 0$ and $S(P,Q) = 1$. In this case, the larger the value of $S(P,Q)$, the higher the similarity between P and Q.

When the values in intervals P or Q has the same value, which can be assumed as $P = [p,p]$ and $Q = [q,q]$, then based on equation (4):

$$\begin{aligned} S(P,Q) &= S([p,p],[q,q]) \\ &= 1 - \frac{|p-q| + |p-q|}{2} \\ &= 1 - |p-q|. \end{aligned}$$

Next, let A and B be two interval-valued fuzzy sets in the universe of discourse of X, where:

$$X = \{x_1, x_2, \dots, x_n\},$$

$$A = \frac{[a_{11}, a_{12}]}{x_1} + \frac{[a_{21}, a_{22}]}{x_2} + \cdots + \frac{[a_{n1}, a_{n2}]}{x_n},$$

$$B = \frac{[b_{11}, b_{12}]}{x_1} + \frac{[b_{21}, b_{22}]}{x_2} + \cdots + \frac{[b_{n1}, b_{n2}]}{x_n}.$$

Denote $[a_{i1}, a_{i2}]$ as the grade of membership of x_i belonging to the interval-valued fuzzy set A and $[b_{i1}, b_{i2}]$ as the grade of membership of x_i belonging to the interval-valued fuzzy set B such that $0 \leq a_{i1} \leq a_{i2} \leq 1$, $0 \leq b_{i1} \leq b_{i2} \leq 1$ and $1 \leq i \leq n$.

Based on the matrix representation method, the interval-valued fuzzy sets A and B can also be represented by the matrices $\bar{A}$ and $\bar{B}$ respectively, as follows:

$$\bar{A} = \langle [a_{11}, a_{12}], [a_{21}, a_{22}], \dots, [a_{n1}, a_{n2}] \rangle,$$
$$\bar{B} = \langle [b_{11}, b_{12}], [b_{21}, b_{22}], \dots, [b_{n1}, b_{n2}] \rangle.$$

In the case where $A = B$ implies that $a_{ij} = b_{ij}$ such that $1 \leq i \leq n$ and $1 \leq j \leq 2$, then, $\bar{A} = \bar{B}$.

This section aim is to obtain the similarity function, T that can be used to measure the degree of similarity between the interval-valued fuzzy sets A and B, denoted as $T(\bar{A}, \bar{B})$ can be achieved by applying equation (4), as below:

$$T(\bar{A}, \bar{B}) = \frac{\sum_{i=1}^{n} S([a_{i1}, a_{i2}], [b_{i1}, b_{i2}])}{n} = \frac{\sum_{i=1}^{n} \left(1 - \frac{|a_{i1} - b_{i1}| + |a_{i2} - b_{i2}|}{2}\right)}{n} \tag{5}$$

where $T(\bar{A}, \bar{B}) \in [0,1]$. In this case, the larger the value of $T(\bar{A}, \bar{B})$, the higher the similarity between the interval-valued fuzzy sets A and B.

3 Interval-Valued Fuzzy Sets in Evaluating Students' Answerscripts

Firstly, the universe of discourse, X is defined, where $X = \{0\%, 20\%, 40\%, 60\%, 80\%, 100\%\}$. The five fuzzy linguistic hedges used in this study is based on study conducted by Biswas [1]. In this study, standard Fuzzy Sets which are denoted by **E** (Excellent), **V** (Very Good), **G** (Good), **S** (Satisfactory), and **U** (Unsatisfactory) of the universe of discourse X was defined as follows:

$$\boldsymbol{E} = \frac{0}{0\%} + \frac{0}{20\%} + \frac{0.8}{40\%} + \frac{0.9}{60\%} + \frac{1}{80\%} + \frac{1}{100\%},$$
$$\boldsymbol{V} = \frac{0}{0\%} + \frac{0}{20\%} + \frac{0.8}{40\%} + \frac{0.9}{60\%} + \frac{0.9}{80\%} + \frac{0.8}{100\%},$$
$$\boldsymbol{G} = \frac{0}{0\%} + \frac{0.1}{20\%} + \frac{0.8}{40\%} + \frac{0.9}{60\%} + \frac{0.4}{80\%} + \frac{0.2}{100\%},$$
$$\boldsymbol{S} = \frac{0.4}{0\%} + \frac{0.4}{20\%} + \frac{0.9}{40\%} + \frac{0.6}{60\%} + \frac{0.2}{80\%} + \frac{0}{100\%},$$
$$\boldsymbol{U} = \frac{1}{0\%} + \frac{1}{20\%} + \frac{0.4}{40\%} + \frac{0.2}{60\%} + \frac{0}{80\%} + \frac{0}{100\%}.$$

These five standard fuzzy sets can equivalently be represented by interval-valued fuzzy sets $\tilde{E}$, $\tilde{V}, \tilde{G}, \tilde{S}$, and $\tilde{U}$ respectively [2].

$$\tilde{E} = \frac{[0,0]}{0\%} + \frac{[0,0]}{20\%} + \frac{[0.8,0.8]}{40\%} + \frac{[0.9,0.9]}{60\%} + \frac{[1,1]}{80\%} + \frac{[1,1]}{100\%},$$
$$\tilde{V} = \frac{[0,0]}{0\%} + \frac{[0,0]}{20\%} + \frac{[0.8,0.8]}{40\%} + \frac{[0.9,0.9]}{60\%} + \frac{[0.9,0.9]}{80\%} + \frac{[0.8,0.8]}{100\%},$$
$$\tilde{G} = \frac{[0,0]}{0\%} + \frac{[0.1,0.1]}{20\%} + \frac{[0.8,0.8]}{40\%} + \frac{[0.9,0.9]}{60\%} + \frac{[0.4,0.4]}{80\%} + \frac{[0.2,0.2]}{100\%},$$

,

$$\tilde{S} = \frac{[0.4, 0.4]}{0\%} + \frac{[0.4, 0.4]}{20\%} + \frac{[0.9, 0.9]}{40\%} + \frac{[0.6, 0.6]}{60\%} + \frac{[0.2, 0.2]}{80\%} + \frac{[0, 0]}{100\%}$$
$$\widetilde{U} = \frac{[1, 1]}{0\%} + \frac{[1, 1]}{20\%} + \frac{[0.4, 0.4]}{40\%} + \frac{[0.2, 0.2]}{60\%} + \frac{[0, 0]}{80\%} + \frac{[0, 0]}{100\%}$$

Then, the standard interval-valued fuzzy sets $\widetilde{E}$, $\tilde{V}$, $\tilde{G}$, $\tilde{S}$, and $\widetilde{U}$ can be transform into matrices $\bar{E}$, $\bar{V}$, $\bar{G}$, $\bar{S}$, and $\bar{U}$, respectively, as follows:

$$\bar{E} = \langle [0, 0], [0,0], [0.8,0.8], [0.9, 0.9], [1, 1], [1, 1] \rangle,$$
$$\bar{V} = \langle [0, 0], [0,0], [0.8,0.8], [0.9, 0.9], [0.9, 0.9], [0.8, 0.8] \rangle,$$
$$\bar{G} = \langle [0, 0], [0.1, 0.1], [0.8,0.8], [0.9, 0.9], [0.4, 0.4], [0.2, 0.2] \rangle,$$
$$\bar{S} = \langle [0.4, 0.4], [0.4, 0.4], [0.9, 0.9], [0.6, 0.6], [0.2, 0.2], [0, 0] \rangle,$$
$$\bar{U} = \langle [1, 1], [1, 1], [0.4, 0.4], [0.2, 0.2], [0, 0], [0, 0] \rangle.$$

Assume that “$\boldsymbol{A}$”, “$\boldsymbol{B}$”, “$\boldsymbol{C}$”, “$\boldsymbol{D}$” and “$\boldsymbol{E}$” are letter grades, where $90 \leq \boldsymbol{A} < 100$, $70 \leq \boldsymbol{B} < 90$, $50 \leq \boldsymbol{C} < 70$, $30 \leq \boldsymbol{D} < 50$ and $0 \leq \boldsymbol{E} < 30$. An interval-valued fuzzy grade sheet is used to obtain the interval-valued fuzzy sets for each question in the answerscript. The general sample of an interval-valued fuzzy grade sheet is shown in Table 1.

Table 1 Interval-valued fuzzy grade sheet

Question No.	Interval-valued fuzzy mark						Derived fuzzy letter grade
	0%	20%	40%	60%	80%	100%	
Q_1	[0, 0.1]	[0.2, 0.3]	[0.4, 0.5]	[0.6, 0.7]	[0.8, 0.9]	[1, 1]	
Q_2							
Q_3							
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Q_n							
							Total mark =

The interval-valued fuzzy mark indicated the degree of satisfaction of the evaluator towards the answer to each question answered by student. From the second row of Table 3.1 (assumed to be an evaluation of an evaluator to question Q_1), the interval-valued fuzzy marks [0, 0.1], [0.2, 0.3], [0.4, 0.5], [0.6, 0.7], [0.8, 0.9], and [1, 1] awarded to the answer for question Q_1 indicated the degree of the evaluator’s satisfaction for that answer are 0%, 20%, 40%, 60%, 80%, and 100% respectively.

Afterwards, the interval-valued fuzzy mark of the answer to each question, $Q_1, Q_2, \dots, Q_n$ are denoted by $\widetilde{M}_i$ of the universe of discourse X such that,

$$X = \{0\%, 20\%, 40\%, 60\%, 80\%, 100\%\}.$$

Hence, the interval-valued fuzzy marks for question Q_1, in Table 1 can be denoted as $\widetilde{M}_1$, in the universe of discourse X, where

$$\widetilde{M}_1 = \frac{[0, 0.1]}{0\%} + \frac{[0, .2, 0.3]}{20\%} + \frac{[0.4, 0.5]}{40\%} + \frac{[0.6, 0.7]}{60\%} + \frac{[0.8, 0.9]}{80\%} + \frac{[1, 1]}{100\%},$$

which can be represented by a matrix $\bar{M}_1$, as follows:

$$\bar{M}_1 = \langle [0, 0.1], [0.2, 0.3], [0.4, 0.5], [0.6, 0.7], [0.8, 0.9], [1, 1] \rangle.$$

The interval-valued fuzzy evaluation method (IVFEM) generally involved two major steps, which are:

Step 1: Obtain the derived fuzzy letter grade, $\tilde{g}_i$ for each question Q_i $(1 \le i \le n)$ in the answerscript.

Step 2: Calculate the total marks of the student.

3.1 Step 1: Obtain the derived fuzzy letter grade, $\tilde{g}_i$

The tasks in this step need to be performed repeatedly by the examiner to obtain the derived fuzzy letter grades for each question existed in the answerscripts using the interval-valued fuzzy grade sheet in Table 1.

Firstly, an interval-valued fuzzy mark, denoted as $\widetilde{M}_i$ awarded by examiner to each question Q_i by the evaluator's judgement. The $\widetilde{M}_i$ is represented by an interval-valued fuzzy set in the universe of discourse $X = \{0\%, 20\%, 40\%, 60\%, 80\%, 100\%\}$. Each cell in Table 1 is filled up for the first seven columns up to the ith row, where $1 \le i \le n$. Then, let $\bar{M}_i$ be the matrix representation of the interval-valued fuzzy mark $\widetilde{M}_i$ of question Q_i such that $1 \le i \le n$.

The next task is where the degrees of similarity between the two interval-valued fuzzy set are calculated by using equation (4). The degrees of similarity are $T(\bar{E}, \bar{M}_i)$, $T(\bar{V}, \bar{M}_i)$, $T(\bar{G}, \bar{M}_i)$, $T(\bar{S}, \bar{M}_i)$ and $T(\bar{U}, \bar{M}_i)$, where $\bar{E}$, $\bar{V}$, $\bar{G}$, $\bar{S}$ and $\bar{U}$ are the matrix representations of the standard fuzzy sets $\tilde{E}$ (excellent), $\tilde{V}$ (very good), $\tilde{G}$ (good), $\tilde{S}$ (satisfactory) and $\widetilde{U}$ (unsatisfactory), respectively. Then, assume that $T(\bar{E}, \bar{M}_i) = \mu_{i1}$, $T(\bar{V}, \bar{M}_i) = \mu_{i2}$, $T(\bar{G}, \bar{M}_i) = \mu_{i3}$, $T(\bar{S}, \bar{M}_i) = \mu_{i4}$ and $T(\bar{U}, \bar{M}_i) = \mu_{i5}$, where $\mu_{ij} \in [0, 1]$, $1 \le i \le n$ and $1 \le j \le 5$.

The standard fuzzy sets $\tilde{E}$, $\tilde{V}$, $\tilde{G}$, $\tilde{S}$ and $\widetilde{U}$ correspond to the letter grades "$\boldsymbol{A}$", "$\boldsymbol{B}$", "$\boldsymbol{C}$", "$\boldsymbol{D}$" and "$\boldsymbol{E}$" respectively. Therefore, the derived fuzzy letter grade $\tilde{g}_i$ of question Q_i can be represented by a fuzzy set shown as follows:

$$\tilde{g}_i = \frac{\mu_{i1}}{\boldsymbol{A}} + \frac{\mu_{i2}}{\boldsymbol{B}} + \frac{\mu_{i3}}{\boldsymbol{C}} + \frac{\mu_{i4}}{\boldsymbol{D}} + \frac{\mu_{i5}}{\boldsymbol{E}},$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, $\boldsymbol{D}$ and $\boldsymbol{E}$ are the letter grades, while $T(\bar{E}, \bar{M}_i) = \mu_{i1}$, $T(\bar{V}, \bar{M}_i) = \mu_{i2}$, $T(\bar{G}, \bar{M}_i) = \mu_{i3}$, $T(\bar{S}, \bar{M}_i) = \mu_{i4}$ and $T(\bar{U}, \bar{M}_i) = \mu_{i5}$, $\mu_{ij} \in [0, 1]$, $1 \le i \le n$ and $1 \le j \le 5$.

3.2 Step 2: Calculate the total mark of student

To achieve the final objective of this study, a formula is used, where

$$Total\ Mark = \frac{1}{100} \times \sum_{i=1}^{n} [R(Q_i) \times K(\tilde{g}_i)] \qquad (6)$$

The $R(Q_i)$ in equation (6) denotes the mark allocated to the question Q_i in the question paper, $\tilde{g}_i$ denotes the fuzzy letter grade awarded to Q_i (from step 1) and $K(\tilde{g}_i)$ denotes the derived grade point of the derived fuzzy letter grade $\tilde{g}_i$ based on the index of optimism $\lambda \in [0, 1]$ determined by the evaluator during assessing the answerscripts. The index of optimism shown in Table 2 could be affected by various factors including personal problems like fatigue and stress.

Table 2 Index of optimism of evaluator

Evaluator Behaviour	Index of optimism, λ
Pessimistic	$0 \leq \lambda < 0.5$
Normal	$\lambda = 0.5$
Optimistic	$0.5 < \lambda \leq 1$

Based on the letter grades in Table 3.2, the derives grade point $K(\tilde{g}_i)$ from equation (6) is calculated as the formula below:

$$K(\tilde{g}_i) = \{ \mu_{i1} \cdot [(1-\lambda) \times 90 + \lambda \times 100] + \mu_{i2} \cdot [(1-\lambda) \times 70 + \lambda \times 90] + \mu_{i3} \cdot [(1-\lambda) \times 50 + \lambda \times 70] + \mu_{i4} \cdot [(1-\lambda) \times 30 + \lambda \times 50] + \mu_{i5} \cdot [(1-\lambda) \times 0 + \lambda \times 30] \}/(\mu_{i1} + \mu_{i2} + \mu_{i3} + \mu_{i4} + \mu_{i5}) \quad (7)$$

where $\lambda \in [0, 1]$ is the index of optimism of the evaluator.

4 Experimental Results

The method of implementing this interval-valued fuzzy evaluation method for students' answerscripts evaluation is illustrated as an experimental result of this study. The total marks to the student's answerscript in an examination is considered a total of 100 marks, where there are five questions to be answered in total. The marks allocated to each question is denoted as $R(Q_i)$ where $1 \leq i \leq 5$, as follows:

Q_1 carries 10 marks, implies that $R(Q_1) = 10$,
Q_2 carries 20 marks, implies that $R(Q_2) = 20$,
Q_3 carries 20 marks, implies that $R(Q_3) = 20$,
Q_4 carries 25 marks, implies that $R(Q_4) = 25$,
Q_5 carries 25 marks, implies that $R(Q_5) = 25$,
$TOTAL\ MARK = 100$.

The interval-valued fuzzy grade sheet is filled up by the evaluator since there are five questions existed, as shown in Table 3. The index of optimism λ during the judgement process is determined based on evaluator's behaviour (Table 2). During the judgement, the evaluator is assumed to be optimistic with the index of 0.65 ($\lambda = 0.65$).

Table 3 Interval-valued fuzzy grade sheet for a student

Question No.	Interval-valued fuzzy mark						Derived fuzzy letter grade
	0%	20%	40%	60%	80%	100%	
Q_1	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0.8, 0.9]$	$[1, 1]$	
Q_2	$[0, 0]$	$[0, 0]$	$[0.6, 0.7]$	$[0.8, 0.9]$	$[1, 1]$	$[0.7, 0.8]$	
Q_3	$[0, 0]$	$[0, 0]$	$[0, 0]$	$[0.4, 0.5]$	$[0.7, 0.8]$	$[1, 1]$	
Q_4	$[0, 0]$	$[0.4, 0.5]$	$[0.7, 0.8]$	$[1, 1]$	$[0, 0]$	$[0, 0]$	
Q_5	$[0, 0]$	$[1, 1]$	$[0.8, 0.9]$	$[0.5, 0.6]$	$[0, 0]$	$[0, 0]$	
							Total mark =

From Table 3, the interval-valued fuzzy marks of the questions Q_1, Q_2, Q_3, Q_4 and Q_5 can be represented by an interval-valued fuzzy sets $\widetilde{M}_1$, $\widetilde{M}_2$, $\widetilde{M}_3$, $\widetilde{M}_4$ and $\widetilde{M}_5$, respectively, as follows:

,

,

,

$$\widetilde{M}_1 = \frac{[0,0]}{0\%} + \frac{[0,0]}{20\%} + \frac{[0,0]}{40\%} + \frac{[0,0]}{60\%} + \frac{[0.8,0.9]}{80\%} + \frac{[1,1]}{100\%}$$
$$\widetilde{M}_2 = \frac{[0,0]}{0\%} + \frac{[0,0]}{20\%} + \frac{[0.6,0.7]}{40\%} + \frac{[0.8,0.9]}{60\%} + \frac{[1,1]}{80\%} + \frac{[0.7,0.8]}{100\%}$$
$$\widetilde{M}_3 = \frac{[0,0]}{0\%} + \frac{[0,0]}{20\%} + \frac{[0,0]}{40\%} + \frac{[0.4,0.5]}{60\%} + \frac{[0.7,0.8]}{80\%} + \frac{[1,1]}{100\%}$$
$$\widetilde{M}_4 = \frac{[0,0]}{0\%} + \frac{[0.4,0.5]}{20\%} + \frac{[0.7,0.8]}{40\%} + \frac{[1,1]}{60\%} + \frac{[0,0]}{80\%} + \frac{[0,0]}{100\%},$$
$$\widetilde{M}_5 = \frac{[0,0]}{0\%} + \frac{[1,1]}{20\%} + \frac{[0.8,0.9]}{40\%} + \frac{[0.5,0.6]}{60\%} + \frac{[0,0]}{80\%} + \frac{[0,0]}{100\%}.$$

4.1 Step 1: Obtain the derived fuzzy letter grades

Firstly, the standard interval-valued fuzzy sets $\tilde{E}$, $\tilde{V}$, $\tilde{G}$, $\tilde{S}$ and $\tilde{U}$ are represented by the matrices $\bar{E}$, $\bar{V}$, $\bar{G}$, $\bar{S}$ and $\bar{U}$, respectively as follows:

$$\bar{E} = \langle [0,0], [0,0], [0.8,0.8], [0.9,0.9], [1,1], [1,1] \rangle,$$
$$\bar{V} = \langle [0,0], [0,0], [0.8,0.8], [0.9,0.9], [0.9,0.9], [0.8,0.8] \rangle,$$
$$\bar{G} = \langle [0,0], [0.1,0.1], [0.8,0.8], [0.9,0.9], [0.4,0.4], [0.2,0.2] \rangle,$$
$$\bar{S} = \langle [0.4,0.4], [0.4,0.4], [0.9,0.9], [0.6,0.6], [0.2,0.2], [0,0] \rangle,$$
$$\bar{U} = \langle [1,1], [1,1], [0.4,0.4], [0.2,0.2], [0,0], [0,0] \rangle.$$

Then, represent the interval-valued fuzzy sets $\widetilde{M}_1$, $\widetilde{M}_2$, $\widetilde{M}_3$, $\widetilde{M}_4$ and $\widetilde{M}_5$ by the matrices $\overline{M}_1$, $\overline{M}_2$, $\overline{M}_3$, $\overline{M}_4$ and $\overline{M}_5$, respectively, where

$$\overline{M}_1 = \langle [0,0], [0,0], [0,0], [0,0], [0.8,0.9], [1,1] \rangle,$$
$$\overline{M}_2 = \langle [0,0], [0,0], [0.6,0.7], [0.8,0.9], [1,1], [0.7,0.8] \rangle,$$
$$\overline{M}_3 = \langle [0,0], [0,0], [0,0], [0.4,0.5], [0.7,0.8], [1,1] \rangle,$$
$$\overline{M}_4 = \langle [0,0], [0.4,0.5], [0.7,0.8], [1,1], [0,0], [0,0] \rangle,$$
$$\overline{M}_5 = \langle [0,0], [1,1], [0.8,0.9], [0.5,0.6], [0,0], [0,0] \rangle.$$

The degree of similarity between the five standard IVFS and interval-valued fuzzy mark of question Q_1 is obtained by applying equation (5) shown as follows:

$$\begin{aligned} T(\bar{E}, \overline{M}_1) = \frac{1}{6}\Bigg[& \left(1 - \frac{|0-0| + |0-0|}{2}\right) + \left(1 - \frac{|0-0| + |0-0|}{2}\right) \\ & + \left(1 - \frac{|0.8-0| + |0.8-0|}{2}\right) + \left(1 - \frac{|0.9-0| + |0.9-0|}{2}\right) \\ & + \left(1 - \frac{|1-0.8| + |1-0.9|}{2}\right) + \left(1 - \frac{|1-1| + |1-1|}{2}\right)\Bigg] \\ = {} & 0.692. \end{aligned}$$

$$
\begin{aligned}
T(\bar{V},\bar{M}_1) = \frac{1}{6}\Bigg[&\left(1-\frac{|0-0|+|0-0|}{2}\right)+\left(1-\frac{|0-0|+|0-0|}{2}\right)\\
&+\left(1-\frac{|0.8-0|+|0.8-0|}{2}\right)+\left(1-\frac{|0.9-0|+|0.9-0|}{2}\right)\\
&+\left(1-\frac{|0.9-0.8|+|0.9-0.9|}{2}\right)+\left(1-\frac{|0.8-1|+|0.8-1|}{2}\right)\Bigg]\\
= 0.675.&
\end{aligned}
$$

$$
\begin{aligned}
T(\bar{G},\bar{M}_1) = \frac{1}{6}\Bigg[&\left(1-\frac{|0-0|+|0-0|}{2}\right)+\left(1-\frac{|0.1-0|+|0.1-0|}{2}\right)\\
&+\left(1-\frac{|0.8-0|+|0.8-0|}{2}\right)+\left(1-\frac{|0.9-0|+|0.9-0|}{2}\right)\\
&+\left(1-\frac{|0.4-0.8|+|0.4-0.9|}{2}\right)+\left(1-\frac{|0.2-1|+|0.2-1|}{2}\right)\Bigg]\\
= 0.492.&
\end{aligned}
$$

$$
\begin{aligned}
T(\bar{S},\bar{M}_1) = \frac{1}{6}\Bigg[&\left(1-\frac{|0.4-0|+|0.4-0|}{2}\right)+\left(1-\frac{|0.4-0|+|0.4-0|}{2}\right)\\
&+\left(1-\frac{|0.9-0|+|0.9-0|}{2}\right)+\left(1-\frac{|0.6-0|+|0.6-0|}{2}\right)\\
&+\left(1-\frac{|0.2-0.8|+|0.2-0.9|}{2}\right)+\left(1-\frac{|0-1|+|0-1|}{2}\right)\Bigg]\\
= 0.342.&
\end{aligned}
$$

$$
\begin{aligned}
T(\bar{U},\bar{M}_1) = \frac{1}{6}\Bigg[&\left(1-\frac{|1-0|+|1-0|}{2}\right)+\left(1-\frac{|1-0|+|1-0|}{2}\right)\\
&+\left(1-\frac{|0.4-0|+|0.4-0|}{2}\right)+\left(1-\frac{|0.2-0|+|0.2-0|}{2}\right)\\
&+\left(1-\frac{|0-0.8|+|0-0.9|}{2}\right)+\left(1-\frac{|0-1|+|0-1|}{2}\right)\Bigg]\\
= 0.375.&
\end{aligned}
$$

Since the standard interval-valued fuzzy sets $\tilde{E}$, $\tilde{V}$, $\tilde{G}$, $\tilde{S}$ and $\tilde{U}$ corresponded to the letter grades "***A***", "***B***", "***C***", "***D***" and "***E***", respectively, the derived fuzzy letter grade $\tilde{g}_1$ of question Q_1 can be represented by a fuzzy set, shown as follows:

$$\tilde{g}_1 = \frac{0.692}{\boldsymbol{A}}+\frac{0.675}{\boldsymbol{B}}+\frac{0.492}{\boldsymbol{C}}+\frac{0.342}{\boldsymbol{D}}+\frac{0.375}{\boldsymbol{E}}.$$

As stated earlier, the index of optimism λ of the evaluator is 0.65. Then, the derived grade point, $K(\tilde{g}_1)$ of question Q_1 can be obtained by applying equation (7).

$$
\begin{aligned}
K(\tilde{g}_1) = \{&0.692\times[(1-0.65)\cdot 90+(0.65)\cdot 100]\\
&+0.675\times[(1-0.65)\cdot 70+(0.65)\cdot 90]\\
&+0.492\times[(1-0.65)\cdot 50+(0.65)\cdot 70]\\
&+0.342\times[(1-0.65)\cdot 30+(0.65)\cdot 50]\\
&+0.375\times[(1-0.65)\cdot 0+(0.65)\cdot 30]\}\\
&/\,(0.692+0.675+0.492+0.342+0.375)
\end{aligned}
$$

$$= \frac{0.692 \cdot 96.5 + 0.675 \cdot 83 + 0.492 \cdot 63 + 0.342 \cdot 43 + 0.375 \cdot 19.5}{2.576}$$
$$= \frac{66.788 + 56.025 + 30.996 + 14.706 + 7.313}{2.576}$$
$$= 68.256.$$

The same process is repeated for $\widetilde{M}_2$, $\widetilde{M}_3$, $\widetilde{M}_4$ and $\widetilde{M}_5$ to obtain their own derived grade point where $K(\tilde{g}_2) = 71.689$, $K(\tilde{g}_3) = 71.267$, $K(\tilde{g}_4) = 60.780$ and $K(\tilde{g}_5) = 55.880$, respectively. Based on equation (7), the total mark of the student can be calculated, as follows:

$$Total\ Mark$$
$$= \frac{1}{100}[R(Q_1) \cdot K(\tilde{g}_1) + R(Q_2) \cdot K(\tilde{g}_2) + R(Q_3) \times K(\tilde{g}_3) + R(Q_4) \cdot K(\tilde{g}_4) + R(Q_5) \cdot K(\tilde{g}_5)]$$
$$= \frac{[10(68.256) + 20(71.689) + 20(71.267) + 25(60.780) + 25(55.880)]}{100}$$
$$= 64.582$$
$= 65$ (assuming no half mark is given in the total mark).

The total mark of this student is 65 with the assumption of no half mark is given in the total mark of this type of answerscript. The evaluator is said to be optimistic during the judgement process. The letter grade of this student is a "$\boldsymbol{C}$" since 65 is in $50 \leq \boldsymbol{C} < 70$.

5 Conclusion

In this study, we have presented the method of evaluating students' answerscripts based in the interval-valued fuzzy sets. This method has achieved the purpose of an educational institutions which is to provide the students a more flexible and intelligent evaluation system.

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