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Analytical Treatment for Fractional Caputo-Fabrizio Casson Fluid Flow over a Vertical Riga Plate

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Abstract
An innovation that aids the regulation of fluid flow, the Riga plate is an integral innovation in the fields of mechanical engineering, marine engineering and biomedical sciences. Whereas fractional derivative, a concept where the n^{th} order of a derivative would be an arbitrary number, has been proven to introduce a new spectrum of solutions when applied to groundwater flow problems, the growth rate of cancer cells and the spread of an infectious disease. This study aims to investigate the analytical treatment of a fractional Caputo-Fabrizio derivative applied on a Casson fluid, flowing over a vertical Riga plate. Obtained analytical solutions from this study will be a valuable tool for experimental researchers for validating their investigations. Governing partial differential equations are solved via Laplace and inverse Laplace transform. It is observed from this study that inflating the fractional parameter and the modified Hartmann number, which exists due to the presence of the Riga plate, increases the fluid flow. Meanwhile, as the fractional parameter is increased, the velocity profile of the fluid also increases.
Keywords: Fluid mechanics; Caputo-Fabrizio fractional derivative; Riga plate; Laplace transform

Introduction

In 1961, Gailitis and Lielausis developed an actuator that is composed of electrodes and magnets that are aligned together alternatingly on a flat plane to help regulate fluid flow [1]. The actuator is called a Riga plate. A Riga plate induces Lorentz force that obstructs or aids fluid flow depending on the position of the plate and the direction of fluid flow. Unlike a magnetohydrodynamic (MHD) fluid that only obstructs a fluid flow, the presence of a Riga plate introduces options and variables in controlling the rate of fluid flow. As such, the Riga plate can be seen applied in the development of submarines, thermal nuclear reactors and micro coolers. Analytical study on fluid mechanics within the presence of a Riga plate is very minimal. Loganathan and Deepa [2] pioneered the analytical study of fluid flow with the presence of a Riga plate by investigating the effects of electromagnetic and radiative energy on a Casson fluid flowing over a vertical Riga plate using Laplace transform. A Casson fluid is a non-Newtonian fluid that behaves as solid when the shear stress is lower than the yield stress. Casson fluid is considered ideal for a theoretical fluid mechanics investigation due to its simplicity as well as its wide application in various technology and engineering fields. Other examples of analytical studies on Casson fluid flow with Riga plates include Shamshuddin et al [3], Bilal et al [4] and Asogwa et al [5]. These studies however lack in introducing fractional derivatives into the governing equations of fluid flow.

A fractional derivative is defined as a derivative with an arbitrary number as the order. The concept was first introduced in a letter between L'Hospital and Leibniz [6]. Since then, several definitions have been developed to try and explain fractional derivatives, including the Riemann, Riemann-Liouville, Caputo, Caputo-Fabrizio and Atangana-Baleanu derivatives. Researchers have also successfully applied the concept of fractional derivatives into numerous mathematical problems including groundwater flow, cancer cell growth and the spread of an infectious disease problem. The introduction of fractional derivatives into the study of fluid mechanics is relatively new.

Khan et al [7] introduced the Caputo derivative into the dynamics of a Casson fluid flow and the study showed that it is possible to incorporate fractional derivatives into the study of fluid mechanics. Other studies, including Maiti et al [8], Sheikh et al [9] and Saqib et al [10], showed that when a fractional derivative is introduced, a new spectrum of solutions is obtained. Although the geometrical representation of fractional parameters within these solutions have yet to be discovered, they are essential in validating experimental or numerical studies in the future.

To the best of the authors' knowledge, an analytical study on a Casson fluid flow over a vertical Riga plate with the Caputo-Fabrizio fractional derivative has yet to be done. Thus, motivated by this and past literature, this study aims to provide an analytical solution for the mechanics of Casson fluid flow over a vertical Riga plate with the Caputo Fabrizio fractional derivative and investigate the impact of fractional derivatives and the presence of Riga plate on fluid velocity and temperature.

Problem formulation

Heat transfer of a Casson fluid flow with the presence of thermal radiation over an accelerated permeable semi-infinite vertical Riga plate is considered. At time, $t=0$, the plate and fluid are both at rest with constant temperature, T_∞. When $t \geq 0$, the temperature of the plate is increased to T_W and remained constant thereafter. Also, at $t \geq 0$, the Riga plate begins to move with a uniform velocity of U_0. A permeated uniform thermal radiation, q_r parallel to the y-axis is applied to the fluid. Due to the small Reynold number, effects of the induced magnetic field in fluid flow, it is insignificant enough for it to be ignored. The y coordinate, measured perpendicular to the plate and fluid flow is only considered at $y > 0$ Velocity, U and temperature, T are dependent on space variable, y and time, t.

By applying Roseland's and Boussinesq's approximation, a governing set of partial differential equations (PDEs) is developed. A set of dimensionless parameters is then used to convert the PDEs into a dimensionless form. The Caputo-Fabrizio fractional derivative is then incorporated into the governing PDEs by replacing the classical derivatives, concerning ($\partial / \partial t$), with the Caputo-Fabrizio fractional derivative $D_t^\alpha f(y,t)$, where $D_t^\alpha f(x,t)$ is defined as:

$$D_t^\alpha f(x,t) = \frac{1}{1-\alpha}\int_0^t \frac{\partial f(x,s)}{\partial x} exp\left(-\alpha\frac{t-s}{1-\alpha}\right)ds. \tag{1}$$

and α is the fractional parameter and x as well as s are variable parameters.

Thus, the final dimensionless governing PDEs for a Casson fluid flow over a vertical Riga plate with the Caputo-Fabrizio fractional derivatives can be written as:

$$D_t^\alpha U(y,t) = \beta_0 \frac{\partial^2 U(y,t)}{\partial y^2} + GrT(y,t) + E\exp(-Ly), \tag{2}$$

$$D_t^\alpha T(y,t) = \left(1+\frac{4}{3}N\right)\frac{1}{\Pr}\frac{\partial^2 T(y,t)}{\partial y^2}, \tag{3}$$

bounded by dimensionless initial and boundary conditions:

$$U(y,0)=0, \quad T(y,0)=0, \tag{3}$$
$$U(0,t)=t, \quad T(0,t)=1, \tag{4}$$
$$U(\infty,t)\to 0, \quad T(\infty,t)\to 0. \tag{5}$$

The parameters β_0, Gr, E, L, N and $\Pr$ are the dimensionless Casson parameter, Grashof number, modified Hartmann number, dimensionless parameter for the width of electrodes and magnets, dimensionless thermal radiation parameter and Prandtl number, respectively. The aforementioned parameters can be mathematically defined such as:

$$\beta_0 = 1+\frac{1}{\beta}, \quad Gr = \frac{\nu g \beta_T (T_W - T_\infty)}{U_0^{\,3}}, \quad E = \frac{\pi J_0 M_0 \nu}{U_0^{\,3} 8\rho}, \tag{6}$$

$$L = \frac{\pi\nu}{lU_0}, \qquad N = \frac{4\sigma^* T_\infty^{\,3}}{k^* k}, \qquad \Pr = \frac{\nu\rho C_P}{k}. \tag{7}$$

While the parameters β, ν, g, β_T, J_0, M_0, ρ, l, σ^*, k^*, k and C_P are the Casson parameter, kinematic viscosity, gravitational acceleration, volumetric thermal expansion coefficient, density of electrical current density, magnetization of magnets, fluid density, width of electrodes and magnets, Stefan-Boltzmann constant, mean absorption coefficient and the thermal conductivity parameter, respectively.

Results and discussion

Solving Equations (2) and (3), requires it to first be transformed into the frequency domain through Laplace transform. Using a variation of parameters, solutions in the frequency domain are obtained for both the velocity and temperature profiles. Obtaining the final solutions in the time domain required the solutions in the frequency domain to be transformed using inverse Laplace transform. During the inverse Laplace transform process, the compound function method and the convolution theorem for inverse Laplace transform are utilized. Obtained velocity and temperature profile solutions are written as:

$$U(y,t) = f_2(y,t) + g_1(y,t) + h_1(y,t) + g_2(y,t) + k_1(y,t), \tag{8}$$

$$T(y,t) = f_1(y,t). \tag{9}$$

Where functions $f_i(y,t)$, $g_i(y,t)$, $h_i(y,t)$ and $k_i(y,t)$, such that $i \in \mathbb{Z}^+$, are defined as:

$$f_i(y,t) = \int_0^\infty \frac{\sqrt{c_i}}{2\sqrt{\pi}u^{3/2}} \exp\left(\frac{-c_i}{4u} - uy^2\right)\left[2\Phi(t)-1\right]du \tag{10}$$

$$+\int_0^\infty\int_0^t\int_0^\pi \frac{1}{\pi}\frac{\sqrt{c_i}}{2\sqrt{\pi}u^{3/2}}\frac{\sqrt{d_i uy^2}}{\sqrt{s}}\cos(\theta) \tag{11}$$

$$\exp\left[\frac{-c_i}{4u} - uy^2 - d_i s + \left(2\sqrt{d_i uy^2 s}\right)\cos(\theta)\right]d\theta ds du, \tag{12}$$

$$g_i(y,t) = \int_0^\infty \frac{e_i\sqrt{f_i}}{2\sqrt{\pi}u^{3/2}} \exp\left(\frac{-f_i}{4u} - uy^2\right)(1+g_i t)\left[2\Phi(t)-1\right]du \tag{13}$$

$$+\int_0^\infty\int_0^t\int_0^\pi \frac{e_i}{\pi}\frac{\sqrt{f_i}}{2\sqrt{\pi}u^{3/2}}\left[1+g_i\,(t-s)\right]\frac{\sqrt{g_i uy^2}}{\sqrt{s}}\cos(\theta) \tag{14}$$

$$\exp\left[\frac{-f_i}{4u} - uy^2 - g_i s + \left(2\sqrt{g_i uy^2 s}\right)\cos(\theta)\right]d\theta ds du, \tag{15}$$

$$h_i(y,t) = \int_0^\infty \frac{h_i\sqrt{l_i}}{2\sqrt{\pi}u^{3/2}} \exp\left(\frac{-l_i}{4u} - uy^2\right)\left[\frac{j_i}{k_i} + \left(\frac{k_i - j_i}{k_i}\right)\exp\left(-k_i t\right)\right]\left[2\Phi(t)-1\right]du \tag{16}$$

$$+\int_0^\infty\int_0^t\int_0^\pi \frac{h_i}{\pi}\frac{\sqrt{l_i}}{2\sqrt{\pi}u^{3/2}}\frac{\sqrt{m_i uy^2}}{\sqrt{s}}\exp\left[\frac{-l_i}{4u} - uy^2 - m_i s + \left(2\sqrt{m_i uy^2 s}\right)\cos(\theta)\right] \tag{17}$$

$$\cos(\theta)\left[\frac{j_i}{k_i} + \left(\frac{k_i - j_i}{k_i}\right)\exp\left(-k_i\,(t-s)\right)\right]d\theta ds du, \tag{18}$$

$$k_i(y,t) = n_i\left[\frac{p_i}{r_1} + \left(\frac{r_i - p_i}{r_i}\right)\exp\left(-r_i t\right)\right]\exp\left(s_i y\right), \tag{19}$$

and

$$\begin{gathered} c_1 = f_2 = a_0 \Pr_0, \quad c_2 = f_1 = l_1 = a_0/\beta_0, \quad d_1 = d_2 = g_1 = g_2 = j_1 = m_1 = p_1 = a_1, \\ e_1 = -b_1, \qquad e_2 = b_1, \qquad h_1 = -b_3, \\ k_1 = r_1 = b_2, \qquad n_1 = -b_3, \qquad s_1 = -L. \end{gathered} \tag{20}$$

The values of a_0, a_1, b_1, b_2, b_3 and Pr_0 are dimensionless parameters that are defined as:

$$\mathrm{Pr}_0 = \frac{\mathrm{Pr}}{1+\frac{4}{3}N}, \qquad a_0 = \frac{1}{1-\alpha}, \qquad a_1 = \alpha a_0, \tag{21}$$

$$b_1 = -\frac{Gr}{\beta_0 \,\mathrm{Pr}_0 \, a_0}, \quad b_2 = \frac{L^2 \beta_0 a_1}{L^2 \beta_0 - a_0}, \quad b_3 = -\frac{E}{L^2 \beta_0 - a_0}. \tag{22}$$

Equations are then analyzed graphically by rewriting them in Mathcad-15. An analysis with the presence of Riga plate and factional parameter, α is presented in Figure 1.

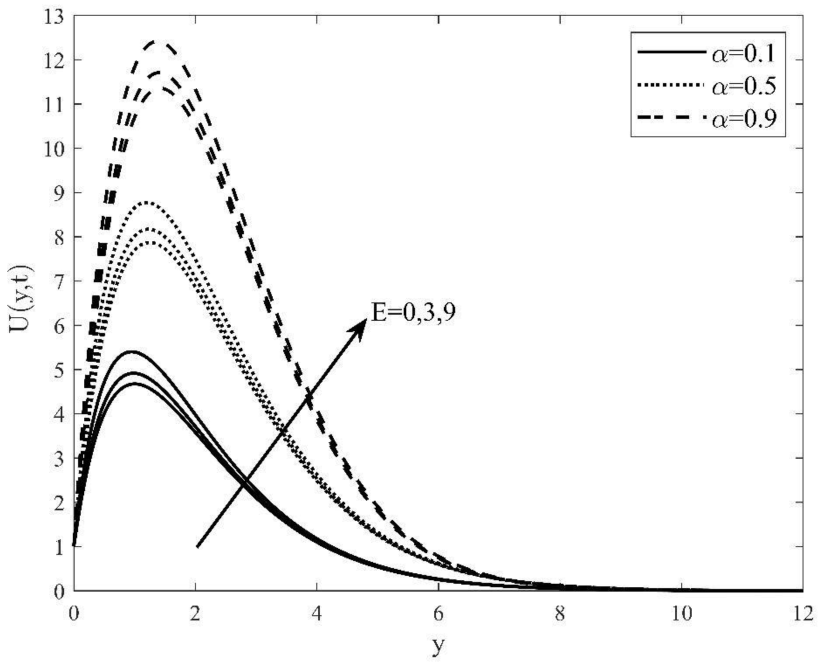

Figure 1 Impact of Riga plate and fractional derivatives on Casson fluid flow.

It is observed from Figure 1 that as the parameter of the modified Hartmann number is increased, the fluid velocity is increased as well. When a Riga plate is introduced into the geometry of the fluid flow, an induced electromagnetic field is considered and it is presented in the solution as the modified Hartmann number, E. Increasing the modified Hartmann number indicates that the induced electromagnetic field is increased. Consequently, increasing the Lorentz force that is generated from the electromagnetic field. The Lorentz force is flowing with the fluid flow, thus aiding the fluid and increasing its velocity.

Meanwhile, it also observed from Figure 1 that an increase in the fractional parameter α increases the velocity profile of the fluid. A geometrical representation of fractional parameters in fluid flow has yet to be discovered. That being said, these new spectra of solutions that are obtained by considering fractional derivatives could be used for validating future experimental or numerical studies.

Conclusion

An analytical solution for a Casson fluid flow over a Riga plate with the Caputo-Fabrizio fractional derivative has been obtained. It can be concluded that the presence of a Riga plate does impact the fluid flow, in this case, aiding it. Meanwhile, by considering fractional derivatives a new spectrum of solutions is obtained that is going to be beneficial in the future for experimental and numerical studies.

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