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The Zero Product Probabilities Associated to Upper Triangular Matrices

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Abstract
In mathematics, mainly in the field of algebra, the study on probability related to groups and rings is a common topic which is widely discussed by many researchers. This study originated from the commutativity degree, which is introduced to find the probability that two elements in a group commute. Many extensions have been done on the commutativity degree of both groups and rings. In this research, variants of probability associated to finite rings are determined for ring of upper triangular matrices. The first probability is called the zero-product probability which is defined as the probability that any two elements of a finite ring have product zero. The second probability is a very closely related probability, called the commuting zero product probability. This probability focuses on any two elements of the finite ring which commute, and their product is zero.
Keywords: zero product probability, commutativity degree, zero divisor, noncommutative rings, ring of matrices

Introduction

In mathematics, mainly in the field of algebra, studies on groups and rings have caught the attention of many researchers. One of the most well-known studies in this field is the study on the probability of certain algebraic properties of the groups or rings itself. For example, Gustafson [1] has published an article in 1973 where the author introduced the commutativity degree in order to determine the probability that two elements of a finite group commute. The formal definition is given as follows.

Definition 1 [1]
Let G be a group of finite order n. The commutativity degree or the probability, Pr(G) that two elements selected at random (with replacement) from G are commutative is:

$$\Pr(G) = \frac{|\{(x,y) \in G \times G \mid xy = yx\}|}{n^2}. \quad (1)$$

This study has then sparked the interest of many more researchers to investigate on this subject. For instance, Sherman [2] studied on the probability that an automorphism fixes a group element for finite groups, Erfanian et al. [3] introduced the relative commutativity degree of a subgroup of a finite group, and Zamri et al. [4] defined the conjugation degree, which is the probability that two elements of a finite group fix a set by conjugation action.

Then, this concept of probability has also attracted the ring theorists too. For instance, MacHale [5] in 1976 determined the probability that two random elements of a finite ring commute for noncommutative rings. Later, Buckley et al. [6] extended the research and found the probability that a random pair of elements in a finite ring commute by isoclinism.

Recently, based on the idea of commutativity degree and zero divisor of a finite ring, Khasraw [7] introduced a new concept of probability in finite rings, which is the probability that two elements of a finite ring have product zero.

Based on the idea given in [7], Zaid et al. [8] defined a new probability which is called the commuting zero probability, focusing not only on the elements of a finite ring that have product zero, but also the commutativity of the elements.

In this research, the concepts introduced in [7] and [8] are the focus and will be further discussed in the next section. These probabilities which are associated to upper triangular matrices are computed using their definitions.

Preliminaries

In this section, definitions related to finite rings as well as the probabilities which are used in this research are presented. The methodology of conducting this study is also provided in this section. First, the definition of a ring is given.

According to Fraleigh [9], a ring is a set (denoted as *R*) with two binary operations, which is addition (+) and multiplication $(\cdot)$. The set R must be an Abelian group by addition, the multiplication operation on R must be associative, and for all elements of R, the left distributive law and right distributive law must hold.

In this research, the ring which is taken into consideration is the ring of 2×2 upper triangular matrices over integer modulo three. First, the zero divisors of the ring are determined based on its definition. If *a* and *b* are two nonzero elements of a ring *R* such that $ab=0$, then *a* and *b* are called the zero divisors of *R* [9].

Next, using the results of the zero divisors, the probability that two elements of a ring have product zero, or later called the zero product probability, is computed. Its definition is given in the following.

Definition 2 [7]

Let *R* be a finite ring. The probability that two elements chosen at random (with replacement) from a ring *R* have product zero is

$$P(R)=\frac{|Ann(R)|}{R\times R}, \tag{2}$$

where $Ann(R)=\{(x,y)\in R\times R \mid xy=0\}$.

Next, using both results from the zero divisor and zero product probability, the commuting zero product probability is determined based on its definition, which is given as follows:

Definition 3 [8]

The probability $P_{comm}(R)$ that a pair of commuting elements chosen at random (with replacement) from a ring *R* have product zero is

$$P_{comm}(R)=\frac{\{(x,y)\in R\times R \mid xy=yx=0\}}{|R\times R|}. \tag{3}$$

In this research, the pairs of commuting elements where the product is zero will be called the commuting annihilators, $Ann_{comm}(R)$. In the next section, the results obtained in this research are presented.

Results and discussions

This section consists of two parts, the first part focuses on determining the zero divisors of the ring, while the second part focuses on determining its probabilities, which are the zero product probability and the commuting zero product probability. The ring *R* mentioned throughout this section is the ring of 2×2 upper triangular matrices over integer modulo three.

In the following lemma, the zero divisors of the ring *R* are found.

Lemma 1

Let $R=\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a,b,c\in\mathbb{Z}_3\right\}$ be a finite ring. Then, *R* has 14 zero divisors.

Proof

Let $R=\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a,b,c\in\mathbb{Z}_3\right\}$. Then, $|R|=3^3=27$. By using its definition, the zero divisors of *R* are determined. The set of zero divisors of *R* are given in the following:

$$Z(R)=\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix},\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix},\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\right.$$
$$\left.\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\right\}.$$

Lemma 1 is used to find the zero product and commuting zero product probabilities given in Proposition 1 and 2.

Proposition 1

Let $R=\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a,b,c\in\mathbb{Z}_3\right\}$. Then, *R* has 76 pairs of annihilators and the zero product probability of *R* is $\frac{76}{729}$.

Proof

Let $R=\left\{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a,b,c\in\mathbb{Z}_3\right\}$. Referring to Definition 2, the set of annihilators of *R* is defined as $Ann_R=\left\{(x,y)\in R\times R \mid xy=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$ for all *x* and *y* in *R*. By using this definition with the assistance of the zero divisors obtained in Lemma 1, the annihilators of *R* are determined. It is found that the ring *R* has 76 pairs of annihilators, such as the pairs $\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}\right),\left(\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix},\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\right)$ and $\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\right)$. Hence, by Definition 2, the zero product probability of *R* is $P(R)=\frac{76}{27\times 27}=\frac{76}{729}$. □

Based on the zero product probability found in Proposition 1, the zero product property of the ring *R* is known. From all 729 possible pairs in the ring, only 76 pairs satisfy the zero product property. The probability found is only $\frac{76}{729}\approx 0.1043$ which is a very small part of the ring.

Proposition 2

Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a,b,c \in \mathbb{Z}_3 \right\}$. Then, R has 26 pairs of commuting annihilators and the commuting zero product probability of R is $\frac{26}{729}$.

Proof

Let $R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| a,b,c \in \mathbb{Z}_3 \right\}$. In this case, pairs of commuting elements with product zero are considered. The set of such pairs, which is called the commuting annihilators is defined as $Ann_{comm}(R) = \left\{ (x,y) \in R \times R \mid xy = yx = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ for all x and y in the ring R. Based on the annihilators which have been determined in Proposition 1, the commuting pairs are singled out as the commuting annihilators. Hence, it is found that there are a total of 26 commuting annihilators available in the ring R, which includes the pairs $\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \right), \left(\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right)$ and $\left(\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix} \right)$. Therefore, by Definition 3, the commuting zero product probability of R, $P_{comm}(R) = \frac{26}{729}$. □

Based on the result in Proposition 2, only $\frac{26}{729} \approx 0.0357$ of all possible pairs of elements in R satisfies both the zero product property and the commuting property. From all 76 pairs with the zero product property as found in Proposition 1, only $\frac{26}{76} \approx 0.3421$ of all pairs also comply with the commuting property.

Conclusion

In this study, the zero divisors are determined for the ring of 2×2 upper triangular matrices over integer modulo three. Then, the results on the zero divisor are used to determine the zero product probability and commuting zero product probability of the ring. It is found that the zero product probability of R is $\frac{76}{729}$, while the commuting zero product probability of R is $\frac{26}{729}$. From these probabilities, the zero product characteristic, as well as the commuting characteristic of the ring R are recognized.

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