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The First Zagreb Index of Zero-Divisor Type Graph for Some Rings of Integers Modulo *n*

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Abstract

The zero-divisor type graph for ring of integers modulo n, Z_n denoted as $\Gamma^T[Z_n]$ has vertices d where d is the nontrivial divisors of n. Two distinct vertices d_i and d_j are adjacent if and only if $d_i \cdot d_j = 0$. In this research, the ring of integers modulo $p^a q$, $Z_{p^a q}$, with distinct primes p and q is considered. One of the degree-based topological indices, namely the first Zagreb index of $\Gamma^T[Z_{p^a q}]$, $M_1(\Gamma^T[Z_{p^a q}])$, is computed based on the degree of all vertices in $\Gamma^T[Z_{p^a q}]$ that was obtained.

Keywords: zero-divisor graph; zero-divisor type graph; first Zagreb index

Introduction

The zero-divisor type graph is a modification of the concept of the zero-divisor graph by compressing the vertices of the zero-divisor graph. The zero-divisor graph of a ring R, $\Gamma(R)$ has been defined by Beck in [1]. In [1], $\Gamma(R)$ is introduced in order to determine the graph coloring and the vertex set of the graph consists of all elements of the ring R. Anderson and Livingston in [2] restricted the vertex set of the zero-divisor graph of R to be only nonzero zero-divisor of R, denoted as $Z^*(R)$. The two vertices in $\Gamma(R)$, x and y are adjacent if and only if $x \cdot y = 0$.

One of the graph modifications including the compression of the graph's vertices has been introduced by Mulay in [3]. In [3], some useful theorems on cycles of zero-divisor graph based on equivalence class of zero-divisor of ring R denoted as $\Gamma_E(R)$ are published rather than the zero-divisor itself. Another important features in $\Gamma_E(R)$ is the associated primes of ring R. Therefore, Spiroff and Wickham in [4] investigated how to identify the associated primes of R.

In 2016, Smith in [5] introduced the zero-divisor type graph for the ring of integers modulo n, Z_n. In this graph, Smith partitioned the zero-divisors of Z_n , $Z^*(Z_n)$ based on the greatest common divisor between $Z^*(Z_n)$ and divisor of n, d. Therefore, the vertices of the graph has been defined as d and two distinct vertices d_i and d_j are adjacent if and only if $d_i \cdot d_j = 0$. In [5], the zero-divisor type graph of Z_n served as an approach to determine the perfectness of the zero-divisor graph of Z_n. Later Pirzada *et. al* in [6] investigated the compressed zero-divisor graph of a ring of integers modulo p^a where $a \in N$. In [6], various graph properties are computed including the order of the graph, clique number, degree of the vertices and the Wiener index of the graph.

Meanwhile, the topological indices of a graph are mathematical formulas that represent the molecular structure of the graph. Nowadays there are many topological indices that have been applied to any graph such as Zagreb indices [7], multiplicative Zagreb indices [8] and exponential Zagreb indices [9] which are calculated using the degrees of vertices.

One of the degree-based topological indices is known as the first Zagreb index, introduced by Gutman and Trinajestic in [7]. The first Zagreb index of a graph G, denoted as $M_1(G)$ is the sum of squares of the degrees of the vertices in G.

The objectives of this research is to compute the first Zagreb index of the zero-divisor type graph of $Z_{p^a q}$.

Preliminaries

In this section, some definitions and basic concepts related to the graph theory, the first Zagreb index and the zero-divisor type graph which are used in this research are presented.

Definition 1 [8]

The degree of a vertex, v of a graph G denoted as $deg_G(v)$ is the number of edges adjacent to v.

Definition 2 [8]

The clique of a graph G denoted as $cl(G)$ is a complete subgraph of G.

Definition 3 [7]

The first Zagreb index of a graph G, $M_1(G)$ is the sum of squares of the degrees of the vertices in G, written as

$$M_1(G) = \sum_{v \in V(G)} deg(v)^2. \tag{1}$$

Next, the formal definition and some results related to the zero-divisor type graph of Z_n are presented.

Definition 4 [5]

The zero-divisor type graph for the ring of integers modulo n, Z_n denoted as $\Gamma^T[Z_n]$, has vertices d where d is the nontrivial divisor of n given as

$$d = \{x \in Z^*(Z_n) |\ gcd(x, n) = d\}. \tag{2}$$

Two distinct vertices d_i and d_j are adjacent if and only if $i \cdot j = 0$.

Proposition 1 [5]

Let x and y be the vertices of zero-divisor type graph of $Z_{p^a q}$, $\Gamma^T[Z_{p^a q}]$ and suppose $x \neq y$. Therefore, x and y can be written as $x = p^r q^s$ and $y = p^m q^n$. Then, x and y are adjacent if and only if $r + m \geq a$ and $s + n \geq 1$.

From Proposition 1, the vertices of $\Gamma^T[Z_{p^a q}]$ are divided into two sets:

$$A_i = \{p^i \text{ } where \text{ } i = 1,2,3, \dots, a\} \text{ } and \tag{3}$$

$$B_j = \{p^j q \text{ } where \text{ } j = 0,1,2, \dots, a-1\}. \tag{4}$$

Proposition 2 [6]

Every vertex v_i in $\Gamma_E[Z_{p^a}]$ is adjacent to $v_{a-1}, v_{a-2}, \dots, v_{a-i}$ and the vertices $v_{[\frac{a}{2}]}$, $v_{[\frac{a}{2}]} + 1$, $v_{[\frac{a}{2}]} + 2, \dots, v_{a-1}$ form a clique, where $[x]$ denotes the greatest integer function. Any B_j is connected to $B_{a-1}, B_{a-2}, \dots, B_{a-j}$. Then, there exist two sub cases in B which are:

Case 1: For **even** a, $B_{\frac{a}{2}}$, $B_{\frac{a}{2}}+1, \dots, B_{a-1}$ form a clique. (5)

Case 2: For **odd** a, $B_{\frac{a+1}{2}}$, $B_{\frac{a+1}{2}} + 1, \dots, B_{a-1}$ form a clique. (6)

Next, using Proposition 1 and Proposition 2, the zero-divisor type graph of $Z_{p^a q}$ is constructed. In the next section, the results obtained in this research are presented.

Results and discussion

In this section, the first Zagreb index of $\Gamma^T[Z_{p^a q}]$, $M_1(\Gamma^T[Z_{p^a q}])$ is computed from the degree of the vertices obtained. Firstly, an example of $\Gamma^T[Z_{p^a q}]$ is presented.

Example 1 Let $\Gamma^T[Z_{96}]$ be the zero-divisor type graph of Z_{96}. Then, $\Gamma^T[Z_{96}]$ is a simple undirected graph with 10 vertices.

Figure 1. Zero-divisor type graph of Z_{96}

Since $96 = 2^5 \times 3$, therefore $p = 2$, $q = 3$ and $a = 5$. In Figure 1, every vertex is labelled based on the power of p for every nontrivial divisor of 96. A is represented by vertex $1,2,3,4$ and 5 on the right side of the graph. Meanwhile, B is represented by vertex $0,1,2,3$ and 4 on the left side of the graph. In this graph, where $i = j = 1,2,3,4,5$ there are five observations can be concluded:

1. A_i *is adjacent to* $B_{a-1}, B_{a-2}, \dots, B_{a-j}$,
2. B_{j-1} is adjacent to $A_{a-1}, A_{a-2}, \dots, A_{a-i}$,
3. B_0 is not adjacent to any B_j,
4. B_1 is adjacent to B_{5-1}, therefore $B_1 \leftrightarrow B_4$ and
5. B_2, B_3 and B_4 form a clique.

In general, the degree of every vertices of zero-divisor type graph of $Z_{p^a q}$ the following lemma. The lemma will be used later in finding the first Zagreb index of $\Gamma^T[Z_{p^a q}]$.

Lemma 1. Let $\Gamma^T[Z_{p^a q}]$ be the zero-divisor type graph of $Z_{p^a q}$ with distinct primes p and q and a is a positive integer. Each degree of vertices of $\Gamma^T[Z_{p^a q}]$ is stated as follows:

$$deg(A_i) = i$$

and when a is **even**,

$$for\ 1 \le j \le a, \quad deg(B_{j-1}) = \begin{cases} 2j-1, & if\ 1 \le j \le \frac{a}{2}, \\ 2j-2, & if\ \frac{a}{2}+1 \le j \le a, \end{cases} \tag{7}$$

when a is **odd**,

$$for\ 1 \le j \le a, deg(B_{j-1}) = \begin{cases} 2j-1, & if\ 1 \le j \le \frac{a+1}{2}, \\ 2j-2, & if\ \frac{a+1}{2}+1 \le j \le a \end{cases} \tag{8}$$

From Lemma 1 and Definition 3, the first Zagreb index of $\Gamma^T[Z_{p^a q}]$ is obtained in the following theorem.

Theorem 1. Let $\Gamma^T[Z_{p^a q}]$ be the zero-divisor type graph of $Z_{p^a q}$ with distinct primes p and q and a is a positive integer. The first Zagreb index of $\Gamma^T[Z_{p^a q}]$ is stated as follows:

$$M_1(\Gamma^T[Z_{p^a q}]) = \begin{cases} \dfrac{a(5a^2+1)}{3} - a^2, & if\ a\ is\ even, \quad (9) \\ \dfrac{(a-1)[5a^2+5a+3]}{3} - a^2 + 2a + 1, & if\ a\ is\ odd. \quad (10) \end{cases}$$

Conclusion

In this research, the degree for each vertex in $\Gamma^T[Z_{p^a q}]$ is obtained. All values of a is considered since the vertices of the graph are represented by the greatest integer function. From the degree of the graph obtained and definition of first Zagreb index, the first Zagreb index of $\Gamma^T[Z_{p^a q}]$ is then determined.

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