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The Dynamics of *b*-Bistochastic-Volterra Quadratic Stochastic Operators

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Abstract
Quadratic Stochastic Operator, QSO is one of the simplest non-linear Markov operators which was often studied on singleton finite dimension of all probability distribution, S^{n-1}. However, the study of QSOs in general setting is rather tricky compared to linear operators. Therefore, several classes of QSOs were introduced such as Volterra QSOs, strictly non-Volterra QSOs, Orthogonal preserving QSOs, Centered QSOs and etc. In this paper, a new class of QSO named *b*-bistochastic-Volterra QSO, or in short, bV-QSO is introduced and defined on one-dimensional simplex, $S^1 \times S^1$. Note that, the main problem in the non-linear operator theory is to study their dynamics. Thus, the set of all fixed points of bV-QSOs are provided. Next, a linear Lyapunov function is constructed which is then applied in order to investigate the dyamical behavior of bV-QSO. The results obtained in this study contribute to advancement of knowledge in the theory of non-linear operators.
Keywords: Quadratic Stochastic Operators, Markov operators, *b*-Bistochastic QSOs, Volterra QSOs

Introduction

A *quadratic stochastic operators* denoted as V was originally introduced by (Bernstein, 1924) which usually arise from the problems of population genetics (see also (Lyubich, Akin, Karpov, & Vulis, 1992)). QSOs become one of the main sources of analysis in studying dynamical properties and modelling in a system which requires many interactions. For the sake of comprehension, consider the following biological ambiance. Assume that each individual in this population belongs precisely to one of the species (trait) which denoted by $I = 1,2,\dots,n$. The probability of an individual is denoted by $P_{ij,k}$, where an individual in i^{th} species and j^{th} species to cross-fertilize and produce an individual from k^{th} species. These coefficients $P_{ij,k}$ are known as *heredity coefficients* which define a QSO, V. Provided initial probability distribution of the species, $x^{(0)} = (x_1^{(0)},\dots,x_n^{(0)})$, the probability distribution of the first generation, $x^{(1)} = (x_1^{(1)},\dots,x_n^{(1)})$ can be found by applying the QSO as a total probability that is,

$$x_k^{(1)} = \sum_{i,j=1}^{n} P_{ij,k}x_i^{(0)}x_j^{(0)} =: V(x)_k, \qquad for\ any \quad k \in \{1,\dots,n\}. \tag{1}$$

Hence, these iteration produce a dynamical system as $\{x^{(0)}, x^{(1)}, \dots,\}$ which describes the species' dispersion on the long run.

Studying QSOs in general is challenging unlike the linear case. Therefore, classes of QSOs were introduced by researchers such as QSOs on Banach Lattices, Volterra QSOs, *b*-bistochastic QSOs, centered QSOs, Orthogonal preserving QSOs, Lebesgue QSOs, QSOs corresponding to permutations and etc (for example see (Badocha & Bartoszek, 2018; Ganikhodzhaev, Mukhamedov, & Rozikov, 2011; Mukhamedov & Embong, 2015 (1); Bartoszek, Domsta, & Pułka, 2019; Mukhamedov & Taha, 2015; Karim, Hamzah, & Ganikhodjaev, 2021; Jamilov, Khudoyberdiev, & Ladra, 2020)).

However, these classes do not yet cover the whole set of QSO. The introduction of this new class of QSO is to contribute knowledge in the theory of non-linear operators. The book by (Mukhamedov & Ganikhodjaev, 2015) serves as a comprehensive reference in the theory of QSOs. Recent achievement of QSOs could be further read in (Saburov & Saburov, 2020; Abdulghafor, et al., 2020; Abdurakhimova & Rozikov, 2021) and the references therein.The concept of majorization was first introduced by Lorenz in (Lorenz, 1905) and further investigated by Hardy et al. in (Hardy, Littlewood, & Pólya, 1952). This new order majorization generalizes the classical majorization. Besides, it is indeed a partial order on sequence which is an advantage compared to classical majorization. In this paper, majorization is considered as b-order which is denoted as $\leq^{b}$. A QSO, namely bistochastic QSO, also called as doubly stochastic is defined in terms of classical majorization (Ganikhodzhaev R. N., 1993), where $V(x) \prec x$, for all x from $n-1$ dimensional simplex.

Most well-studied class of QSOs is known as Volterra QSO. Biological meaning of this operator is that: *the child could inherit the trait from their parents only*. In the study of the Volterra dynamical systems (acting on finite dimensional simplex) for a given biological population, the following question may arise: *what kind of genotypes will preserve and which of them will disappear?* There are many papers published on the investigations of discrete Volterra operators (Mukhamedov, Khakimov, & Embong, 2020; Rozikov & Shoyimardonov, 2019; Ganikhodzhaev R. N., 1993). Note that most of the studies in the theory of QSOs were done by considering V that maps from S^{n-1} into itself.

The objectives of this paper include to provide a full description of bV-QSO defined on one dimensional simplex, to list all the fixed points obtained, to construct Lyapunov functions and determine its dynamical behavior.

Materials and methods

There are studies on the most well-known class of QSO which is Volterra QSO denoted as V were defined on two-sex population (Rozikov & Zhamilov, 2011), where the mapping is from $S^{n-1} \times S^{v-1}$ to itself. Motivated from this idea, a new class of QSOs is introduced to define the population genetics for two-sex population named *b*-bistochastic-Volterra quadratic stochastic operator, or simply bV-QSO. Since stability is the fundamental concept that can be specified in terms of fixed points, this study also investigates the fixed points of the system under QSO and their behavior.

Let S^{n-1} be the set of all probability distribution i.e,

$$S^{n-1} = \left\{ x = (x_1, x_2, \dots, x_n) \in R^n | x_i \geq 0, \sum_{i=1}^{n} \quad x_i = 1 \right\}, \tag{2}$$

where $n \in N^* = \{1,2,\dots,n\}$. S^{n-1} is called as a *simplex.* Now, define an operator V that maps from $S^{n-1} \times S^{v-1}$ where ${P_{ij,k}}^{(f)}$ and ${P_{ij,l}}^{(m)}$ are its coefficients of inheritance. In biological point of view, these coefficients respresent as the probability of a female offspring being type k and, respectively a male offspring being type l, when the parental pair is i, j $(i, k = 1,2,\dots, n$; and $j, l = 1,2,\dots, v)$. The following is obtained:

$$P_{ij,k}^{(f)} \geq 0, \quad \sum_{k=1}^{n} \quad P_{ij,k}^{(f)} = 1, \quad P_{ij,k}^{(m)} \geq 0, \quad \sum_{k=1}^{n} \quad P_{ij,k}^{(m)} = 1. \tag{3}$$

Using these coefficients, the operator V can be defined as follows: Let $x = (x_1, \dots, x_n) \in S^{n-1}$ and $y = (y_1, \dots, y_n) \in S^{v-1}$

$$V(x, y) = \{V_x = \left(\sum_{i,j=1}^{n,v} \quad P_{ij,k}^{(f)} x_i y_j \right)_{k=1}^{n} \quad V_y = \left(\sum_{i,j=1}^{n,v} \quad P_{ij,l}^{(m)} x_i y_j \right)_{l=1}^{v}. \tag{4}$$

One may check that V_x and V_y is stochastic, i.e., V maps from $S^{n-1} \times S^{v-1}$ into itself. This paper is limited to the value $n = 2,$ and $v = 2$. The following definition is applied to describe bV-QSO where $V: S^1 \times S^1 \to S^1 \times S^1$.

Definition 1: *A QSO V is called bV-QSO if*

$$V_x \leq^b x, \tag{5}$$

and the heredity coefficients for V_y satisfy $P_{ij,k}^{(m)} = 0$ for any $k \notin \{i, j\}$.

Results and discussion

1. **Description of bV-QSO on 1D Simplex**

This section aims to give a full description bV-QSOs on $S^1 \times S^1$. One can see that if $n = 2$, then the simplex is reduced to:

$$S^1 = \{x = (x_1, x_2) \in R^2 | x_1, x_2 \geq 0, x_1 + x_2 = 1\}. \tag{6}$$

Theorem 1: *Let V be a QSO defined on $S^1 \times S^1$. The operator V is a bV-QSO if and only if*

$$V(x, y) = \{x' = axy\; y' = xy + b(x - 2xy + y),$$

where $a = {P_{11,1}}^{(f)}$, $b = {P_{12,1}}^{(m)} = {P_{21,1}}^{(m)}$, and ${P_{11,1}}^{(m)} = 1$.

Corollary 1 *Let V be a bV-QSO defined on $S^1 \times S^1$. Then, the following properties hold:*

i. ${P_{12,1}}^{(f)} = {P_{21,1}}^{(f)} = {P_{22,1}}^{(f)} = 0$.
ii. ${P_{11,2}}^{(m)} = {P_{22,1}}^{(m)} = 0$.
iii. ${P_{11,1}}^{(m)} = 1$.

2. **Fixed Points**

In this section, the fixed points of bV-QSO defined on one-dimensional simplex are obtained as follows:

Theorem 2: *Let V be a bV-QSO defined on $S^1 \times S^1$, then one has the following statements:*

i. $(0,0)$ is always the fixed point.
ii. If $a < 1$ and $b = 1$, $(0, y)$ is the fixed point for any $y \in (0,1]$.
iii. If $b < 1$, $a = 1$, then $(1,1)$ is the fixed points.
iv. If $a = 1$ and $b = 1$, then $(0, y)$ and $(x, 1)$ are the fixed points.

Corollary 2 *Let V be a bV-QSO. If $a < 1$ and $b < 1$, then $(0,0)$ is a unique fixed point.*

3. **The Lyapunov Function of bV-QSO**

This section provides the Lyapunov function of bV-QSO defined on one-dimensional simplex which then will be applied to prove the limiting behavior of such operators.

Proposition 1 *Let V be a bV-QSO. Define a functional $\varphi: S^1 \times S^1 \to R$ as follows:*

$$\varphi(x, y) = x + y. \tag{7}$$

Then, φ is a Lyapunov functions for $V(x, y)$ if $a + b \leq 1$.

4. Limiting Behavior of bV-QSO

To study the trajectory, consider the boundaries as follows

$$\text{Side 1: } x = 0;\ \text{Side 2: } y = 0;\ \text{Side 3: } x = 1;\ \text{Side 4: } y = 1. \quad (8)$$

Theorem 3: *Let V be bV-QSO defined on $S^1 \times S^1$. If the initial point (x, y) is taken from the sides given in (6), then V is converged. Moreover, the trajectories are described as follows:*

i. Side 1: $V^{(n)}(0, y) = \{(0,0),\ b < 1, (0, y),\ b = 1.$

ii. Side 2: $V^{(n)}(x, 0) = \{(0,0),\ b < 1, (0, x),\ b = 1.$

iii. Side 3: $V^{(n)}(1, y) = \{(0,1),\ a < 1, b = 1, \quad (y, 1),\ a = 1, b = 1, \quad (0,0),\ a < 1\ or\ y < 1, b = 0, \quad (1,1),\ a = 1\ and\ y = 1, b < 1.$

iv. Side 4: $V^{(n)}(x, 1) = \{(0,1),\ a < 1, b = 1, \quad (x, 1),\ a = 1, b = 1, \quad (0,0),\ a < 1\ or\ x < 1, b = 0, \quad (1,1),\ a = 1\ and\ x = 1, b < 1.$

Next, the trajectory of the internal region is investigated and the following theorem is obtained:

Theorem 4: *If $a + b < 1$, then*

$$x^{(n)} = y^{(n)} = 0. \quad (9)$$

Conclusion

The study of non-linear Markov operator specifically quadratic stochastic operators (QSOs) are tricky in general setting. Therefore, many classes of QSOs were introduced. This paper introduces a new class of QSO namely *b*-bistochastic-Volterra QSO (bV-QSO) defined on one-dimensional simplex. Full description of the considered class of QSO is provided. By using the canonical form, all fixed points are listed in this paper. Then, the limiting behavior of bV-QSO is obtained by applying the Lyapunov function. Note that, a QSO could have various dynamical behavior such as non-ergodic, periodic and regular. Since this study focus on a specific operator which is bV-QSO on one-dimensional simplex, this work can be extended and generalised into n-dimensional simplex.

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