Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia

Vol. 6, 2022, Page 90-93

The Total Perfect Codes of The Non-Zero Divisor Graph of Ring of Integers Modulo 10

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Abstract
In recent years, research on graph associated to rings has been extensively done. Let $\Gamma(R)$ be the non-zero divisor graph with vertices of all non-zero elements of a ring, R such that two distinct elements x and y are adjacent if and only if $xy \neq 0$. In this research, the main focus is to determine the total perfect code of a non-zero divisor graph of a commutative ring. A code is said to be a total perfect code in a graph if every vertex of the graph has exactly one neighbor. The non-zero divisors were found first in order to establish the non-zero divisor graph. Finally, the total perfect code of the graph constructed is obtained. The non-zero divisor graph, $\Gamma(\mathbb{Z}_{10})$ is found to be a simple, undirected graph with nine vertices and 32 edges. This graph also admits a total perfect code.
Keywords: non-zero divisors; commutative ring; non-zero divisor graph; total perfect code

Introduction
After its discovery in the late 19th century, zero-divisor graph has been an important object of study in the field of graph theory. The zero-divisor graph was first introduced by Beck [1] in 1988. The main focus was on the graph coloring where the author found the chromatic and clique number of the graph. A few decades later, the concept was further explored by researchers using different types of groups, rings and fields. Redmond [2] defined the undirected zero divisor graph of a non-commutative ring. The vertices of this zero-divisor graph are all the non-zero zero divisors of the ring and for two distinct vertices, a and b, there is an edge connecting them if and only if $ab = 0$ or $ba = 0$. In the following year, the zero-divisor graph of a commutative Von Neumann regular ring is investigated by Anderson *et al.* [3]. Von Neumann regular ring is a ring in which for each element of x in the ring, there exist an element y in the ring such that $x = x^2y$. In 2007, Akhbari and Mohammadian [4] studied and defined the zero-divisor graphs of matrix rings and group rings. In recent years, Anderson and Weber [5] studied the zero-divisor graphs of a commutative ring without identity.

Total perfect codes have been studied extensively by many researchers and the concept has appeared in the literature under various names such as efficient open domination sets, total domination sets and exact transversals. The first research on the total perfect code was conducted by Cockayne *et al.* [6] in 1980 where the authors found total dominating set of a graph or known as the set of neighborhood elements of a vertex. In 2008, Abay-Asmerom *et al.* [7] found the total perfect codes in tensor products of graphs. Zhou [8] continued the study of this concept by studying the total perfect codes in Cayley graphs. The author considered the partition of a graph to find the perfect code. There are few other perfect codes that have been explored by other researchers such as the *e*-error-correcting code and perfect *t*-code. In 2021, Zaid *et al.* [9] studied the perfect codes of commuting zero divisor graph of some matrices of dimension two. The authors found that $\Gamma_{comm}(R_1)$ has no 1-perfect code. Mudaber *et al.* [10] extended the concept of perfect codes by finding the perfect codes over induced subgraphs of unit graphs of ring of integers modulo n.

In this research, the main focus of this research is to determine the perfect code of non-zero divisor graph of a commutative ring. The zero divisors of the ring are established and then, the non-zero divisor graph is constructed with the assistance of MAPLE Software. Finally, the perfect code of the graph is determined.

Preliminaries

In this section, some definitions and properties related to the main topic are presented.

Definition 1 [11] Zero Divisors

Zero divisors of a ring are the elements that have product zero when multiplied with each other. If $\boldsymbol{p}$ and $\boldsymbol{q}$ are two nonzero elements of a ring $\boldsymbol{R}$ such that $\boldsymbol{pq} = \mathbf{0}$, then $\boldsymbol{p}$ and $\boldsymbol{q}$ are divisors of $\mathbf{0}$.

Next, some definitions related to graph theory are presented. Graph theory is one of the oldest and most geometric branches of topology. It deals with the presentation of graphic drawing and is based on the data structures which actually come from mathematics.

Definition 2 [12] Graph

A finite graph, denoted as $\boldsymbol{G}$ is an object with two sets, which are the edge set, $\boldsymbol{E}(\boldsymbol{G})$ and the non-empty vertex set, $\boldsymbol{V}(\boldsymbol{G})$. The $\boldsymbol{E}(\boldsymbol{G})$ may be empty, but otherwise its elements are two-element subsets of the vertex set.

Definition 3 [13] Non-Zero Divisor Graph

The non-zero divisor graph of $\boldsymbol{R}$, denoted by $\boldsymbol{\Gamma}(\boldsymbol{R})$, is a simple graph with vertices of all non-zero elements of a ring, $\boldsymbol{R}$ such that two distinct elements $\boldsymbol{x}$ and $\boldsymbol{y}$ are adjacent if and only if $\boldsymbol{xy} \neq \mathbf{0}$.

In this research, some properties of the graph known as the total perfect code are further found from the construction of the non-zero divisor graph. The definition of total perfect code of a graph is given as follows.

Definition 4 [7] Total Perfect Code

A code $\boldsymbol{C}(\boldsymbol{G})$ is said to be a total perfect code in $\boldsymbol{G}$ if every vertex of $\boldsymbol{G}$ has exactly one neighbor in $\boldsymbol{C}(\boldsymbol{G})$, that is, $|\boldsymbol{N}(\boldsymbol{v}) \cap \boldsymbol{C}(\boldsymbol{G})| = \mathbf{1}$ for all $\boldsymbol{v} \in \boldsymbol{V}(\boldsymbol{G})$, where $\boldsymbol{N}(\boldsymbol{v})$ is the open neighbourhood of a vertex $\boldsymbol{v}$ in $\boldsymbol{G}$.

Results and discussions

In this section, non-zero divisor and its non-zero divisor graph of some finite commutative rings are discussed. The scope of this research is the ring of integers modulo 10, denoted as $\mathbb{Z}_{\mathbf{10}}$. The non-zero divisor graph is then constructed for this ring. The total perfect code of the graph is then determined.

Proposition 1. The ring of integers modulo 10, $\mathbb{Z}_{\mathbf{10}}$ has nine non-zero divisors.

Proof. To find the pairs of elements where the multiplication of these non-zero elements is not equal to zero, we first find the zero divisors using Definition 1. The zero divisors of $\mathbb{Z}_{\mathbf{10}}$ are determined. The set of zero divisors of $\mathbb{Z}_{\mathbf{10}}$ are $\{\mathbf{2}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{8}\}$ since $\mathbf{2} \cdot \mathbf{5} = \mathbf{5} \cdot \mathbf{2}$, $\mathbf{4} \cdot \mathbf{5} = \mathbf{5} \cdot \mathbf{4}$, $\mathbf{6} \cdot \mathbf{5} = \mathbf{5} \cdot \mathbf{6}$ and $\mathbf{8} \cdot \mathbf{5} = \mathbf{5} \cdot \mathbf{8}$. Since there are only nine non-zero elements in $\mathbb{Z}_{\mathbf{10}}$, the total pairs of elements are $\mathbf{9} \cdot \mathbf{9} = \mathbf{81}$. Therefore, the pairs of elements where the multiplication of these pairs of those non-zero elements is not equal to zero is 81 − 8 (pairs of elements from zero divisors) = 73.

Next, the non-zero divisor graph of the commutative ring is constructed based on the non-zero divisors found in Proposition 1.

Proposition 2. Let $\boldsymbol{\Gamma}(\mathbb{Z}_{\mathbf{10}})$ be the non-zero divisor graph of $\mathbb{Z}_{\mathbf{10}}$, then, $\boldsymbol{\Gamma}(\mathbb{Z}_{\mathbf{10}})$ is an undirected graph with nine vertices and 32 edges.

Proof. Based on Proposition 1, it can be seen that the total pair of elements $(\boldsymbol{x}, \boldsymbol{y})$ in which $\boldsymbol{x} \cdot \boldsymbol{y} \neq \mathbf{0}$ is equal to 73. To prevent the existence of loops in the non-zero divisors graph, the edges $\boldsymbol{x} \cdot \boldsymbol{x} \neq \mathbf{0}$ are excluded as in the definition. The non-zero divisor graph is constructed as in Figure 1.

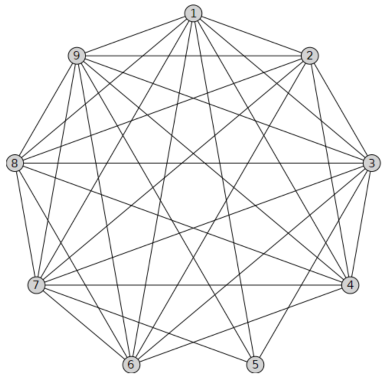

Figure 1 The non-zero divisor graph of $\mathbb{Z}_{\mathbf{10}}$.

Based on $\Gamma(\mathbb{Z}_{10})$, the set of elements adjacent of its corresponding vertices are determined and the total perfect code is obtained.

Proposition 3. Let $\boldsymbol{\Gamma}(\mathbb{Z}_{\mathbf{10}})$ be the non-zero divisor graph of $\mathbb{Z}_{\mathbf{10}}$. Then, $\boldsymbol{\Gamma}(\mathbb{Z}_{\mathbf{10}})$ has a total perfect code.
Proof. Based on the Definition 4, the set of all neighborhood elements of $\mathbb{Z}_{\mathbf{10}}$ are, when $\boldsymbol{e} = \mathbf{1}$, the set $\boldsymbol{S_1}(\boldsymbol{x})$, which is the set of all neighborhood elements with distance less than or equal to one for all $\boldsymbol{x}$ in $\boldsymbol{V}(\boldsymbol{\Gamma}(\mathbb{Z}_{\mathbf{10}}))$. The sets of all neighborhoods are listed as follows:

- $\boldsymbol{S_1}(\mathbf{1}) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$
- $\boldsymbol{S_1}(\mathbf{2}) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$
- $\boldsymbol{S_1}(\mathbf{3}) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$
- $\boldsymbol{S_1}(\mathbf{4}) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$
- $\boldsymbol{S_1}(\mathbf{5}) = \{\mathbf{1}, \mathbf{3}, \mathbf{5}, \mathbf{7}, \mathbf{9}\}$
- $\boldsymbol{S_1}(\mathbf{6}) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$
- $\boldsymbol{S_1}(\mathbf{7}) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$
- $\boldsymbol{S_1}(\mathbf{8}) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$
- $\boldsymbol{S_1}(\mathbf{9}) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}, \mathbf{7}, \mathbf{8}, \mathbf{9}\}$

Let $C(\mathbb{Z}_{10}) = \{1\}$ be a code. Based on Definition 4, since for all $x \in \mathbb{Z}_{10,}\ |S_1(x) \cap C(\mathbb{Z}_{10})| = 1$, therefore $\Gamma(\mathbb{Z}_{10})$ admits a total perfect code.

Conclusion

In this research, the non-zero divisor graph of the commutative ring of integers modulo 10 is found. The non-zero divisors are first determined using their definition. The non-zero divisor graph of the commutative ring is found to be a simple, undirected graph with nine vertices and 32 edges. Furthermore, the total perfect code of the graph is determined with the code $\{1\}$.

Acknowledgement

The first author would like to acknowledge Universiti Teknologi Malaysia (UTM) and Public Service Department (JPA) for the financial support through Program Pelajar Cemerlang (PPC). The authors would also like to thank UTM for the funding of this research through UTM Fundamental Research Grant (UTMFR) Vote Number 20H70 and Fundamental Research Grant Scheme (FRGS1/2020/STG06/UTM/01/2).

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