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The Non-Normal Subgroup Graph for Generalized Quaternion Groups

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Abstract
In this research, the generalized quaternion groups are represented geometrically by their subgroup structure. The non-normal subgroup graph of a non-normal subgroup *H* in a group *G* is defined as a directed graph with the vertex set are elements of *G* and two distinct elements *x* and *y* are adjacent if $xy \in H$. The non-normal subgroups graphs are found for all generalized quaternion groups and stated in two cases for $n \in N$. The first case is when n is odd and the second one is when n is even.
Keywords: non-normal subgroup; subgroup graph; directed

Introduction
The application of graph theory in a finite group has become active research recently in many ways, for instance, to associate a graph with a finite group. There are many types of graphs associated with a finite group that have been studied and each graph is constructed by using elements or subgroups as a set of vertices. The two vertices are connected by an edge if they satisfy certain properties or relations defined for the finite group.

In this paper, the generalized quaternion groups are represented geometrically by their subgroup structure. One of the graphs that are related to its subgroups is the cyclic subgroup graph, defined by Devi and Singh [1] using the cyclic subgroups as vertices and two distinct subgroups are adjacent if one of them is a subset of the other. Two other graphs using subgroups as their vertices are the stable subgroup graph defined by Tolue in [2], and a normal subgroup-based power graph defined by Bhuniya and Bera in [3].

Most of the research on graphs associated with finite groups are simple undirected graphs. However, there is some research on the directed graph related to group theory. For instance, the order graph was defined by Al-Hasanat and Al-Hasanat [4] as a directed graph whose vertices are the elements of the group order classes. Next, the subgroup graph was formally defined in 2015 by Kakeri and Erfanian [5]. The subgroup graph is a simple directed graph associated with subgroups of a finite group and this graph was extended to a non-normal subgroup graph defined by Rahin *et al*. in [6]. In this research, the non-normal subgroup graph is found for the generalized quaternion group for all n. The presentation of the generalized quaternion group, Q_{4n} is given as follows:

$$Q_{4n} \cong \langle a, b : a^{2n} = b^4 = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle, n \geq 2, n \in N.$$

The formal definition of the non-normal subgroup graph is stated in the following.

Definition 1 [6] The Non-Normal Subgroup Graph

Let G be a group and H a non-normal subgroup of G. The non-normal subgroup graph $\Gamma_H^{NN}(G)$ is a directed graph with vertex set G; such that x is the initial vertex and y is the terminal vertex of an edge if and only if $x \neq y$ and $xy \in H$.

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The following section includes the main results of this research.

Results and discussion

In this section, the non-normal subgroup graph for the generalized quaternion group, Q_{4n} are stated in two theorems, for the case *n* odd and *n* even.

Theorem 1

Let G be the generalized quaternion group, $Q_{4n} = \{1, a, a^2, \dots, a^{2n-1}, b, ab, a^2b, \dots, a^{2n-1}b\}$ where n is an odd integer and H is a cyclic, non-normal subgroup of G with $H_i = \langle a^i b\rangle = \{1, a^i b, a^n, a^{n+i}b\}$, $1 \leq i \leq 2n - 1$. Then the non-normal subgroup graph for H , $\Gamma_H^{NN}(Q_{4n})$ is a directed graph with $14n - 2$ edges.

An example is provided in the following to illustrate the theorem above.

Example 1

Let $n = 3$, then $Q_{12} = \{1, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$. Let the non-normal subgroup of Q_{12} be $H_i = \langle a^i b\rangle = \{1, a^i b, a^n, a^{n+i}b\}$ and $1 \leq i \leq 5$. Consider for $i = 0$, therefore, the subgroup is $\langle b\rangle = \{1, b, a^3, a^3b\}$. From definition 1, the coset of this subgroup gives result $1, b, a^3$ and a^3b are needed. The possibilities of cosets are listed and the directed edges for each coset are determined as follows.

(i) $a^r \cdot (a^r)^{-1} = 1$

$a \to a^5, a^2 \to a^4, a^3 \to a^3, a^4 \to a^2, a^5 \to a.$

The edge $a^3 \to a^3$ need to be removed since it is mapped to the same vertex.

(ii) $a^r \cdot a^{-r+n} = a^n$

$a \to a^2, a^2 \to a, a^3 \to 1, a^4 \to a^5, a^5 \to a^4$

(iii) $a^r \cdot a^{-r+i}b = a^i b$

$a \to a^5b, a^2 \to a^4b, a^3 \to a^3b, a^4 \to a^2b, a^5 \to ab$

(iv) $a^r \cdot a^{-r+n+i}b = a^{n+i}b$

$a \to a^2b, a^2b \to ab, a^3 \to b, a^4 \to a^5b, a^5 \to a^4b$

(v) $1 \cdot a^i b = a^i b$

$1 \to b$

(vi) $1 \cdot a^{n+i}b = a^{n+i}b$

$1 \to a^3b$

$1 \cdot a^n = a^n$

$1 \to a^3$

(vii) $b \cdot a^n b = 1$

$b \to a^3b$

(viii) $b \cdot a^{2n-i} = a^i b$

$b \to 1$

(ix) $b \cdot a^{-i+n} = a^{n+i}b$

$b \to a^3$

(x) $a^r b \cdot a^{-n+r}b = 1$

$ab \to a^4b, a^2b \to a^5b, a^3b \to b, a^4b \to ab, a^5b \to a^2b$

(xi) $a^r b \cdot a^{r-n-i} = a^{n+i} b$

$$ab \to a^4, a^2 b \to a^5, a^3 b \to 1, a^4 b \to a, a^5 b \to a^2$$

(xii) $a^r b \cdot a^{r-i} = a^i b$

$$ab \to a, a^2 b \to a^2, a^3 b \to a^3, a^4 b \to a^4, a^5 b \to a^5$$

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Thus, the non-normal subgroup graph, $\Gamma_{H_0}^{NN}(Q_{12})$ is illustrated in Figure 1.

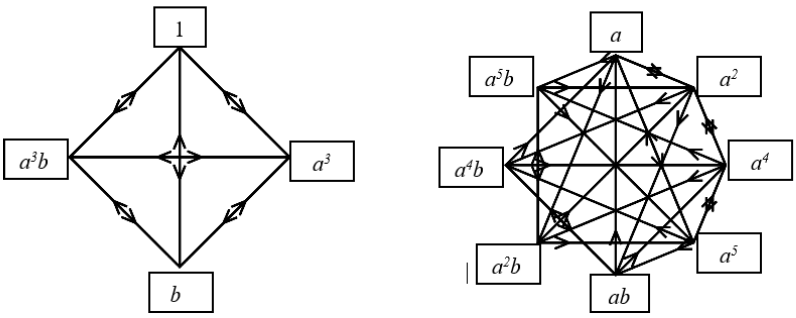

Figure 1 The non-normal subgroup graph of Q12, $\Gamma_{H_0}^{NN}(Q_{12})$

Therefore, there exists $14n - 2 = 14(3) - 2 = 40$ of directed edges for this graph. The graph is a disconnected directed graph with a complete digraph, K_{12}.

Next, the non-normal subgroup graph of the generalized quaternion group is found for n is an even integer.

Theorem 2

Let G be the generalized quaternion group, $Q_{4n} = \{1, a, a^2, \dots, a^{2n-1}, b, ab, a^2 b, \dots, a^{2n-1} b\}$ where n is an even integer and H is a cyclic, non-normal subgroup of G with $H_i = \langle a^i b\rangle = \{1, a^i b, a^n, a^{n+i} b\}, 1 \le i \le 2n - 1$. Then the non-normal subgroup graph for H, $\Gamma_H^{NN}(G)$ is a directed graph with $14n - 4$ edges.

An example is provided as follows to illustrate the theorem above.

Example 2

Let $n = 4, Q_{16} = \{1, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2 b, a^3 b, a^4 b, a^5 b, a^6 b, a^7 b\}$. Assume $H_i = \langle a^i b\rangle = \{1, a^i b, a^n, a^{n+i} b\}$ and 1≤ $i \le 7$. Consider for $i = 0$, therefore, the subgroup is $\langle b\rangle = \{1, b, a^4, a^4 b\}$. From the definition 1, the coset of this subgroup gives results $1, b, a^4$ and $a^4 b$ are needed. The possibilities of the cosets are listed and the directed edges for each coset are determined as follows.

(i) $a^r . (a^r)^{-1} = 1$

$a \to a^7, a^2 \to a^6, a^3 \to a^5, a^4 \to a^4, a^5 \to a^3, a^6 \to a^2, a^7 \to a.$

The edge $a^4 \to a^4$ need to be removed since it is mapped to the same vertex.

(ii) $a^r . a^{-r+n} = a^n$

$a \to a^3, a^2 \to a^2, a^3 \to a, a^4 \to 1, a^5 \to a^7, a^6 \to a^6, a^7 \to a^5$

(iii) $a^r.a^{-r+i}b = a^i b.$

$$a \to a^7b, a^2 \to a^6b, a^3 \to a^5b, a^4 \to a^4b, a^5 \to a^3b,\ a^6 \to a^2b, a^7 \to ab$$

(iv) $a^r.a^{-r+n+i}b = a^{n+i}b$

$$a \to a^3b, a^2 \to a^2b, a^3 \to ab, a^4 \to b, a^5 \to a^7b, a^6 \to a^6b, a^7 \to a^5b$$

(v) $1.a^i b = a^i b$

$1 \to b$

(vi) $1.a^{n+i}b = a^{n+i}b$

$1 \to a^4b$

$1.a^n = a^n$

$1 \to a^4$

(vii) $b.a^n b = 1$

$b \to a^4b$

(viii) $b.a^{2n-i} = a^i b$

$b \to 1$

(ix) $b.a^{-i+n} = a^{n+i}b$

$$b \to a^4$$

(x) $a^r b.a^{-n+r}b = 1$

$ab \to a^5b, a^2b \to a^6b, a^3b \to a^7b, a^4b \to b, a^5b \to ab, a^6b \to a^2b, a^7b \to a^3b$

(xi) $a^r b.a^{r-n-i} = a^{n+i}b$

$$ab \to a^5, a^2b \to a^6, a^3b \to a^7, a^4b \to 1, a^5b \to a, a^6b \to a^2, a^7b \to a^3$$

(xii) $a^r b.a^{r-i} = a^i b$

$$ab \to a, a^2b \to a^2, a^3b \to a^3, a^4b \to a^4, a^5b \to a^5, a^6b \to a^6, a^7b \to a^7$$

Thus, the non-normal subgroup graph, $\Gamma_{H_0}^{NN}(Q_{16})$ is illustrated in Figure 2.

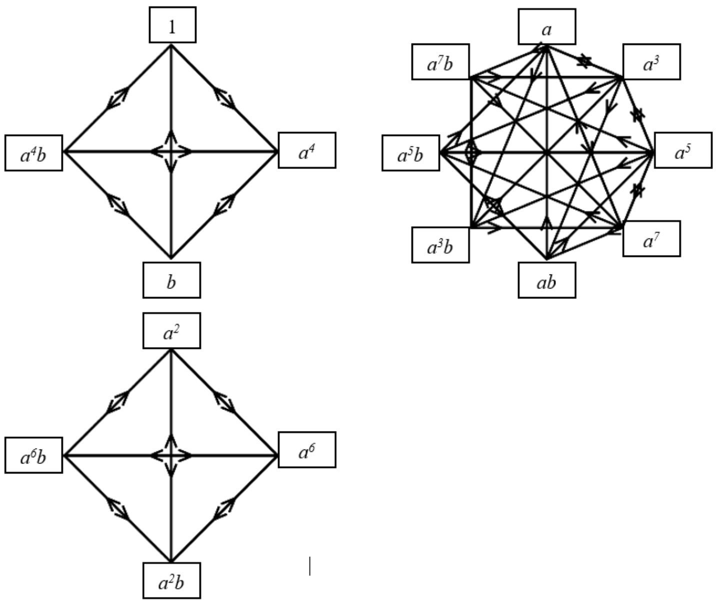

Figure 2 The non-normal subgroup graph of Q_{16}, $\Gamma_{H_0}^{NN}(Q_{16})$

There exists $14n - 4 = 14(4) - 4 = 52$ of directed edges for this graph. The graph is a disconnected directed graph with two complete digraphs, K_{12}

Conclusion

In this research, the non-normal subgroup graph for generalized quaternion groups has been determined. The graph is found to be a directed graph with the edges depending on the order of the generalized quaternion groups.

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