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On Dynamics of ℓ-Volterra Quadratic Stochastic Operators on 1-Dimensional Simplex

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Abstract The purpose of this study is to find the canonical form, fixed point, the stability of fixed point and to study the trajectory of ℓ-Volterra quadratic stochastic operators (QSOs) in 1-dimensional simplex. ℓ-Volterra quadratic stochastic operator was defined on (n - 1)-dimensional simplex, where ℓ ∈ {0, 1,…, n}. The ℓ-Volterra operator is a Volterra operator if and only if ℓ = n. In this study, we are going to use $\ell = 1$ and $\ell = 2$. By utilize the technique of Jacobian, we study stability of the fixed points. Moreover, we fully study the dynamical behaviour of Volterra QSO defined on S^1.

Keywords: Quadratic stochastic operator; fixed point; Volterra; simplex

1 Introduction

A quadratic stochastic operator (QSO) has meaning of a population evolution operator, which arises as follows. Consider a population consisting of n species. Let $x^0 = (x_1^0, \dots, x_n^0)$ be the probability distribution of species in the initial generations, and P_{ijk} the probability that individuals in the ith and jth species reproduction to produce an individual k. Then, the probability distribution $(x) = (x_1', \dots, x_n')$ (the state) of the species in the first generation can be found by the total probability i.e.

$$\mathrm{V(x)}_k = \sum_{ij,k}^{n} P_{ijk}\, x_i x_j, \qquad k = 1, \dots, n \qquad 1.1$$

This means that the association $x \rightarrow V$ defines a map V called the evolution operator. The population evolves by starting from an arbitrary state x^0 then passing to the state $x' = V(x)$ (in the next "generation"), then to the state $x'' = V(V(x))$ and so on. Note that V (defined by (1.1)) is a non-linear (quadratic) operator, and it is higher-dimensional if $n \geq 3$. Higher-dimensional dynamical systems are important but here are relatively few dynamical phenomena that are currently understood (see [1,2,20]).

$$x, \qquad x' = (x), \qquad x'' = ((x)), \qquad x''' = V^3(x), \qquad \dots \qquad 1.2$$

In other words, each QSO describes the evolution of generations in terms of probabilities distribution.

Ganikhodzhaev et al. (2011) has provided a self-contained exposition of the recent achievements and open problems in the theory of the QSO. What's left to study in the nonlinear operator theory is the behaviour of nonlinear operators. Unfortunately, there is a lack of study on the utmost vital part, which is the dynamical phenomena on higher dimensional systems that are currently comprehendible. In case of QSOs, the degree of difficulty depends on the given cubic matrix$(P_{ijk})_{ij,k=1}^{m}$. An asymptotic behaviour of the QSOs is complicated, even for the small dimensional simplex (N. Ganikhodzhaev and Zanin, 2004; Mukhamedov, Saburov and Qaralleh, 2013; Mukhamedov, Qaralleh and Rozali, 2014; Vallander, 1972; Zakharevich, 1978). Many researchers introduced a certain class of QSO and studied their behaviour, for example Volterra-QSO, permutated Volterra-QSO, Quasi-Volterra-QSO, and ℓ-Volterra-QSO to solve this problem.

2 Preliminaries

Let V be a mapping on the (n-1) dimensional

$$\mathrm{S}^{n-1} = \{(x_1, x_2, \dots, x_n) \in \mathrm{R}^n, x_i \geq 0, \textstyle\sum_{i=1}^{n} x_i = 1\},$$

maps into itself, $V : S^{n-1} \to S^{n-1}$. V has such a form

$$(\mathrm{x})_k = \textstyle\sum_{i=1}^{n} P_{ij,k} x_i y_j \ , \mathrm{k} = 1,2, \dots, \mathrm{n}, \qquad 2.1$$

where $P_{ij,k}$ are coefficient of heredity and satisfy

$$\mathrm{P}_{ijk} \geq 0, \mathrm{P}_{ijk} = \mathrm{P}_{jik}, \textstyle\sum \mathrm{P}_{ijk} = 1, \qquad \mathrm{i, j, k} = 1,2, \dots, \mathrm{n}. \qquad 2.2$$

Then, V is called Quadratic Stochastic Operators (QSOs). Next, i going to recall the definition of ℓ-Volterra Quadratic Stochastic Operators.

Definition for Volterra

Definition 1.1 The QSO defined by (3.2.2), (3.2.3), is called an ℓ -Volterra QSO if

$$\mathrm{P}_{ijk} = 0 \text{ for } \mathrm{k} \notin \{\mathrm{i, j}\}, \ \mathrm{k} = 1, \dots, \ell, \ \mathrm{i, j} = 1, \dots, \mathrm{n}; \qquad 2.3$$

$$P_{ijk} > 0 \qquad 2.4$$

for at least one set $i, j, k, i \neq k, j \neq k$ for any $k \in \{\ell + 1, \dots, n\}$.

Denote by V_ℓ the set of all ℓ-Volterra QSOs

Remark 1.2

1. The condition (2.4) guarantees that $V_{\ell_1} \cap V_{\ell_2} = \emptyset$ for any $\ell_1 = \ell_2$.
2. Note that ℓ-Volterra QSO is Volterra if and only if $\ell = n$.
3. Quasi-Volterra operators are particular case of ℓ-Volterra operators.
4. The class of ℓ-Volterra QSO for a given ℓ does not coincide with a class of non-Volterra

QSOs.

Definition 1.3 Let f be mapping from set X to set X again. If any c ∈ X and f(c) = c then c is a fixed point of f.

Definition 1.4 A fixed point x_0 for $F : R^n \to R^n$ is called hyperbolic if all the eigenvalues of the Jacobian matrix J of the mapping F at the point x_0 are not equal to 1.

There are three types of hyperbolic fixed points:

1. P is an attracting fixed point if all of the eigenvalues of J(P) are less than one in absolute value.
2. P is a repelling fixed point if all of the eigenvalues of J(P) are greater than one in absolute value.
3. P is a saddle point otherwise.

In this project, i going to consider $n = 2$. Therefore, $\ell \in \{1,2\}$. In what follows, i consider 1-Volterra and 2-Volterra. In case 2-Volterra, i simply Volterra QSO.

3 Volterra Quadratics Stochastic Operators

First to study the Canonical form of Volterra's discrete model.

3.1 Canonical form of Volterra's QSO

In this section, we are going describe the canonical form of Volterra acting on one dimensional simplex where $n = 2$ on S^1.
Recall the definition for Volterra,

$$P_{ij,k} = 0 \text{ for } \mathrm{k} \neq \{\mathrm{i,j}\}, \text{ where } \mathrm{k} = 1, \dots, \mathrm{n}, \quad \mathrm{i,j} = 1, \dots, \mathrm{n}. \tag{3.1}$$

Recall S^1(i.e., see 3.2) reduce to

$$\mathrm{S}^1 = \{(\mathrm{x}_1, \mathrm{x}_2),\ \mathrm{x}_1, \mathrm{x}_2 \geq 0,\ \mathrm{x}_1 + \mathrm{x}_2 = 1\}\,. \tag{3.2}$$

Furthermore, $V(x) = (V(x_1), V(x_2))$ given by (2.2) can be written as

$$\mathrm{V(x)}_1 = \mathrm{P}_{11,1}\mathrm{x}_1^2 + 2\mathrm{P}_{12,1}\mathrm{x}_1\mathrm{x}_2 + \mathrm{P}_{22,1}\mathrm{x}_2^2, \tag{3.3}$$
$$\mathrm{V(x)}_2 = \mathrm{P}_{11,2}\mathrm{x}_1^2 + 2\mathrm{P}_{12,2}\mathrm{x}_1\mathrm{x}_2 + \mathrm{P}_{22,2}\mathrm{x}_2^2, \tag{3.4}$$

where $\mathrm{P}_{11,1}, \mathrm{P}_{12,1}, \mathrm{P}_{22,1}, \mathrm{P}_{11,2}, \mathrm{P}_{12,2}, \mathrm{P}_{22,2} \epsilon [0,1)$.

Because of stochasticity (i.e. $P_{ij,1} + P_{ij,2} = 1$), so i has

$$P_{11,1} + P_{11,2} = 1$$
$$P_{22,1} + P_{22,2} = 1$$

From (3.6), i has

$$P_{11,2} = 0, \qquad P_{22,1} = 0$$

Therefore, i denote

$$P_{11,1} = 1, P_{22,2} = 1 \tag{3.5}$$

Substitute (3.5) into (3.3) and (3.4), then I has

$$\begin{aligned} \mathrm{V(x)}_1 &= \mathrm{x}_1^2 + 2\mathrm{P}_{12,1}\mathrm{x}_1\mathrm{x}_2 \\ \mathrm{V(x)}_2 &= 2\mathrm{P}_{12,2}\mathrm{x}_1\mathrm{x}_2 + \mathrm{x}_2^2 \end{aligned} \tag{3.6}$$

where $\mathrm{P}_{12,1}, \mathrm{P}_{12,2}$ are unknown.

Therefore i taking into account $\mathrm{P}_{12,1} + \mathrm{P}_{12,2} = 1$, I has $\mathrm{P}_{12,2} = 1 - \mathrm{P}_{12,1}$, Let $\mathrm{P}_{12,1} = \mathrm{a}$

Therefore, $P_{12,2} = 1 - a$.
Subtitute $P_{12,2} = 1 - a$ into (4.4) then equation will become

$$V(x_1) = x_1^2 + 2ax_1 x_2$$

$$V(x_2) = 2(1-a)x_1x_2 + x_2^2 \qquad 3.7$$

By using properties $x_2 = 1 - x_1$, then

$$\begin{aligned} V(x)_1 &= x_1^2 + 2ax_1(1 - x_1), \\ &= x_1^2 + 2ax_1 - 2ax_1^2, \\ &= x_1^2 + 2ax_1 - 2ax_1^2, \\ &= 2ax_1 + (1-2a)x_1^2. \end{aligned}$$

The canonical form is $V(x)_k = 2ax_k + (1-2a)x_k^2$.

3.2 Fixed Point of Volterra QSO

In this section, i going to solve all fixed point of Volterra QSO. As i know, the fixed point of Volterra on 1-dimensional simplex are (1,0), (0,1).
Proof., The fixed point of volterra Quadratic Stochastics Operator in S^1.

$$\begin{gathered} x = (2a-1)(x_1^2 - x_1), \\ 0 = (2a-1)x_1(x_1 - 1), \\ x_1 = 0, \qquad x_1 - 1 = 0, \\ x_1 = 0 \text{ or } 1, \\ x_1 + x_2 = 1, \\ x_2 = 1 - x_1. \end{gathered}$$

Sub x_1 into x_2,

$$x_2 = 0 \text{ or } 1.$$

Therefore, $V(x_1, x_2) = (0,1)$ or $V(x_1, x_2) = (1,0)$
Proven.

3.3 Stability of Fixed Point of Volterra QSO

In this subsection, i going to study stability of fixed point.

$$V(x) = x' = x^2 + 2ax(1-x),$$

$$y' = 1 - \left(x^2 + 2ax(1-x)\right). \qquad 3.8$$

i) First, i going to consider fixed point for $x = 1$

$$\begin{gathered} f(x) = x^2 + 2ax(1-x), \\ f(x) = x^2 + 2ax - 2ax^2, \\ f'(x) = 2x + 2a - 4ax, \\ |f'(1)| = |2 + 2a - 4a|, \\ = |2 - 2a|, \\ = |2(1-a)|, \\ = 2(1-a). \end{gathered}$$

Suppose $2(1-a) < 1$, then I has

$$(1-a) < \frac{1}{2} \quad \Rightarrow \quad -a < -\frac{1}{2}$$

$$a > \frac{1}{2}$$

From this, i conclude that
If $a \in \left(\frac{1}{2}, 1\right]$ then (1,0) attracting.
If $a \in \left[0, \frac{1}{2}\right]$ then (1,0) repelling.
If $a = \frac{1}{2}$ then (1,0) non − hyperbolic fixed point.

ii) Secondly, i going to consider fixed point for $x = 0$;

$$f'(0) = 2a.$$

Suppose

$$2a < 1 \implies a < \frac{1}{2}.$$

From this, i can get;
If $a \in \left[0, \frac{1}{2}\right)$ then (0,1) repelling.
If $a \in \left(\frac{1}{2}, 1\right]$ then (0,1) attracting.
If $a = \frac{1}{2}$ then (0,1) non − hyperbolic fixed point.

$$V(x) = x_1' = x^2 + 2ax(1-x) \qquad 0 \leq x \leq 1,$$
$$x_2' = 1 - \left(x^2 + 2ax(1-x)\right) \quad 0 \leq x \leq 1.$$

3.4 Dynamics of Volterra QSO

To study dynamics of V(x) it enough to study x_1' since $x_2' = 1 - x_1'$

$$\begin{aligned} f(x) &= x^2 + 2ax(1-x) \\ &= (1-2a)x^2 + 2ax \end{aligned} \qquad 3.9$$

Here, i going to completely describe dynamics of volterra QSO on one dimensional simplex by considering all the posibble cases. I considering 3 cases as the folowings :

$$(1)\ \ a \in \left[0, \frac{1}{2}\right)$$
$$(2)\ \ a = \frac{1}{2}$$
$$(3)\ \ a \in \left(\frac{1}{2}, 1\right]$$

First Case

I divide first case into two subcases: subcase 1: $(a = 0)$, subcase 2: $(0 < a < \frac{1}{2})$

Subcase 1: i let $a = 0$. Now define

$$f(x) = x^2 \qquad 3.10$$

I assume for $\forall\, x \in [0,1], f(x) \in [0,1]$. Now, let me prove the assumption.

Proposition 4.1 Let $f(x) = x^2$ given by (3.15), then

$$\forall\, x \in [0,1], f(x) \in [0,1]$$

Proof. Consider $x \in [0,1]$ and compute $f'(x) = 2x$. It is obvious that $f'(x)$ takes positive values only on interval [0, 1]. Thus f(x) is increasing on interval [0,1]. Then I has

$$0 \le x \le 1 \Rightarrow f(0) \le f(x) \le f(1) \Rightarrow 0 \le f(x) \le a \Rightarrow f(x) \in [0,1]$$

Thus $f(x) \in [0,1]$.

It suggests that iterations of $x \in [0,1]$ will converge to 0. Furthemore, $|f'(0)| = 0 < 1$ implies that x_1 is a super attracting fixed point and $|f'(1)| = 2 > 1$ implies that x_2 is a repelling fixed point. Let me find the recurrent equation for $f(x) = x^2$. First of all, compute $\mathrm{f}^2(\mathrm{x})$ and $\mathrm{f}^3(\mathrm{x})$.

$$f^2(x) = f\big(f(x)\big) = f(x^2) = x^4 = x^{2^2}$$
$$f^3(x) = f\big(f^2(x)\big) = f(x^4) = x^8 = x^{2^3}$$

I see that the trajectory of f takes the following form: $\mathrm{f}^n(\mathrm{x}) = \mathrm{x}^{2^n}$. I prove this trajectory by the following proposition :

Proposition 4.2. Let $f(x) = x^2$ given by (3.15) then

$$\mathrm{f}^n(\mathrm{x}) = \mathrm{x}^{2^n}$$

Proof. I will use technique of induction. Let $f(x) = x^2$ and assume that $f^n(x) = \mathrm{x}^{2^n}$. Firstly, i test the assumption by n = 1. I has

$$\mathrm{f}^1(\mathrm{x}) = \mathrm{x}^{2^1} = x^2$$

Thus, it is true for n=1. Secondly, assume that it is also true for $n = k$, so I has

$$\mathrm{f}^k = \mathrm{x}^{2^k}$$

Lastly, i prove that it is also true for $k + 1$.

$$\begin{aligned} \mathrm{f}^{k+1}(\mathrm{x}) &= \mathrm{f}\left(\mathrm{f}^k(\mathrm{x})\right) \\ &= \mathrm{f}\left(\mathrm{x}^{2^k}\right) \\ &= (\mathrm{x}^{2^k})^2 \\ &= \mathrm{x}^{2(2^k)} \\ &= \mathrm{x}^{2^{k+1}} \end{aligned}$$

This complete the induction. The assumption is also true for k=1, thus $f^n(x) = \mathrm{x}^{2^n}$ is proven.

As n goes to infinity, I has

$$\lim_{n\to\infty} \mathrm{f}^n(\mathrm{x}) = \lim_{n\to\infty} \mathrm{x}^{2^n} = 0$$

For example, if the initial point $\mathrm{x} = \frac{1}{2}$, then I has,

$$\lim_{n\to\infty} \left(\frac{1}{2}\right)^{2^n} = \lim_{n\to\infty} \frac{1}{2^{2^n}} = 0$$

Conclusion:

$$V_n(x) = \left(\mathrm{x}^{2^n}, 1 - \mathrm{x}^{2^n}\right)$$

Where $V_n(x)$ is given by (2.1). As $n \to \infty$ for $\forall\, x \in [0,1]$

$$\lim_{n\to\infty} f^n(x) = 0$$
$$\lim_{n\to\infty} V^n(x) = (0,1)$$

Subcase 2: I let $a \in (0,\frac{1}{2})$. Let

$$f(x) = (1-2a)x^2 + 2ax \tag{3.11}$$

and my aim is to study the dynamics of this function.

Solving the fixed point, one has

$$x = (1-2a)x^2 + 2ax \Longrightarrow x_1 = 0, x_2 = \frac{a}{2a-1}$$

Clearly that in this case $1-2a > 0$. Then, $f(x)$ must be concave up. One also can check $x^2 \in [0,1]$ by the followings :

$$1-2a > 0 \Longrightarrow \frac{a}{2a-1} > 0$$

The derivative of f(x) is given by

$$f'(x) = 2(1-2a)x + 2a$$

Then, solve $f'(x) = 0$. Thus, critical point is given has

$$f'(x) = 0 \quad \Longrightarrow x = \frac{a}{(2a-1)}$$

Thus, one sees that $x < 0$, therefore there is no critical point in $[0,1]$. Clearly for any $x \in [0,1], f'(x) > 0$. Therefore, f(x) must be increasing function. Hence,

$$f(0) \leq f(x) \leq f(1)$$

Which gives $0 \leq f(x) \leq 1$ for any $x \in [0,1]$.

Proposition 3.3 Let $f(x) = (1-2a)x^2 + 2ax$ given by (3.17), then

$$\lim_{n\to\infty} f^n(x) = \frac{a}{2a-1} \text{ for } \forall x \in [0,1].$$

Take all $x \in [0,1]$ and let

$$h(x) = f(x) - x = (1-2a)x^2 + 2ax - x$$

Then, solve for $h(x) = 0$

$$0 = (1-2a)x^2 + (2a-1)x \quad \Longrightarrow x_1 = 0, \quad x_2 = \frac{2a-1}{2a-1} = 1$$

Because of $x \geq 0, (1-2a) > 0$ and $(x-1) < 0,$ therefore $h(x) \leq 0$ implies that $f(x) - x \leq 0$. Since $f(x)$ increasing, then the followings hold

$$\begin{gathered} f(x) \leq x \\ f(x)^2 \leq f(x) \\ f(x)^3 \leq f(x)^2 \\ \vdots \\ f^n(x) \leq f^{n-1}(x) \end{gathered}$$

For all n ϵ ℕ. The sequence of $f, f^2, f^3, \ldots, f^n$, ... is monotone decreasing and bounded below for $\forall\, x \in [0,1]$. Therefore, $\{\mathrm{f}^{\mathrm{n}}(\mathrm{x})\}_{\mathrm{n}=1}^{\infty}$ converges. Thus, the limit exist. Next, compute the followings

$$\lim_{n\to\infty} \mathrm{f}^{\mathrm{n}}(\mathrm{x}) = \mathrm{x}^*, \mathrm{x}^* = \lim_{n\to\infty} \mathrm{f}^{\mathrm{n}+1}(\mathrm{x}) = \lim_{n\to\infty} \mathrm{f}(\mathrm{f}^{\mathrm{n}}(\mathrm{x})) = \mathrm{f}\left(\lim_{n\to\infty} \mathrm{f}^{\mathrm{n}}(\mathrm{x})\right) = \mathrm{f}(\mathrm{x}^*)$$

This shows that x^* is a fixed point either $\mathrm{x}^* = 0$ or $\mathrm{x}^* = 1$.

In conclusion

$$\begin{gathered} \lim_{n\to\infty} \mathrm{f}^{\mathrm{n}}(\mathrm{x}) = 0 \\ \lim_{n\to\infty} \mathrm{V}^{\mathrm{n}}(\mathrm{x}) = (0,1) \end{gathered}$$

Second Case

In this case, i let $a = \frac{1}{2}$. Let

$$\mathrm{f}(\mathrm{x}) = \left(1 - 2\left(\frac{1}{2}\right)\right)\mathrm{x}^2 + 2\left(\frac{1}{2}\right)\mathrm{x} = \mathrm{x}$$

and my aim is to study the dynamics of this function.

I assume for $\forall\, x \in [0,1], f(x) \in [0,1]$. Now, let me prove the assumption.
By using Proposition 3.1. Let $\mathrm{f}(\mathrm{x}) = x$ then

$$\forall\, x \in [0,1], f(x) \in [0,1]$$

Proof. Consider $x \in [0,1]$ and compute $f\,'(x) = 1$. It is obvious that $f'(x)$ takes positive values only on interval [0, 1]. Thus f(x) is increasing on interval [0,1]. Then I has

$$0 \leq x \leq 1 \Rightarrow f(0) \leq f(x) \leq f(1) \Rightarrow 0 \leq f(x) \leq \frac{1}{2} \Rightarrow f(x) \in [0,1]$$

Thus $f(x) \in [0,1]$.

It suggests that iterations of $x \in [0,1]$ will converge to 0. Furthemore, $|f'(0)| = 0 < 1$ implies that x is a non − hyperbolic fixed point. Let me find the recurrent equation for $f(x) = x$. First of all, compute $\mathrm{f}^2(\mathrm{x})$ and $\mathrm{f}^3(\mathrm{x})$.

$$\begin{gathered} f^2(x) = f\big(f(x)\big) = f(x) = x^4 = x \\ f^3(x) = f\big(f^2(x)\big) = f(x) = x^8 = x \end{gathered}$$

I see that the trajectory of *f* takes the following form : $f^n(x) = x$.

Therefore, if $a = \frac{1}{2}$, $\lim_{n\to\infty} f^n(x) = x \to \lim_{n\to\infty} V^n(X) = (x_1, 1 - x_1)$ where X=(x,y)

I note that in this case all points are fixed point.
In conclusion

$$\lim_{n\to\infty} f^n(x) = 0$$
$$\lim_{n\to\infty} V^n(x) = (0,1)$$

Third Case

Let $a \in (\frac{1}{2}, 1]$. I divide into two subcase; subcase 1: $(a = 1)$, subcase 2: $\left(\frac{1}{2} < a < 1\right)$

Subcase 1
i let $a = 1$. Let

$$f(x) = -x^2 + 2x$$

and my aim is to study the dynamics of this function.
Solve for the fixed point :

$$x = -x^2 + 2x \ \in \ [0,1] \Longrightarrow \ x_1 = 0, x_2 = 1$$

The derivative of f(x) is given by

$$f'(x) = -2x + 2$$

Then, solve $f'(x) = 0$. Thus, critical point is given has

$$f'(x) = 0 \ \ \Longrightarrow x = 1$$

Thus, one sees that $x = 1$, therefore there is critical point in $[0,1]$. Clearly for $x \in [0,1]$, $f'(x) \leq 0$. Therefore, f(x) must be decreasing function. Hence,

$$f(1) \leq f(x) \leq f(0)$$

Which gives $1 \leq f(x) \leq 0$ for any $x \in [0,1]$.
This completes the proof.

Proposition 3.3 Let $f(x) = -x^2 + 2x$ given by (3.18), then

$$\lim_{n\to\infty} f^n(x) = -x^2 + 2x \quad \text{for } \forall x \in [0,1].$$

Take $\forall x \in [0,1]$ and let

$$h(x) = f(x) - x = -x^2 + 2x \ - x$$

Then, solve for $h(x) = 0$

$$0 = -x^2 + x \quad \Longrightarrow x_1 = 0, \quad x_2 = 1$$

Because of $x \geq 0$, $-x^2 + 2x < 0$ and $(x-1) \leq 0$, therefore $h(x) \geq 0$ implies that $f(x) - x \geq 0$. Since (x) decreasing, then the followings hold

$$\begin{gathered} f(x) \geq x \\ f(x)^2 \geq f(x) \\ f(x)^3 \geq f(x)^2 \\ \vdots \\ f^n(x) \geq f^{n-1}(x) \end{gathered}$$

For all n ϵ ℕ. The sequence of $f, f^2, f^3, \ldots, f^n, \ldots$ is monotone increasing and bounded above for $\forall\, x \in [0,1]$. Therefore, $\{f^n(x)\}_{n=1}^{\infty}$ converges. Thus, the limit exist. Next, compute the followings

$$\lim_{n\to\infty} f^n(x) = x^*, x^* = \lim_{n\to\infty} f^{n+1}(x) = \lim_{n\to\infty} f(f^n(x)) = f\left(\lim_{n\to\infty} f^n(x)\right) = f(x^*)$$

This shows that x^* is a fixed point either $x^* = 0$ or $x^* = 1$.

In conclusion

$$\begin{gathered} \lim_{n\to\infty} f^n(x) = 0 \\ \lim_{n\to\infty} V^n(x) = (0,1). \end{gathered}$$

Subcase 2

I let $a \in (\frac{1}{2}, 1)$. Let

$$f(x) = (1-2a)x^2 + 2ax$$

Clearly that in this case $1 - 2a < 0$. Then f(x) must be concave down.
Solving the fixed point, one has

$$\begin{gathered} x = (1-2a)x^2 + 2ax, \\ x = 0, \qquad x = \frac{2a-1}{2a-1} = 1. \end{gathered}$$

The derivative of f(x) is given by

$$f'(x) = 2(1-2a)x + 2a$$

Thus, critical point is given has

$$x = \frac{-2a}{2(1-2a)}$$

One sees that $x > 0$, therefore there is no critical point in [0,1]
Clearly for any $x \in [0,1]$, $f'(x) > 0$
Therefore, f(x) must be decreasing function. Hence,

$$f(0) \leq f(x) \leq f(1)$$

Which gives

$$0 \leq f(x) \leq 1$$

for any $x \in [0,1]$.
Now, let me study the trajectory of f(x). Define

$$\begin{aligned} h(x) &= f(x) - x \\ h(x) &= (1-2a)x^2 + 2ax - x \\ &= (1-2a)x^2 + (2a-1)x \\ &= (1-2a)x^2 + (1-2a)x \end{aligned}$$

$$= x(1-2a)(x-1)$$

Because of $x \geq 0, (1-2a) > 0$ and $(x-1) < 0$, therefore

$$\mathrm{h(x)} \leq 0$$

Hence

$$\mathrm{f(x) - x} \leq 0$$
$$\mathrm{f(x)} \leq \mathrm{x}$$

Due to increasing property of f then

$$\mathrm{f^n(x)} \leq \mathrm{f^{n-1}(x)}$$

For all n ϵ ℕ. The sequence of

$$\mathrm{f, f^2, f^3, \dots, f^n, \dots}$$

is montone decreasing and bounded below. Thus, the limit exist. Since limit point must fixed point, then

$$\lim_{n\to\infty} \mathrm{f^n(x)} = 0.$$

In conclusion

$$\lim_{n\to\infty} \mathrm{f^n(x)} = 0$$
$$\lim_{n\to\infty} \mathrm{V_n(x)} = (0,1).$$

Example

I let $a = \frac{3}{4}$. Let

$$f(x) = -\frac{1}{2}x^2 + \frac{3}{2}x$$

Clearly that in this case $-\frac{1}{2} < 0$. Then f(x) must be concave down.
Solving the fixed point, one has

$$x = -\frac{1}{2}x^2 + \frac{3}{2}x,$$
$$0 = -\frac{1}{2}x^2 + \frac{3}{2}x - x = -\frac{1}{2}x^2 + \frac{1}{2}x,$$
$$\mathrm{x} = 0, \qquad \mathrm{x} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

The derivative of f(x) is given by

$$f'^{(x)} = -x + \frac{3}{2}$$

Thus, critical point is given has

$$\mathrm{x} = \frac{-\frac{3}{2}}{-1} = \frac{3}{2}$$

One sees that $\mathrm{x} > 0$, therefore there is no critical point in [0,1]
Clearly for any x ϵ [0,1], $\mathrm{f'(x)} > 0$
Therefore, f(x) must be decreasing function. Hence,

$$\mathrm{f(0)} \leq \mathrm{f(x)} \leq \mathrm{f(1)}$$

Which gives

$$0 \leq \mathrm{f(x)} \leq 1$$

for any x ϵ [0,1].
Now, let us study the trajectory of f(x). Define

$$\mathrm{h(x) = f(x) - x}$$

$$\begin{aligned} \mathrm{h(x)} &= -\frac{1}{2}x^2 + \frac{3}{2}x - x \\ &= -\frac{1}{2}x^2 + \frac{1}{2}x \end{aligned}$$

Because of $x \geq 0, (1-2a) > 0$ and $(x-1) < 0$, therefore

$$\mathrm{h(x)} \leq 0$$

Hence

$$\mathrm{f(x) - x \leq 0}$$
$$\mathrm{f(x) \leq x}$$

Due to increasing property of f then

$$\mathrm{f^n(x) \leq f^{n-1}(x)}$$

For all n ϵ ℕ. The sequence of

$$\mathrm{f, f^2, f^3, \ldots, f^n, \ldots}$$

is montone decreasing and bounded below. Thus, the limit exist. Since limit point must fixed point, then

$$\lim_{\mathrm{n}\to\infty} \mathrm{f^n(x)} = 0.$$

In conclusion

$$\lim_{\mathrm{n}\to\infty} \mathrm{f^n(x)} = 0$$
$$\lim_{\mathrm{n}\to\infty} \mathrm{V_n(x)} = (0,1).$$

4 ℓ-Volterra Quadratic Stochastic Operators

4.1 Canonical Form of ℓ-Volterra QSO

In this section, I going describe the canonical form of ℓ-Volterra acting on one dimensional simplex where $n = 2$ on S^1.
Recall the definition for ℓ-Volterra,

$$\mathrm{P}_{ijk} = 0 \text{ for } \mathrm{k} \notin \{\mathrm{i,j}\},\ \mathrm{k} = 1, \ldots, \ell,\ \mathrm{i,j} = 1, \ldots, \mathrm{n}; \quad 4.1$$

$$P_{ijk} > 0 \text{ for at least one set } i,j,k, i \neq k, j \neq k \text{ for any } k \in \{\ell + 1, \ldots, n\}. \quad 4.2$$

Recall S^1(i.e., see 3.2) reduce to

$$\mathrm{S^1} = \{(\mathrm{x_1, x_2}),\ \mathrm{x_1, x_2} \geq 0,\ \mathrm{x_1 + x_2} = 1\}. \quad 4.3$$

Furthermore, $V(x) = (V(x_1), V(x_2))$ given by (3.2) can be written as

$$\mathrm{V(x)_1 = P_{11,1}x_1^2 + 2P_{12,1}x_1x_2 + P_{22,1}x_2^2}, \quad 4.4$$
$$\mathrm{V(x)_2 = P_{11,2}x_1^2 + 2P_{12,2}x_1x_2 + P_{22,2}x_2^2}, \quad 4.5$$

where $\mathrm{P_{11,1}}$, $\mathrm{P_{12,1}}$, $\mathrm{P_{22,1}}$, $\mathrm{P_{11,2}}$, $\mathrm{P_{12,2}}$, $\mathrm{P_{22,2}}$ ϵ [0,1).

The canonical form is $V(x)_k = 2ax_k + (1-2a)\mathrm{x_k}$. Now let me consider $n = 2$.
In this case $\ell = 1$,

$$P_{22,1} = 0,\ P_{22,2} = 1,\ P_{11,2} > 0. \quad 4.6$$

Because of stochasticity (i.e. $P_{ij,1} + P_{ij,2} = 1$), so I has

$$P_{11,1} + P_{11,2} = 1,$$
$$P_{22,1} + P_{22,2} = 1.$$

Based on definition of ℓ-Volterra, I has

$$P_{11,2} > 0, \qquad P_{22,1} = 0$$

Therefore, I denote

$$P_{22,2} = 1. \quad 4.7$$

Substitute (4.7) into (4.5) and (4.6), then I has

$$V(x)_1 = P_{11,1}x_1^2 + 2P_{12,1}x_1x_2, \qquad 4.8$$
$$V(x)_2 = P_{11,2}x_1^2 + 2P_{12,2}x_1x_2 + x_2^2,$$

where $P_{11,1}, P_{12,1}, P_{12,2}$ are unknown.

Therefore i taking into account $P_{12,1} + P_{12,2} = 1$ and $P_{11,1} + P_{11,2} = 1$, I has $P_{12,2} = 1 - P_{12,1}$, Let $P_{12,1} = b$, $P_{11,1} = a$
Therefore, $P_{12,2} = 1 - b, P_{11,2} = 1 - a$.
Subtitute $P_{12,2} = 1 - a$ into (4.6) then equation will become

$$V(x_1) = ax_1^2 + 2bx_1x_2,$$
$$\qquad 4.9$$
$$V(x_2) = (1-a)x_1^2 + 2(1-b)x_1x_2 + x_2^2.$$

By using properties $x_2 = 1 - x_1$, then

$$\begin{aligned} V(x)_1 &= ax_1^2 + 2bx_1x_2, \\ &= ax_1^2 + 2bx_1(1-x_1) \\ &= ax_1^2 + 2bx_1 - 2bx_1^2, \\ &= ax_1^2 + 2bx_1 - 2bx_1^2, \\ &= 2bx_1 + (a-2b)x_1^2. \end{aligned}$$

Thus,

$$V(x_1) = ax^2 + 2bx(1-x)$$
$$V(x_2) = (1-a)x^2 + 2(1-b)x_1(1-x) + (1-x)^2.$$

4.2 Fixed Point of ℓ-Volterra QSO.

In this section, i going to find all the fixed point of ℓ-volterra quadratic stochastic operators on 1-dimensional simplex.

From previous section, i get $V(x) = (V(x)_1, V(x)_2)$. Since $V(x)_1 + V(x)_2 = 1$, thus it is enough to study $V(x)_1$ only.

Let

$$f(x) = \alpha x^2 + 2\beta x(1-x)$$

Solve for the fixed point :

$$\begin{gathered} \alpha x^2 + 2\beta x(1-x) = x \\ \alpha x^2 + 2\beta x - 2\beta x^2 = x, \\ \alpha x^2 - 2\beta x^2 + 2\beta x - x = 0, \\ (\alpha - 2\beta)x^2 + (2\beta - 1)x = 0, \\ x[(\alpha - 2\beta)x + (2\beta - 1)] = 0, \\ x = 0, \qquad x = \frac{-(2\beta - 1)}{\alpha - 2\beta}. \end{gathered}$$

This show that x is a fixed point either $x = 0$ or $x = \frac{-(2\beta-1)}{\alpha-2\beta}$.
I know that $V(x)_2 = 1 - V(x)_1$.
So,

$$1 - x = 1 - \frac{-(2\beta - 1)}{\alpha - 2\beta}$$

$$= \frac{\alpha - 2\beta}{\alpha - 2\beta} - \frac{-2\beta + 1}{\alpha - 2\beta}$$
$$= \frac{\alpha-1}{\alpha-2\beta}.$$

Therefore, the fixed point $V(x) = \left(\frac{-(2\beta-1)}{\alpha-2\beta}, \frac{\alpha-1}{\alpha-2\beta}\right)$, (0,1).

Now, i want to check whether $x_0 = \frac{-(2\beta-1)}{\alpha-2\beta}$ inside [0,1] or not. If $\alpha = 1$, then i will get $x_0 = 1$.
Suppose that $x_0 = \frac{(1-2\beta)}{\alpha-2\beta} < 1$, then I has two cases

i) Case 1: $(\alpha - 2\beta) > 0$, thus
$$(1 - 2\beta) < \alpha - 2\beta$$
$$1 < \alpha$$
Which is a contradiction (since $\alpha \le 1$). Therefore, in this case $x_0 > 1$. Thus, x_0 is not in range [0,1].

ii) Case 2: $\alpha - 2\beta < 0$, thus
$$1 - 2\beta > \alpha - 2\beta$$
$$1 > a$$

Therefore, if $\alpha - 2\beta < 0$, then $x_0 < 1$. To make sure $x_0 \ge 0$, we impose $1 - 2\beta \le 0$
Or equivalently $\beta \ge \frac{1}{2}$. I can conclude the following:
The fixed point of ℓ-Volterra QSO is

i. (0,1) for any ℓ- Volterra QSO
ii. $\left(\frac{1-2\beta}{\alpha-2\beta}, \frac{\alpha-1}{\alpha-2\beta}\right)$ if $\alpha - 2\beta < 0$ and $\beta \ge \frac{1}{2}$

4.3 Stability of Fixed Point of ℓ-Volterra QSO.

In this subsection, I going to study stability of fixed point.
$$x' = \alpha x^2 + 2\beta x(1 - x),$$
$$y' = (1 - \alpha)x_1^2 + 2(1 - \beta)x(1 - x) + (1 - x)^2. \quad 4.10$$

i) Firstly, I going to consider fixed point for $x = 0$;
$$f(x) = \alpha x^2 + 2\beta x - 2\beta x^2.$$
$$f'(x) = 2\alpha x + 2\beta - 4\beta x,$$
$$f'(0) = 2\beta$$

Suppose
$$2\beta < 1 \implies \beta < \frac{1}{2}$$
From this, I can conclude that
If $\beta \in \left(0, \frac{1}{2}\right)$ then (0,1] attracting.
If $\beta \in \left(\frac{1}{2}, 1\right]$ then (0,1] repelling.
If $\beta = \frac{1}{2}$ then (0,1) non − hyperbolic fixed point

ii) Secondly, i going to consider fixed point for $x = \frac{1-2\beta}{\alpha-2\beta}$;
Let $x_0 = \frac{1-2\beta}{\alpha-2\beta}$
$$f(x) = \alpha x^2 + 2\beta x - 2\beta x^2.$$
$$f'(x) = 2\alpha x + 2\beta - 4\beta x,$$

$$
\begin{aligned}
f'(x_0) &= 2\alpha(x_0) + 2\beta - 4\beta(x_0) \\
&= x_0(2\alpha - 4\beta) + 2\beta \\
&= 2\left(\frac{1-2\beta}{\alpha-2\beta}\right)(\alpha - 2\beta) + 2\beta \\
&= 2(1-2\beta) + 2\beta \\
&= 2 - 2\beta = 2(1-\beta)
\end{aligned}
$$

Since $\beta \geq \frac{1}{2}$, then $-\beta \leq -\frac{1}{2}$

$1 - \beta \leq 1 - \frac{1}{2} = \frac{1}{2}$. Thus,

$$f'(x_0) = 2(1-\beta)\theta \leq 2\left(\frac{1}{2}\right) = 1$$

Hence, $|f'(x_0)| \leq 1$. Therefore, if $\frac{1}{2} < \beta \leq 1$, then the fixed point $(\frac{1-2\beta}{\alpha-2\beta}, 1 - \frac{1-2\beta}{\alpha-2\beta})$ is attracting. If $\beta = \frac{1}{2}$, the fixed point $(\frac{1-2\beta}{\alpha-2\beta}, 1 - \frac{1-2\beta}{\alpha-2\beta})$ is non-hyperbolic.

5 Reference

[1] M. Dzhurabayev, Topological classification of fixed and periodic points of quadratic stochastic operators, *Uzbek Math*. J. 5–6 (2000) 12–21.
[2] Bernstein, S. N. (1924). Solution of a mathematical problem related to the theory of I nheritance. *Uch. Zap. n.-i. kaf. Ukrainy*, 1, 83-115.
[3] Camacho L M, Gmez J R, Omirov B A and Turdibaev R M (2013) *Linear and Multilinear Algebra* 61, no.309-322.
[4] Ganikhodzhaev, R., Mukhamedov, F., Pirnapasov A., Qaralleh, I. (2017) *GENETIC VOLTERRA ALGEBRAS AND THEIR DERIVATIONS.*
[5] M. I. Zakharevich, The behavior of trajectories and the ergodic hypothesis for quadratic mappings of a simplex, *Russian Math. Surv.* 33 (1978) 207–208.
[6] Narziev N.B., Ganikhodzhaev R.N., Shahidi F. (2012) Algebraic structures of Volterra type population genetics, In book: *Proc. 2nd Inter. Conf. Math. Appl. in Engin*.IIUM Press, pp.410-416.
[7] N. N. Ganikhodjaev and R. T. Mukhitdinov, On a class of non-Volterra quadratic operators, *Uzbek Math*. (3–4) (2003) 65–69
[8] N. N. Ganikhodjaev and U. A. Rozikov, On quadratic stochastic operators generated by Gibbs distributions, *Regular Chaot. Dynam*. 11(4) (2006) 467–473.
[9] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I* (Springer-Verlag, 1979).
[10] Akin, E., Losert, V. Evolutionary dynamics of zero-sum games. *J. Math. Biology* **20,** 231–258 (1984).
[11] R. N. Ganikhodzhaev, Quadratic stochastic operators, Lyapunov functions and tournaments,*Russian Acad. Sci. Sb. Math*. 76 (1993) 489–506.
[12] N. Ganikhodzhaev, A chart of fixed points and Lyapunov functions for a class of discrete dynamical systems, *Math. Notes* 56 (1994) 1125–1131.
[13] N. Ganikhodzhaev and A. M. Dzhurabaev, The set of equilibrium states of quadratic stochastic operators of type V *Π.Uzbek. Mat. Zh*. 3 (1998) 23–27.
[14] N. Ganikhodzhaev and R. E. Abdirakhmanova, Description of quadratic auto-morphisms of a finite-dimensional simplex, *Uzbek. Mat. Zh*.1 (2002) 7–13.
[15] Akbar Zada and Syed Omar Shah, ON DYNAMICS OF QUADRATIC STOCHASTIC OPERATORS: A SURVEY, Surveys in Mathematics and its Applications. Volume 12 (2017), 117-164
[16] A.Rozikov and A. Zada, On ℓ -Volterra quadratic stochastic operators, *Dokl. Math*. 79 (2009) 32–34.
[17] U. A. Rozikov and A. Zada, On dynamics of ℓ-Volterra Quadratic Stochastic Operators, *International Journal of Biomathematics*. 3(2) (2010) 143-159.
[18] U. A. Rozikov, U. U. Zhamilov, Volterra quadratic stochastic operators of a two-sex population, *Ukrainian Math. J.* 63 (2011), no. 7, 1136-1153

[19] U. A. Rozikov and N. B. Shamsiddinov, On non-Volterra Quadratic Stochastic Operators generated by a *Product Measure*, *Stoch*, *Anal. Appl*., Vol. 27, No. 2, (2009), 9 pages.