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Fractional Casson Fluid Flow in the Cylinder with Slip Velocity Effect

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Abstract
Most of the studies solved numerically on the Casson fluid flow in the cylindrical domain by considering no-slip velocity effect. Due to the lack of fractional analytical approaches, this present study is trying to obtain the analytical solution for the fractional time derivative model of Casson fluid flow in the cylinder with slip velocity effect. The momentum governing equation was expressed by using the Caputo-Fabrizio fractional derivative. The joint methods of Laplace transform and finite Hankel transform are applied to obtain an analytical solution of fluid flow velocity. The results showed that the velocity profiles increased when the Casson parameter, fractional parameter and slip velocity parameter were increased. This study is significant to explore more on fluid behavior with fractional-derivative approach model and the obtained results can be applied in biomedical applications.
Keywords: Casson fluid; cylinder; Caputo-Fabrizio; finite Hankel transform; slip velocity

Introduction
Casson fluid is one of the non-Newtonian fluids with unique behavior. It behaves like an elastic solid and initiates the flow depending on the applied shear stress. Examples of Casson fluid are blood, honey, jelly, tomato sauce and others (Alderman and Pipelines, 1977). Nowadays, researchers are interested in studying the Casson fluid with fractional derivative models such as Caputo, Riemann-Liouville, Caputo-Fabrizio and Atangana-Baleanu. The concept of the fractional derivative is to discuss the *n*-notation of the derivative if it is a fractional or complex number. It is an important tool to describe physical memory in many applications such as fluid mechanics, biological materials and others (Zheng, 2017; Ray *et al.*, 2014). However, some fractional derivative models have limitations and difficulty in modelling the physical problems due to the power law kernel. Thus, the most appropriate for modelling the fluid flow with a non-singular kernel is introduced, which is Caputo-Fabrizio fractional derivative model (Shaikh *et al.*, 2019). Motivated with this derivative, Maiti *et al*., 2020 studied analytically Casson fluid flow in the cylinder with the Caputo-Fabrizio fractional derivative model. Then, Jamil *et al.*, 2021extended the problem with the Casson fluid flow in an inclined cylinder. All of them solved the problems analytically by using Laplace transform and finite Hankel transform methods.

Most of the previous studies on Casson fluid flow in the cylinder consider no-slip velocity effect. However, slip velocity at the boundary plays a vital role to study since it influences fluid velocity and exists in real-life applications such as blood flow in the arteries and the oil and gas drilling process. Slip velocity can be defined as a finite velocity of fluid occurring at the boundary. In another word, it exists when a velocity gradient occurs between two mediums which are the surface and the adjacent particles of the fluid flow freely on the surface (Rao and Rajagopal, 1999; Nubar, 1971). Due to its importance, many researchers are attracted to study the slip effect on the fluid flow such as Padma *et al*., 2019. They explored the Jeffrey fluid flow behavior in the cylinder with the presence of slip and no-slip effect. They obtained an analytical solution by using Laplace and finite Hankel transform methods. Then, Jalil and Iqbal, 2021 solved numerically the impact of slip velocity on the Casson fluid in the cylinder. None of them studied the slip effects on the fluid flow in the cylinder with the Caputo-Fabrizio fractional derivative model.

Based on the above literature, many researchers studied analytically the Casson fluid flow in the cylinder with Caputo-Fabrizio fractional derivative approach but all of them consider no-slip boundary conditions. However, based on the cited above, researchers studied the slip effect and solved them analytically on the other fluid models and some of them solved numerically for the Casson fluid model. Thus, the aim of this study is to obtain analytical solutions for an incompressible Casson fluid flow in the cylinder with the Caputo-Fabrizio fractional derivative approach model and slip velocity effect. Besides, the obtained analytical solution of fluid velocity will be analyzed to understand the fluid flow behavior.

In order to achieve the objectives, several steps need to be considered. Firstly, the dimensionless momentum governing equation is transformed into the Caputo-Fabrizio Fractional order derivative. Secondly, the joint transformation techniques of Laplace transform and finite Hankel transform have been used to obtain analytical solutions of fluid velocity. Lastly, the obtained solution of the fluid velocity will be plotted by using Maple software to analyze fluid velocity behavior with the related parameters.

Mathematical Formulation

Consider the unsteady free convection flow of an incompressible Casson fluid in a vertical cylinder of radius, r_0. The axis of the cylinder in a vertical upward is considered as the z-axis and the *r*-axis is taken as normal to it. Initially, at $t^*=0$, both fluid and cylinder are at rest. Then, at $t^*>0$, the fluid begins to flow due to the existence of the velocity gradient between fluid particles and the wall of the cylinder or also known as slip velocity, u_s occurred at the boundary of the cylinder. The fluid velocity is the function of *r* and *t* only. The fluid flow problem can be represented in the schematic diagram as shown in **Figure 1**.

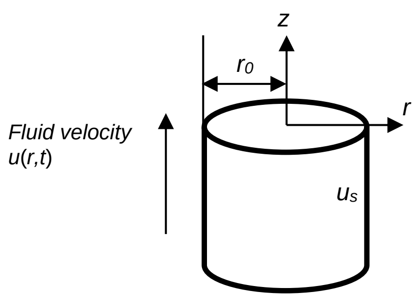

Figure 1. Schematic diagram of the fluid flow

Then, under Boussinesq's approximation, the corresponding partial differential equation of momentum (Padma, Selvi and Ponalagusamy, 2019) is given as

$$\rho\frac{\partial u^*}{\partial t^*}=\mu\left(1+\frac{1}{\beta}\right)\left(\frac{\partial^2 u^*}{\partial r^{*2}}+\frac{1}{r^*}\frac{\partial u^*}{\partial r^*}\right) \tag{1}$$

with the associated initial and boundary conditions [7]

$$\begin{aligned} u^*(r^*,0) &= 0 \quad ;\ r\in[0,r_0]\,, \\ u^*(r_0^*,t^*) &= u_s^* \quad ;\ t^*>0. \end{aligned} \tag{2}$$

Introducing the following dimensionless variables (Padma, Selvi and Ponalagusamy, 2019; Khan *et al.*, 2018)

$$u_s=\frac{u_s^*}{u_0},\ u=\frac{u^*}{u_0},\ r=\frac{r^*}{r_0},\quad t=\frac{t^*v}{r_0^2} \tag{3}$$

where ρ is the density of the fluid, u^* is the velocity component along the *z*-axis, μ is the dynamic viscosity of the fluid, β is the non-Newtonian Casson parameter, u_0 is the average velocity of the fluid, v is the kinematic viscosity of the fluid. Then, substitute equation (3) into equations (1) and (2) for the non-dimensionalization process to obtain the dimensionless partial differential equation of momentum as written below

$$\frac{\partial u}{\partial t}=\beta_1\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{4}$$

with the associated dimensionless initial and boundary conditions

$$\begin{aligned} &u(r,0)=0 \quad ;\ r\in[0,1]\ , \\ &u(1,t)=u_s \quad ;\ t>0 \end{aligned} \tag{5}$$

where $\beta_1=\frac{1}{\beta_0}=\frac{\beta}{\beta+1}$ and $\beta_0=\frac{\beta+1}{\beta}$ are the constant parameters. Employing Caputo-Fabrizio fractional derivative to the above fluid classical model (4), yields

$${}^{CF}D_t^{\alpha}u(r,t)=\beta_1\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right) \tag{6}$$

where ${}^{CF}D_t^{\alpha}u(r,t)=\frac{1}{1-\alpha}\int_0^{\tau}\exp\left(\frac{-\alpha(\tau-t)}{1-\alpha}\right)u'(\tau)dt$ for $0<\alpha<1$ is the definition of the Caputo-Fabrizio fractional derivative (Maiti, Shaw and Shit, 2020). In order to obtain an analytical solution of the fluid velocity, the joint methods of Laplace transform and finite Hankel transform are utilized. Both of the transformations are useful when dealing with the cylindrical domain that is involved with the time, *t*. Applying Laplace transform into momentum equation (4) and boundary condition (5) subjected to the initial condition (5), give

$$\frac{a_0 s\bar{u}(r,s)}{s+a_1}=\beta_1\left[\frac{\partial^2\bar{u}(r,s)}{\partial r^2}+\frac{1}{r}\frac{\partial\bar{u}(r,s)}{\partial r}\right], \tag{7}$$

$$\bar{u}(1,s)=\frac{u_s}{s}, \tag{8}$$

where $a_0=1/1-\alpha$, $a_1=a_0\alpha$, are the fractional constant parameters, $\bar{u}(r,s)$ is the Laplace transform of the function $u(r,t)$ and s is the transform variable. Then, the finite Hankel transform is applied to the equation (7) by using boundary condition (8) and the simplification of the obtained equation can be written as

$$\bar{u}_H(r_n,s)=\frac{J_1(r_n)}{r_n}\left[\frac{u_s}{s}-\frac{a_0u_s}{\left(a_0+\beta_1r_n^2\right)\left(s+\left(a_1\beta_1r_n^2/a_0+\beta_1r_n^2\right)\right)}\right], \tag{9}$$

Where $\bar{u}_H(r_n,s)=\int_0^1 r\bar{u}(r,s)J_0(rr_n)\,dr$ is the finite Hankel transform of the function and r_n, with n=0,1,…are the positive roots of the equation, where J_0 is being the Bessel function of first kind and zero order, J_1 is being the Bessel function of first kind and first order. Next, applying the inverse Laplace transform of equation (9), give

$$\bar{u}_H(r_n,t)=\frac{J_1(r_n)}{r_n}\left[u_s-\frac{a_0u_s}{a_0+\beta_1r_n^2}\exp\left(-\frac{a_1\beta_1r_n^2}{a_0+\beta_1r_n^2}t\right)\right], \tag{10}$$

Finally, the inverse finite Hankel transform is applied to Eq. (10) and the velocity of fluid flow is obtained as:

$$u(r,t)=u_s-2a_0u_s\sum_{n=1}^{\infty}\frac{J_0(rr_n)}{r_nJ_1(r_n)}\frac{\exp\left(-\dfrac{a_1\beta_1r_n^2}{a_0+\beta_1r_n^2}t\right)}{a_0+\beta_1r_n^2}. \qquad (11)$$

Results and Discussions

Figure 2 represents the velocity profile of Casson fluid flow in the cylinder at two different values of slip velocity u_s= 0.2 and 0.4 with three different values of fractional parameter α=0.3,0.7 and 1.0. It shows that fluid velocity increased with the increase of fractional parameters. Fractional parameters play a vital role to control the fluid flow velocity. In this figure at t=1.0, it showed that Casson fluid flow with classical model α=1 is faster than Casson fluid flow with fractional model 0<α<1. It is due to the fact that fractional Casson fluid flow is more realistic compared to the classical Casson fluid flow as time increases. As we can see in Figure **3**, as time increases from t=0.1 to t=1.0, the classical model of fluid flow drastically increases while the fractional model of fluid flow slightly increases. Besides that, the impacts of the slip velocity on the fluid flow behavior can be notified at the wall of the cylinder at r=1. Based on the observations in **Figure 2**, slip velocity effect at the boundary increased will cause fluid velocity at the wall of the cylinder increased. Meanwhile, fluid velocity decreases as it is approaching the center of the cylinder. It is due to the viscosity of the Casson fluid. It needs an additional shear stress or other forces to enhance the fluid flow at the center of the cylinder.

Moreover, the influence of the Casson parameter, β on the fluid flow velocity is exhibited in **Figure 4**. It is clearly seen that an increase in the Casson parameter led to an increase in velocity profiles. It is because yield stress falls and the boundary layer thickness decrease when the Casson parameter increases which results in fluid velocity enhancement.

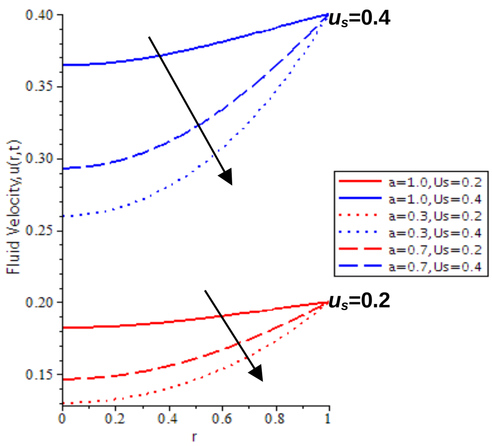

Figure 2 Impact of slip velocity, u_s and fractional parameter, α when β=1, t=1.

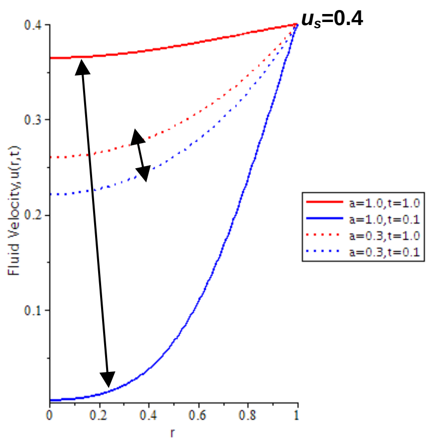

Figure 3 Classical and fractional fluid flow behavior for t=0.1, 1.0

u_s=0.4

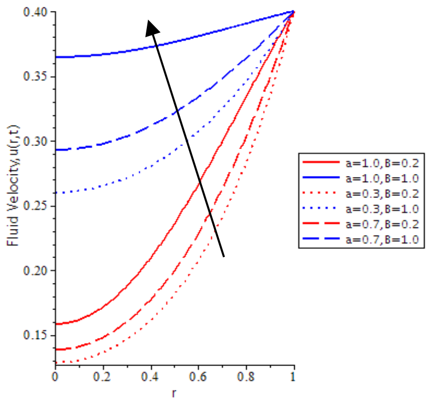

Figure 4 Impact of Casson parameter, β and fractional parameter, α when u_s=0.4, t=1.

Conclusion

In the present study, the unsteadiness of the incompressible Casson fluid model has been considered in the cylindrical domain with the slip velocity effect. The momentum governing equation for Casson fluid has been modeled in a partial differential equation and Caputo-Fabrizio fractional derivative approach model is applied. Then, the analytical solution has been obtained by using the Laplace transform and finite Hankel transform of zeroth-order. Finally, the analytical solution is satisfied with the initial and boundary conditions. Furthermore, the obtained solutions are discussed graphically with the effects of Casson parameter β, fractional parameter α, and slip velocity parameter u_s. The velocity profiles increase when β, α and u_s are increased. Besides, the fluid flow velocity is higher at the boundary of the cylinder and decreases as it is approaching the center of the cylinder. It is significant to study the fluid velocity behavior with the fractional derivative model since it is more realistic. Moreover, these findings are beneficial to analyzing and controlling the fluid flow which is related to the human blood system to overcome the blood disease problems. Besides, this study can be developed in the future with nanofluid and hybrid nanofluid models and add the other effects.

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