Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia

Vol. 7, 2022, Page 25 - 28

Prime Power Cayley Graph of a Cyclic Group

Athirah Zulkarnain[a]*, Nor Haniza Sarmin[a], Hazzirah Izzati Mat Hassim[a], Ahmad Erfanian[b]
[a]Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 Johor Bahru, Johor, Malaysia
[b]Department of Pure Mathematics, Faculty of Mathematical Sciences, Ferdowsi University of Maashhad, Mashhad, Iran
*Corresponding author: athirah5@graduate.utm.my

Abstract

Let *G* be a group and *S* a generating set of *G*. The Cayley graph, *Cay(G,S)* is defined to have a vertex set $V(Cay(G,S)) = G$ and an edge set, $E(Cay(G,S))$. Two vertices *x* and *y* in $V(Cay(G,S))$ are connected if $xy^{-1} \in S$. In this research, the idea of the Cayley graph is extended by introducing a new type of graph called the prime power Cayley graph, in which the vertices of this graph are the elements of *G* and *S* has prime power order. In this paper, the prime power Cayley graph is constructed for a cyclic group of order *p* (*p* is a prime), and its generalization is determined.
Keywords: Cayley graph; Cyclic group; Group theory; Graph theory

Introduction

In 1988, Babai and Seress [1] introduced the Cayley graph of a group. The Cayley graph of a group *G* is constructed based on its subsets *S*, a generating set of *G*. The Cayley graph contains a set of vertices which comprises the elements of *G*. If $xy^{-1} \in S$, then any two vertices, *x* and *y* in *G* are adjacent. Many types of research have been done on the Cayley graph. Kelarev [2] in 2002 explored the relationship between the subsets of the group and the pattern of the Cayley graph. Then, Klotz and Sander [3] in 2007 studied the unitary Cayley graph. The authors focused on the properties of the graphs including the chromatic number, the clique number, the independence number, the diameter, the vertex connectivity and the perfectness of the graph. In 2008, Konstantinova [4] investigated the applications of the Cayley graph in computer science, molecular biology and coding theory.

In 2015, Tolue [5] introduced the prime order Cayley graph which focuses on the subset thatcontains prime order elements in *G*. Later in 2019, Tolue [6] introduced another type of Cayley graph, namely the composite order Cayley graph which focuses on the subset that contains all elements of *G* with composite orders. In the same year, Ida *et al.* [7] explored the topological indices of the composite order Cayley graph of $Z_{p^{n_1}} \times \cdots \times Z_{p^{n_t}}$. In 2020, Zulkarnain *et al.* [8] introduced the p_i-Cayley graph by considering the subset that contains the elements with certain *p*-power order.

In this research, a new graph is introduced, namely the prime power Cayley graph. This graph is constructed for a cyclic group of order *p*, where *p* is a prime number.

Preliminaries

In this section, some basic concepts of group theory and graph theory which are used in this research are included.

Definition 1 (Cyclic Group)

A group *G* is cyclic if there is an element *x* in *G* such that $G = \{x^n | n \in Z\}$.

Definition 2 (Graph of a Group)

A graph of a group *G* denoted as Γ_G, is a finite nonempty set of objects called vertices together with a (possibly empty) set of unordered pairs of distinct vertices of Γ_G called edges. The vertex set of Γ_G is denoted by $V(\Gamma_G)$ is the elements of a group, while the edge set is denoted by $E(\Gamma_G)$ represents the adjacency of the vertices.

Definition 3 (Order of a Group)

The order of Γ_G is equal to the total number of vertices of Γ_G.

Definition 4 (Complete Graph)

A graph Γ_G is complete if all vertices of Γ_G are pairwise adjacent. A complete graph on *n* vertices is denoted as K_n.

Definition 5 (Cayley Graph)

A graph, *Cay* (*G,S*) is a Cayley graph on a group *G* if there is a subset *S* of *G* with $S = S^{-1}$, such that $V(Cay(G,S)) = G$, and two vertices *x* and *y* are adjacent if and only if $xy^{-1} \in S$.

Results and discussion

In this section, a new type of graph is introduced, named as the prime power Cayley graph. The prime power Cayley graph differs from the Cayley graph in such that *S* has prime power order. The formal definition of the prime power Cayley graph is stated in the following definition.

Definition 6 (Prime Power Cayley Graph)

Let *G* be a group and $|G| = p_1^{a_1} \cdot p_2^{a_2} \cdot \ldots \cdot p_k^{a_k}$ where p_i are primes. Let *S* be non-empty subset of *G* in which $S = \{x \in G: |x| = p_i^{k_i}, 1 \leq k_i \leq \alpha_i, i = 1,2,\ldots,k\}$ and $S = S^{-1} := \{s^{-1} | s \in S\}$. The prime power Cayley graph, denoted as $\widetilde{Cay}(G,S)$, is a graph where the set of vertices of the graph are elements of *G*, and two distinct vertices, *x* and *y*, are adjacent if $xy^{-1} \in S$ for all $x, y \in G$.

Next, the prime power Cayley graph is illustrated in the following two examples. Example 1 showed the construction of the prime power Cayley graph of the cyclic group of order 5.

Example 1

Let G_1 be a cyclic group of order 5 and S_5 a non-empty subset of *G* with $S = \{x \in G : |x| = 5\} = \{x, x^2, x^3, x^4\}$ and $S = S^{-1}$. Then, the prime power Cayley graph of *G*, denoted as $\widetilde{Cay}\left(G_1, S\right)$ is K_5 where K_5 is the complete graph of order 5.

Proof

Let G_1 be a cyclic group of order 5 and *S* be non-empty subset of G_1, $S = \left\{x \in G_1 \mid |x| = 5\right\} = \left\{x^i \in G_1 : 1 \leq i \leq 4\right\}$. G_1 is generated by generator *x* with the set of elements $\{e, x, x^2, x^3, x^4\}$. All elements in G_1 are order 5 except for the identity, *e*. By referring to the definition of prime power Cayley graph, the adjacency between two distinct vertices are observed.

The two distinct vertices, x^i and *e* are always adjacent with each other in $\widetilde{Cay}\left(G_1, S\right)$ for $1 \leq i \leq 4$ since $x^i \cdot e^{-1} = x^i \in S$. The same calculations are used to observe the relation between x^i and x^j for $1 \leq i - j \leq 4$. The calculations are divided into two cases. The first case is when *i*=*j* and the second case is when $i \neq j,\ i > j$. The two distinct vertices, x^i and x^j are not adjacent with each other for the first case since $x^i \cdot x^{-j} = e \notin S_5$. Next, for the second case where *i*>*j*, the vertices x^i and x^j are always adjacent with each other since $x^i \cdot x^{-j} \neq e \in S$.

Then, x^i is always adjacent with *e* for $1 \leq i \leq 4$, and x^i and x^j are always adjacent for $i \neq j$, $1 \leq i-j \leq 4$. There is no lope in the same vertices since $x^i \cdot x^{-j} \neq e \in S$ for *i=j*, $1 \leq i-j \leq 4$. Therefore, a complete graph with five vertices is formed since all elements are adjacent with each other as showed in Figure 1.

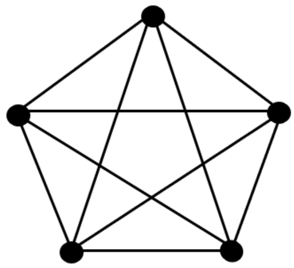

Figure 1 The prime Cayley graph of the cyclic group for order 5

Next, Example 2 showed the construction of the prime power Cayley graph of the cyclic group of order 7.

Example 2
Let *G₂* be a cyclic group of order 7 and *S* a non-empty subset of G_2 with $S = \{x \in G : |x| = 7\} = \{x, x^2, x^3, x^4, x^5, x^6\}$ and $S = S^{-1}$. Then, the prime power Cayley graph of G_2, denoted as $\widetilde{Cay}\left(G_2, S\right)$ is K_7 where K_7 is the complete graph of order 7.

Proof
Let G_2 be a cyclic group of order 7 and *S* be non-empty subset of G_2, $S = \{x \in G \mid |x| = 7\} = \{x^i \in G : 1 \leq i \leq 6\}$. G_2 is generated by generator *x* with the set of elements $\{x, x^2, x^3, x^4, x^5, x^6\}$. All elements in G_2 are order 7 except for the identity, *e*. By referring to the definition of prime power Cayley graph, the adjacency between two distinct vertices are observed.

The two distinct vertices, x^i and *e* are always adjacent with each other in $\widetilde{Cay}\left(G_2, S\right)$ for $1 \leq i \leq 6$ since $x^i \cdot e^{-1} = x^i \in S$. The same calculations are used to observe the relation between x^i and x^j for $1 \leq i-j \leq 6$. The calculations are divided into two cases. The first case is when *i=j* and the second case is when $i \neq j$, *i>j*. The two distinct vertices, x^i and x^j are not adjacent with each other for the first case since $x^i \cdot x^{-j} = e \notin S$. Next, for the second case where *i>j*, the vertices x^i and x^j are always adjacent with each other since $x^i \cdot x^{-j} \neq e \in S$.

Then, x^i is always adjacent with *e* for $1 \leq i \leq 6$, and x^i and x^j are always adjacent for $i \neq j$, $1 \leq i-j \leq 6$. There is no lope in the same vertices since $x^i \cdot x^{-j} \neq e \in S$ for *i=j*, $1 \leq i-j \leq 6$. Therefore, a complete graph with seven vertices is formed since all elements are adjacent with each other as shown in Figure 2.

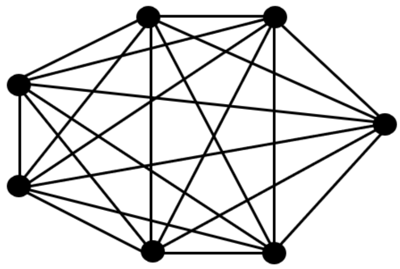

Figure 2 The prime Cayley graph of the cyclic group for order 7

The construction of the prime power Cayley graph of the cyclic group of order *p*, where *p* is a prime number, is shown in the following theorem. The simplest case is used for constructing the graph which is *k* = 1 and $\alpha_1 = 1$.

Main Theorem

Let *G* be a cyclic group with order *p*, where *p* is a prime number. Let *S* be a non-empty subset of *G* in which $S = \{x \in G: |x| = p\}$ and $S = S^{-1}$. Then, the prime power Cayley graph of *G*, $\widetilde{Cay}(G,S)$, is a complete graph of order *p*, K_p.

Proof

Let *G* be a cyclic group of order *p*, |*G*|=*p* and *S* be non-empty subset of *G*, $S = \{x \in G: |x| = p\}$. The group *G* is generated by *x* with the set of elements $\{e, x, x^2, \ldots, x^{p-1}\}$. Since *p* is prime, all elements of *G* are of order *p* except for the identity, *e*. Hence, $S = \{x, x^2, x^3, \ldots, x^{p-1}\}$.

Based on the definition of prime power Cayley graph, the vertices x^i and *e* are always adjacent with each other in $\widetilde{Cay}(G,S)$ for $1 \le i \le p-1$ since $x^i \cdot e^{-1} = x^i$ where $x^i \in S$. The same calculations are used to observe the relation between x^i and x^j for $1 \le i \le p-1$. The calculations are divided into two cases where the first case is when *i* = *j* and the second case is when $i \neq j$ and $i > j$. For the first case where *i* = *j*, x^i and x^j are not adjacent since $x^i \cdot (x^j)^{-1} = x^i \cdot (x^i)^{-1} = x^0 = 1 = e \notin S$. Hence, there is no self-loop edges. For the second case where $i \neq j, i > j$, x^i and x^j are adjacent since $x^i \cdot (x^j)^{-1} = x^{i-j} \neq e \in S$.

From these calculations, x^i and *e* are always adjacent with each other for $1 \le i \le p-1$, and x^i and x^j are always adjacent with each other for $i \neq j, i > j$ and $1 \le i \le p-1$. Therefore, all vertices in prime power Cayley graph of *G*, $\widetilde{Cay}(G,S)$ are always adjacent with each other and form a complete graph with *p* vertices, K_p.

Conclusion

In this paper, a new graph is introduced, which is called the prime power Cayley graph. This graph has been constructed for a cyclic group of order *p*, where *p* is a prime number, and is found to be a complete graph of order *p*, K_p. Two examples have been provided to illustrate this new graph.

Acknowledgement

The Universiti Teknologi Malaysia (UTM) and Ministry of Higher Education Malaysia are funding this project through the Zamalah Scholarship and Graduate Research Assistant Scheme. The authors would like to express their gratitude to the Ministry of Higher Education Malaysia, Universiti Teknologi Malaysia (UTM), and the UTM Research Management Centre (RMC) for providing financial support through the UTM Research University Grant (RUG) Vote Number 16J09, the UTM Fundamental Research Grant Vote Number 20H70 and the Fundamental Research Grant Scheme (FRGS) (FRGS/1/2020/STG06/UTM/01/2).

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