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The Union Power Cayley Graph for Cyclic Groups of Prime Power Order

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Abstract
Various graphs formulated from groups have been investigated and defined over years due to their importance in algebra and many other fields in real life, including the power graphs and Cayley graphs. The power graph of a finite group G is defined as a simplified form of an undirected graph whose vertices are elements of G, in which two distinct vertices are adjacent if one of them can be written as an integral power of the other. Meanwhile, the Cayley graph of a group G with respect to the inverse- closed subset S of G, is a graph whose vertices are the elements of G and two vertices a and b are connected if $a = bs$ or $b = as$ for some $s \in S$. In this paper, a new type of graph, namely union power Cayley graph for cyclic groups of prime power order is introduced by combining two graphs which are the power graph and Cayley graph. In addition, this union power Cayley graph is constructed for cyclic groups of prime power order and its generalization is determined.
Keywords: Power graph; Cayley graph; Cyclic group; Union power Cayley graph.

Introduction
In the last 50 years, the theory of Cayley graphs has grown into a substantial branch in algebraic graph theory. Cayley graph is one of the most important graphs because it has different kinds of structures according to different the subset S, while the power graph is a fixed graph. Besides, Cayley graphs and power graphs have many important applications in life such as in mobile phone networks, scheduling, roadways, and electrical circuits.

The definition of Cayley graph was introduced in [1] to explain the concept of abstract groups which are described by a set of generators. In [2] Cayley graphs were reintroduced under the name Gruppenbild (the graph as a group diagram), which led to today's geometric group theory. In addition, in [3] the degree and diameter of the Cayley graph for cyclic, abelian, and metacyclic groups were investigated. Many problems related to Cayley graphs have been identified by many scholars such as Hamiltonicity or diameter problems, to problem related to computer science and molecular biology such as pancake problem or sorting by reversals, and coding theory such as the vertex reconstruction problem related to error-correcting codes [4]. In addition, the concepts of Cayley graphs have been developed and new types of Cayley graphs have been introduced such as prime Cayley graphs, prime order Cayley graphs, and composite order Cayley graphs [5, 6].

Another graph that has significant applications in group theory is the power graph. The directed power graph of a group was introduced in [7]. Meanwhile, in [8] introduced the undirected or simply the power graph of a finite group and a semigroup. It is shown in [8] that when a group G is finite, the power graph of a subgroup of G is an induced subgraph of the undirected power graph $P(G)$, and $P(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, where p is a prime number and $m \in N$. Moreover, in [8] it was apparent that $P(G)$ with any finite group G is connected graph because the identity element of G is adjacent to all other vertices of $P(G)$.

In this paper a new type of graph, namely the union power Cayley graph is introduced. Besides, the union power Cayley graph for cyclic groups of order nine, Z_9 with a subset of size four is constructed. Finally, the union power Cayley graphs for cyclic groups of prime power order will be generalized for all subsets.

Materials and methods

The construction of the union power Cayley graphs for cyclic groups has been done using definitions of the union power Cayley graphs, the group presentations, and related theorems.

In this section, some basic concepts, definitions, notations and previous results related to Cayley graph and power graph of groups are given.

First, the formal definition of the power graph is given in Definition 1 followed by a related proposition.

Definition 1 [8] (Power Graph)
The undirected power graph $P(G)$ of a finite group G has the vertex set G, and two distinct elements x and y are adjacent if and only if $x = y^n$ or $y = x^n$ for some $n \in N$.

Proposition 1. [8] A finite group has a complete power graph if and only if it is cyclic and has prime power order.

The formal definition of the Cayley graph is given in Definition 2.

Definition 2 [9] (Cayley Graph)
A graph $Cay(G,S)$ is a Cayley graph on a finite group G if there is a subset $S \subseteq G\backslash e$, with $S^{-1} = \{s^{-1} | s \in S\} \subseteq S$, such that $V\big(Cay(G,S)\big) = G$, and two vertices a and b are adjacent if and only if $ba^{-1} \in S$.

The formal definition of the Cyclic Group is given in Definition 3.

Definition 3 [10] (Cyclic Group)
A group G is called a cyclic group if there is an element $x \in G$ that generates it. More specifically, $G = \langle x \rangle = \{x^n : n \in N\}$.

Results and discussion

In this section, the new graph, which is the union power Cayley graph is introduced followed by construction the union power Cayley graph for cyclic groups of order nine with a subset of size four.

Definition 4 (Union Power Cayley Graph) Let G be a group and S be a subset of G such that $e \notin S$ and $S^{-1} \subseteq S$. Then, the union power Cayley graph of G related to S, denoted by $Pow - Cay^{+}(G,S)$ is an undirected simple graph with the vertex set equal to G and two vertices x and y are adjacent if and only if at least one of the following two conditions is satisfied:

- $xy^{-1} \in S$.
- $x = y^n$ or $y = x^n$ for some $n \in N$.

Example 1.
Let $G = Z_9$ be the cyclic group of order nine, where Z_9= $\{e, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8\}$ = $\langle g \rangle$ $\langle g^2 \rangle$ = $\langle g^4 \rangle$ = $\langle g^5 \rangle = \langle g^7 \rangle = \langle g^8 \rangle$, and $S = \{ g, g^3, g^6, g^8\}$ be the subset of Z_9. Then, the Cayley graph of Z_9 with respect to the subset S, $Cay(Z_9, S)$ can be constructed by using the Definition 2, such that the vertex x is connected to y if $yx^{-1} \in S$.

$e - g$ since $e + g^{-1} = g^8 \in S$
$g - g^7$ since $g + (g^7)^{-1} = g^3 \in S$
$e - g^3$ since $e + (g^3)^{-1} = g^6 \in S$
$g^2 - g^3$ since $g^2 + (g^3)^{-1} = g^8 \in S$
$e - g^6$ since $e + (g^6)^{-1} = g^3 \in S$
$g^2 - g^5$ since $g^2 + (g^5)^{-1} = g^6 \in S$
$e - g^8$ since $e + (g^8)^{-1} = g \in S$
$g^2 - g^8$ since $g^2 + (g^8)^{-1} = g^3 \in S$
$g - g^2$ since $g + (g^2)^{-1} = g^8 \in S$
$g^3 - g^4$ since $g^3 + (g^4)^{-1} = g^8 \in S$
$g - g^4$ since $g + (g^4)^{-1} = g^6 \in S$
$g^3 - g^6$ since $g^3 + (g^6)^{-1} = g^6 \in S$
$g^4 - g^5$since $g^4 + (g^5)^{-1} = g^8 \in S$
$g^5 - g^8$ since $g^5 + (g^8)^{-1} = g^6 \in S$
$g^4 - g^7$ since $g^4 + (g^7)^{-1} = g^6 \in S$
$g^6 - g^7$ since $g^6 + (g^7)^{-1} = g^8 \in S$
$g^5 - g^6$since $g^5 + (g^6)^{-1} = g^8 \in S$
$g^7 - g^8$ since $g^7 + (g^8)^{-1} = g^8 \in S$

The Cayley graph of Z_9 with respect to the subset S is illustrated as in the following:

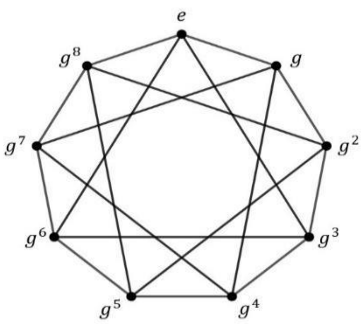

Figure 1 The Cayley graph of Z_9 with the subset S, $Cay(Z_9, S)$.

Besides, the power graph of Z_9, $P(Z_9)$ can be constructed by using Definition 1, such that the vertex x is connected to y if $x = y^n$ for some $n \in N$ or $y = x^m$ for some $m \in N$. Moreover, it can be seen that:

$$\langle e \rangle = \{e\}$$
$$\langle g \rangle = \{e, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8\}$$
$$\langle g^2 \rangle = \{e, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8\}$$
$$\langle g^3 \rangle = \{e, g^3, g^6\}$$
$$\langle g^4 \rangle = \{e, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8\}$$
$$\langle g^5 \rangle = \{e, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8\}$$
$$\langle g^6 \rangle = \{e, g^3, g^6\}$$
$$\langle g^7 \rangle = \{e, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8\}$$
$$\langle g^8 \rangle = \{e, g, g^2, g^3, g^4, g^5, g^6, g^7, g^8\}$$

The power graph of Z_9 is illustrated as in the following:

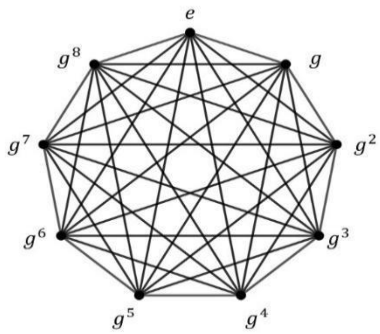

Figure 2 The power graph of Z_9 , $P(Z_9)$.

The union power Cayley graph of Z_9 with the subset S, $Pow - Cay^{+}(Z_9, S)$ can be constructed by using the Definition 4, such that $\{x, y\} \in E(P(Z_9) \cup E(Cay(Z_9, S))$. The union power Cayley graph of Z_9 with respect to the subset S is illustrated as in the following:

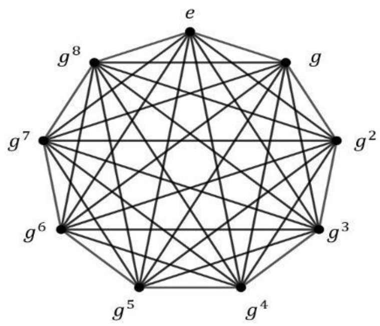

Figure 3 The union power Cayley graph of Z_9 with the subset S, $Pow - Cay^{+}(Z_9, S)$.

The construction of the union power Cayley graph for a cyclic group of prime power order, Z_{p^m}, where p is a prime number and $m \in N$ is shown in the following theorem.

Main Theorem
Let G be a cyclic group of prime power order. Then, the union power Cayley graph of G is a complete graph. In other words, $Pow - Cay^{+}(G, S) \cong K_{p^m}$, where p is a prime number and $m \in N$.

Proof.
By Proposition 1, a finite group has a complete power graph if and only if it is cyclic and has prime power order. Hence, the union of a complete graph with any graph is a complete graph. Thus, by Definition 4, the union of power graph Cayley graph of a cyclic group of prime power order p^m is isomorphic to K_{p^m}.

Conclusion
In this paper, a new type of graph, namely the union power Cayley graph has been introduced. The union power Cayley graph of cyclic groups of prime power order is found to be a complete graph.

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