Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia

Vol. 10, 2022, page 10 - 20

Analytical Solution for Free Convection Flow in Brinkman Type Fluid through Two Vertical Channels by using Laplace Transform

Ain Norfithriyaani Amran, Ahmad Qushairi Mohamad*
Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia
*Corresponding author: ahmadqushairi@utm.my

Abstract

The main purpose of this study is to obtain analytical solutions for the problem of free convection flow in Brinkman type fluid past through two vertical channels. In this paper,the dimensional governing equations of momentum and energy equations are introduced.The appropriate dimensionless variables are used to transform the dimensional governingequations into dimensionless forms. The dimensionless parameters are obtained through this dimensionless process such as Brinkman type fluid, β_1, Prandtl number, Pr, Grashof number, *Gr* and time, *t*. The dimensionless equations with associated initial and boundaryconditions are solved using the Laplace transform method and the mathematical solutionsfor velocity and temperature profiles are obtained. The analytical results for obtained profiles are graphically plotted to illustrate the effects of corresponding dimensionless parameters. It is observed that velocity increases with increasing *Gr*, and *t*, but decreaseswith increasing β_1 and Pr. Furthermore, temperature profiles decrease with increasing Pr,while increasing with increasing *t*. Finally, the other published results were utilized to compare and validate the obtained results, and they provided identical results.

Keywords: Free convection; Brinkman type fluid; two vertical channels; analytical solutions; Laplace Transform method

1. Introduction

According to the researchers, non-Newtonian fluid flow is more important in real-world and industrial applications than Newtonian fluid flow. The Brinkman model is commonly used as the basis for non-Newtonian fluid flow research in a variety of fields, including chemical engineering, pharmaceuticals and cosmetics [1]. The Brinkman model was used to investigate the flow of viscous incompressible fluid through a porous channel [2]. The Brinkman Type Fluid is a dimensionless number related to heat conduction from a wall to a flowing viscous fluid, commonly used in polymer processing. Brinkman type fluid, which is one of the complex models proposed by H.C. Brinkman, is one of the most common non-Newtonian fluids currently that stated by [3]. He demonstrated that a Brinkman type fluid flows through a highly porous medium. The Brinkman model for incompressible flow may properly represent viscous fluid flow in a porous medium. This model features a particular term for viscosity and is useful for fluid flow through a high porous surface.

Free convection is defined as fluid motion induced solely by density changes in the fluid caused by temperature gradients and not by any other external source. Besides, free convection is a method of heat transmission in which fluid masses, buoyancy, and gravity forces are moved from one temperature region to another. According to [4], free convection flow in a boundary layer area is caused by gravity interacting with density changes within a fluid. In many industrial operations, free convection flow in Brinkman type fluid is important. Natural convection flows through a vertical plate are essential in solving several industrial and technical difficulties, such as process filtration and design, drying of porous materials in textile industries, and solar energy collectors [5].

Brinkman model was used by the following researchers during their studies which are [2], [6], and [7] talked more about the natural convection from a vertical plate in a porous medium using Brinkman's model in that article where the Brinkman equation's significance is illustrated by problems with high permeability near the boundary. [8] recently extended these solutions, as well as those corresponding to fluid motion due to a constantly or highly accelerating plate, to fluids of the Brinkman type. Then, [9] used Brinkman model to obtain exact solution by applying Laplace Transformation. Later, [10] utilized the Fourier transform approach to determine the exact solution for velocity and shear stress of unsteady and incompressible Brinkman type fluids enclosed inside a channel, taking into consideration the applied transverse magnetic effect. [11] explored the free convection flow of an incompressible and viscous fluid through a moving vertical plate with the impact of radiation when it is heated. This problem was solved analytically using Laplace transform method. Exact solutions of momentum and energy equations are achieved using the Laplace transform method, according to [12].

In order to obtain the mathematical and analytical solution of free convection flow in Brinkman Type Fluid through two vertical channels, the Laplace Transform will be used. By solving the governing differential equations, the Laplace Transform technique was used to obtain the expression for velocity and temperature fields [13]. The Laplace Transform is a type of integral transform proposed by French mathematician Pierre-Simon Laplace (1749-1827) and systematically improved by British physicist Oliver Heaviside (1850-1925) to solve multiple differential equations that represent physical process easier.

The current research is focused on free convection flow in a Brinkman type fluid through two vertical channels. For the boundary condition, the vertical channel will be fixed because it is not moving and has a constant temperature. The method of Laplace transform is applied in this study to solve that problem analytically with fixed and constant temperature.

2. Methodology

2.1. Problem Formulation

Consider the free convection flow of Brinkman Type Fluid past through two vertical channel plates which is separated by distance *d* with constant temperature. The *x*-axis is in upward direction along the two vertical channel plates and *y*-axis is in normal direction to the plates. Initially at time $t^* \leq 0$, the fluids and plates are both at rest and assumed at the same temperature ${T_d}^*$. Then, at the time $t^* > 0$, plate temperature is raised to ${T_w}^*$ at $y^* = 0$ while plate temperature at $y^* = d$ is remain at constant temperature ${T_d}^*$ and velocity for both plates are not moving $u^*(y,t) = 0$. The corresponding governing equations are obtained in a set of partial differential equations as below

$$\frac{\partial u^*}{\partial t} = v\frac{\partial^2 u^*}{\partial y^2} - \beta^* u' + \rho g \beta (T^* - {T_d}^*) \tag{1}$$

$$\rho C_p \frac{\partial T^*}{\partial t^*} = k\frac{\partial^2 T^*}{\partial y^{*2}} \tag{2}$$

with the associated initial and boundary conditions

$$\begin{aligned} u^*(y^*,0) &= 0\,; 0 \leq y^* \leq d, \\ u^*(0,t^*) &= 0\,;\ t^* > 0, \\ u^*(d,t^*) &= 0; t^* > 0, \end{aligned} \tag{3}$$

and

$$\begin{aligned} T^*(y^*,0) &= {T_d}^*; 0 \leq y^* \leq d, \\ T^*(0,t^*) &= {T_w}^*; t^* > 0, \\ T^*(d,t^*) &= {T_d}^*; t^* > 0. \end{aligned} \tag{4}$$

In order to transform the governing equations (1) and (2) and the corresponding conditions of equation (3) and (4) into dimensionless form, the dimensionless variable are defined as

$$u = \frac{u^* d}{v}, \qquad y = \frac{y^*}{d}, \qquad t = \frac{t^* v}{d^2}, \qquad T = \frac{T^* - T_d{}^*}{T_w{}^* - T_d{}^*} \tag{5}$$

where u^* is the velocity component along *x*-axis, v is the kinematic viscosity of fluid, β^* is the Brinkman parameter, ρ is the density of fluid, *g* is the gravitational acceleration, *β* is the heat transfer coefficient, *T* is temperature of fluid, c_p is the specific heat capacity of fluid at constant temperature, and k is thermal conductivity. Dimensionless variables are unit less values that are used to eliminate units which is involved in governing equations and its corresponding conditions in order to simplify the equations by reducing the number of variables. Dimensionless momentum and energy equations are obtained by dropping out the *notation, substitute equation (1) and (2) with dimensionless variables equations (5) as

$$\frac{\partial^2 u}{\partial y^2} - \left(\beta_1 u + \frac{\partial u}{\partial t}\right) = -GrT, \tag{6}$$

$$\frac{\partial^2 T}{\partial y^2} - \mathrm{Pr}\frac{\partial T}{\partial t} = 0, \tag{7}$$

with the associated dimensionless initial and boundary conditions

$$\begin{aligned} u(y,0) &= 0\,;0 \le y \le 1, \\ u(0,t) &= 0\,;\ t > 0, \\ u(1,t) &= 0; t > 0, \end{aligned} \tag{8}$$

and

$$\begin{aligned} T(y,0) &= 0; 0 \le y \le 1, \\ T(0,t) &= 1; t > 0, \\ T(1,t) &= 0; t > 0. \end{aligned} \tag{9}$$

Reduce number of variables by grouping them into dimensionless parameters which can be obtained as

$$Gr = \frac{d^3 \rho g \beta}{v^2}(T_w{}^* - T_d{}^*), \qquad \beta_1 = \frac{\beta \partial^2}{v}, \qquad \mathrm{Pr} = \frac{\mu C_p}{k}, \tag{10}$$

where Gr is the Grashof number, β_1 is Brinkman Type Fluid number, and Pr is the Prandtl number.

3.2. Problem Solution

Temperature and Velocity Profiles

In order to obtain the analytical solution of equation (6) and (7), apply Laplace transform into equations (6) and (7) subjected to the initial equations (8) and (9), yields

$$\frac{d^2}{dy^2}\bar{u}(y,s) - (\beta_1 + s)\bar{u}(y,s) = -Gr\bar{T}(y,s)\,, \tag{11}$$

$$\frac{d^2}{dy^2}\bar{T}(y,s) - s\mathrm{Pr}\bar{T}(y,s) = 0\,, \tag{12}$$

with the corresponding Laplace transform for boundary conditions

$$\begin{aligned} \bar{u}(0,s) &= 0\,, \\ \bar{u}(1,s) &= 0\,. \end{aligned} \tag{13}$$

and

$$\begin{aligned} \bar{T}(0,s) &= \frac{1}{s}\,, \\ \bar{T}(1,s) &= 0\,. \end{aligned} \tag{14}$$

Therefore, the inverse Laplace transform of equations (11) and (12) by imposed of equations (13) can be expressed as

$$\begin{aligned} u(y,t) = {} & b_5\sum_{n=0}^{\infty}[u_1(y,t)] - b_5\sum_{n=0}^{\infty}[u_2(y,t)] - b_5\sum_{n=0}^{\infty}[u_3(y,t)] + b_5\sum_{n=0}^{\infty}[u_4(y,t)] \\ & -b_5\sum_{n=0}^{\infty}[u_5(y,t)] + b_5\sum_{n=0}^{\infty}[u_6(y,t)] + b_5\sum_{n=0}^{\infty}[u_7(y,t)] - b_5\sum_{n=0}^{\infty}[u_8(y,t)]\,, \end{aligned} \tag{15}$$

$$T(y,t) = \sum_{n=0}^{\infty}\left(erfc\left(\frac{\sqrt{\mathrm{Pr}}(2n+y)}{2\sqrt{t}}\right) - erfc\left(\frac{\sqrt{\mathrm{Pr}}(2n+2-y)}{2\sqrt{t}}\right)\right). \tag{16}$$

where

$$u_1(y,t) = \frac{1}{2}\exp\left(b_6\sqrt{\beta_1}\right)erfc\left(\frac{b_6}{2\sqrt{t}}+\sqrt{\beta_1 t}\right)+\frac{1}{2}\exp\left(-b_6\sqrt{\beta_1}\right)erfc\left(\frac{b_6}{2\sqrt{t}}-\sqrt{\beta_1 t}\right),$$

$$u_2(y,t) = \frac{1}{2}\exp\left(b_7\sqrt{\beta_1}\right)erfc\left(\frac{b_7}{2\sqrt{t}}+\sqrt{\beta_1 t}\right)+\frac{1}{2}\exp\left(-b_7\sqrt{\beta_1}\right)erfc\left(\frac{b_7}{2\sqrt{t}}-\sqrt{\beta_1 t}\right),$$

$$u_3(y,t) = erfc\left(\frac{b_8}{2\sqrt{t}}\right),$$

$$u_4(y,t) = erfc\left(\frac{b_9}{2\sqrt{t}}\right),$$

$$u_5(y,t) = \frac{1}{2}\exp\left(b_4 t + b_6\sqrt{\beta_1+b_4}\right)erfc\left(\frac{b_6}{2\sqrt{t}}+\sqrt{(\beta_1+b_4)t}\right)$$
$$+\frac{1}{2}\exp\left(b_4 t - b_6\sqrt{\beta_1+b_4}\right)erfc\left(\frac{b_6}{2\sqrt{t}}-\sqrt{(\beta_1+b_4)t}\right),$$

$$u_6(y,t) = \frac{1}{2}\exp\left(b_4 t + b_7\sqrt{\beta_1+b_4}\right)erfc\left(\frac{b_7}{2\sqrt{t}}+\sqrt{(\beta_1+b_4)t}\right)$$
$$+\frac{1}{2}\exp\left(b_4 t - b_7\sqrt{\beta_1+b_4}\right)erfc\left(\frac{b_7}{2\sqrt{t}}-\sqrt{(\beta_1+b_4)t}\right),$$

$$u_7(y,t) = \frac{1}{2}\exp\left(b_4 t + b_8\sqrt{b_4}\right)erfc\left(\frac{b_8}{2\sqrt{t}}+\sqrt{b_4 t}\right)$$
$$+\frac{1}{2}\exp\left(b_4 t - b_8\sqrt{b_4}\right)erfc\left(\frac{b_8}{2\sqrt{t}}-\sqrt{b_4 t}\right),$$

$$u_8(y,t) = \frac{1}{2}\exp\left(b_4 t + b_9\sqrt{b_4}\right)erfc\left(\frac{b_9}{2\sqrt{t}}+\sqrt{b_4 t}\right)$$
$$+\frac{1}{2}\exp\left(b_4 t - b_9\sqrt{b_4}\right)erfc\left(\frac{b_9}{2\sqrt{t}}-\sqrt{b_4 t}\right).$$

where

$$b_4 = \frac{\beta_1}{\Pr - 1}, \quad b_5 = \frac{Gr}{\beta_1}, \quad b_6 = (2n+2-y), \quad b_7 = (2n+y),$$

$$b_8 = (2n+2-y)\sqrt{\Pr}, \qquad b_9 = (2n+y)\sqrt{\Pr}.$$

3. Results and discussion

In order to verify the accuracy of the present results in this problem for equation (15), the limiting case of the present result is compared to published results by [14]. This is known as the verification test, and it is carried out by comparing the solution result with equation (11) in [14], which states that no Brinkman type fluid is included in the problem by setting the Casson value to zero. Figure 1 depicts the comparison. It is discovered that the findings are identical. Therefore, the accuracy of the obtained results is confirmed.

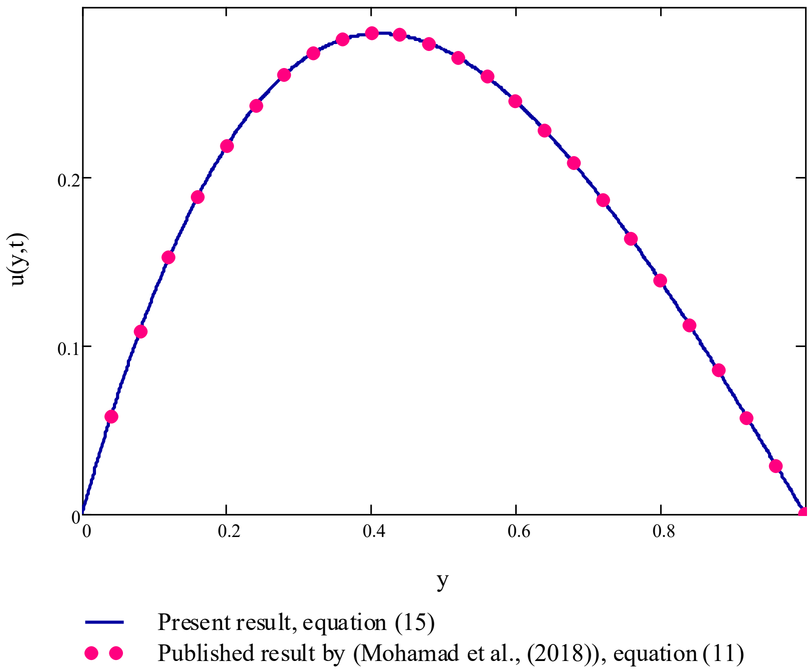

Figure 1 Comparison of velocity profile *u(y,t)* from equation (15) with equation (11) by Mohamad (2018)

Figure 2 and Figure 6 depicts the behaviour of velocity and temperature towards time changes. When the value of time, *t* is increased, the velocity and temperature increases. It is due to the external energy given to the fluid flow, which results in enhanced fluid particle movement as time is increasing. The effect of Grashof number, Gr, on velocity profiles is shown in Figure 3. Grashof number, Gr, is a dimensionless number used in heat transfer research including free or natural convection, according to [15]. Then, a parameter that defines the ratio of buoyancy forces and viscous forces is required, and the Grashof number is a dimensionless number that approximates the ratio of buoyancy force to viscous force flowing on a liquid [16]. As a result, during the free convection process, the buoyancy force is dominating, causing Gr to increase, consequently increasing velocity. Therefore, the result obtained when the Grashof number, Gr increases, and the velocity increased because the density of fluid decreases and small viscous effects in momentum equation which leads to increment in fluid velocity. Next, Figure 4 and Figure 7 illustrate the velocity and temperature profiles with different values of Prandtl number, Pr. Prandtl number, Pr is defined as the ratio of momentum diffusivity (kinematic viscosity) to thermal diffusivity. The increases of Prandtl number, Pr in the fluid flow will reduce thermal conductivity and increase fluid viscosity. The fluid becomes thick and increase in viscous force which results to decrease in fluid velocity. Thus, the velocity decreases as Pr are increased. The thermal boundary layer thickness decreases since decreasing fluid thermal conductivity with increasing Prandtl number which leads to decrease in temperature profiles. According to [17], the problem was formulated as a linear boundary value problem with an exact analytical solution in non-dimensional form. It turns out that the solutions for the non-dimensional velocity and temperature variables are affected by the fluid's Prandtl number, Pr. It also observed that the fluid velocity formula is not uniformly valid for all values of Pr. There are two different solutions, one valid for fluids with Prandtl numbers, Pr other than unity and the other for fluids with Prandtl number, Pr equal to unity that have been found separately. The graph in Figure 5 depicts the effect of the Brinkman type fluid parameter, β_1 on the velocity profiles. According to the observations, the velocity decreases in the boundary region as the value of the Brinkman type fluid parameter increases. The results obtained when the fluid velocity decreases as Brinkman parameter increases due to high viscosity of fluid. The velocity decrease with increasing values of Brinkman type fluid parameter, β_1.

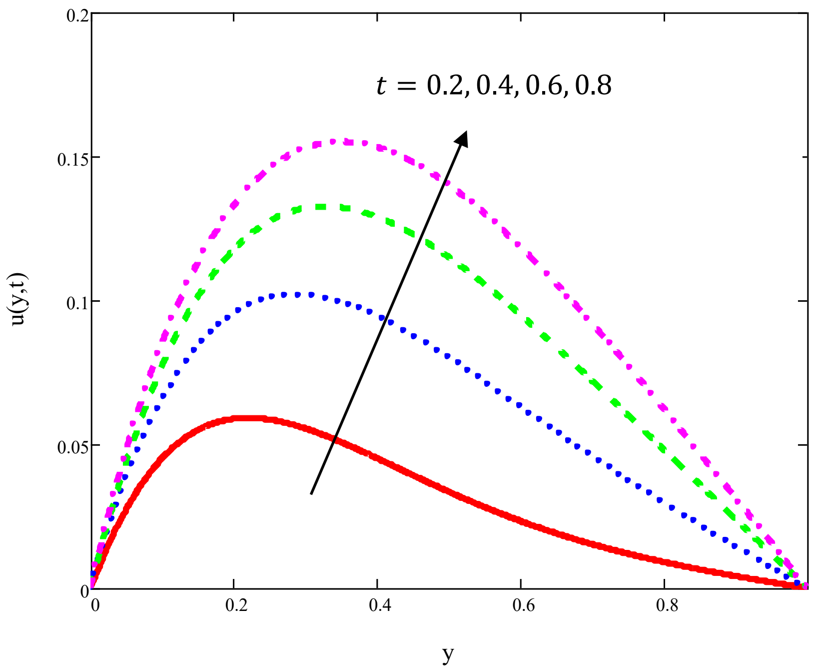

Figure 2 Velocity profiles for different values of time, t with $\beta_1 = 5.0,\ \mathrm{Pr} = 6.0,$ and $Gr = 5.0$.

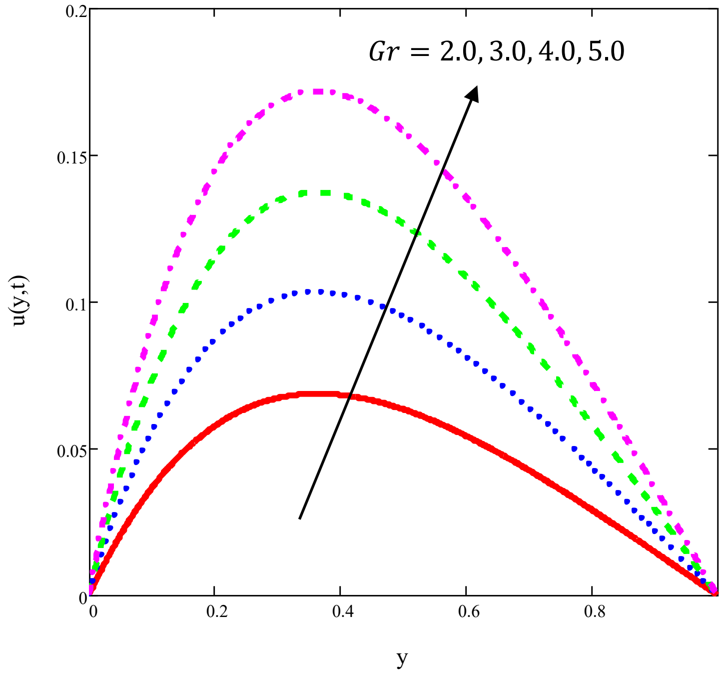

Figure 3 Velocity profiles for different values of Grashof number, *Gr* with $\beta_1 = 5.0,\ \mathrm{Pr} = 6.0,\ \text{and}\ t = 1.0.$

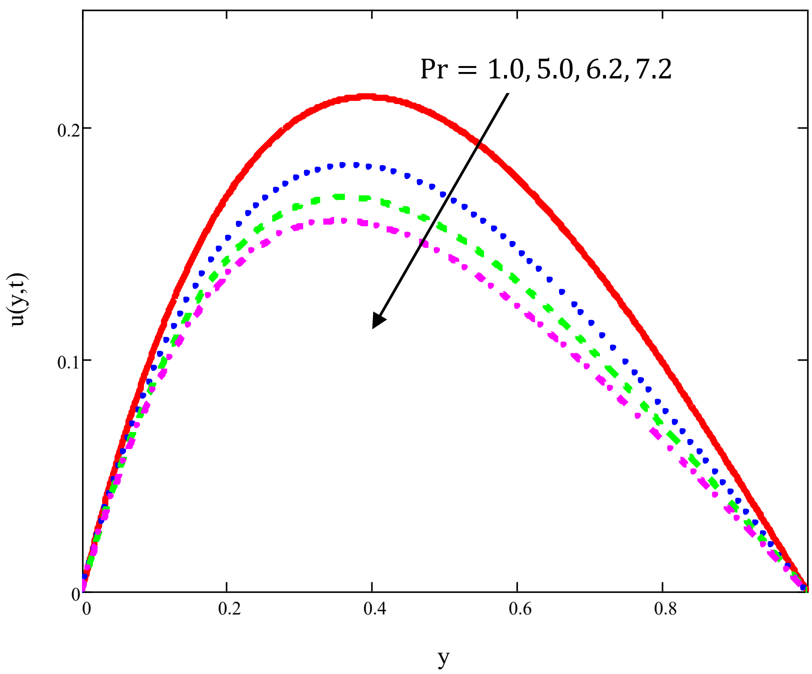

Figure 4 Velocity profiles for different values of Prandtl number, Pr with $\beta_1 = 5.0,\ Gr = 5.0,\ \text{and}\ t = 1.0.$

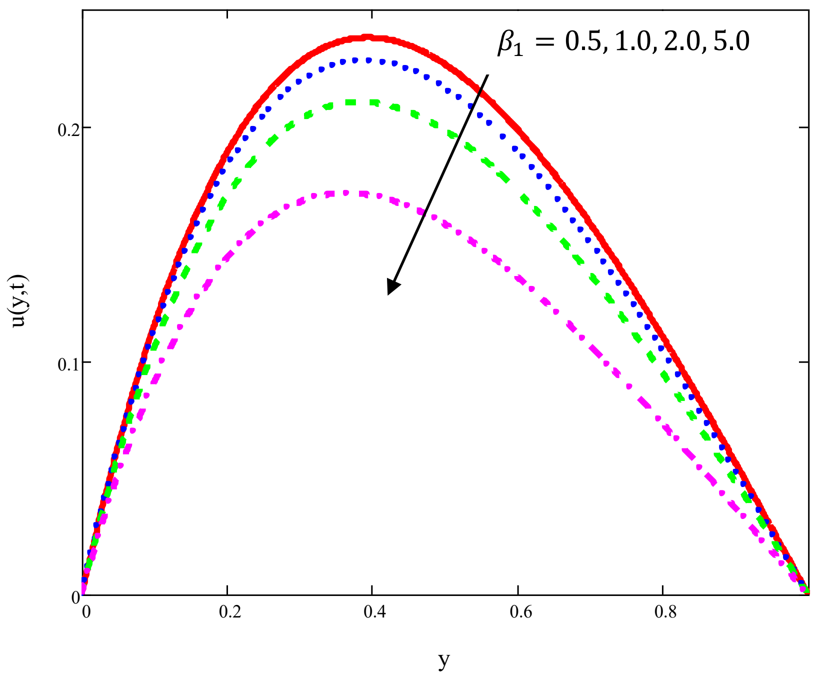

Figure 5 Velocity profiles for different values of Brinkman type fluid parameter, β_1 with $\mathrm{Pr} = 6.0,\ Gr = 5.0,$ and $t = 1.0$.

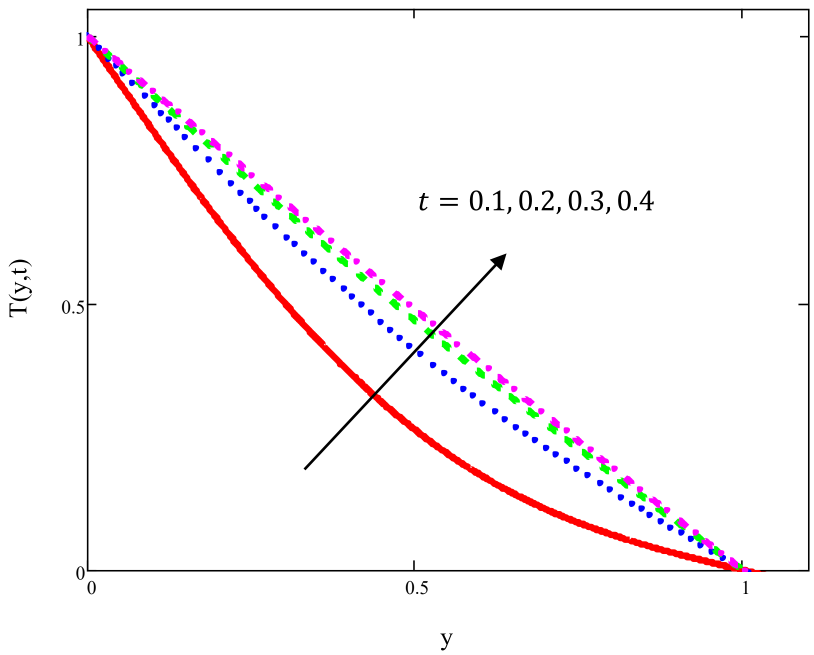

Figure 6 Temperature profiles for different values of time, *t* with $\mathrm{Pr} = 1.0$.

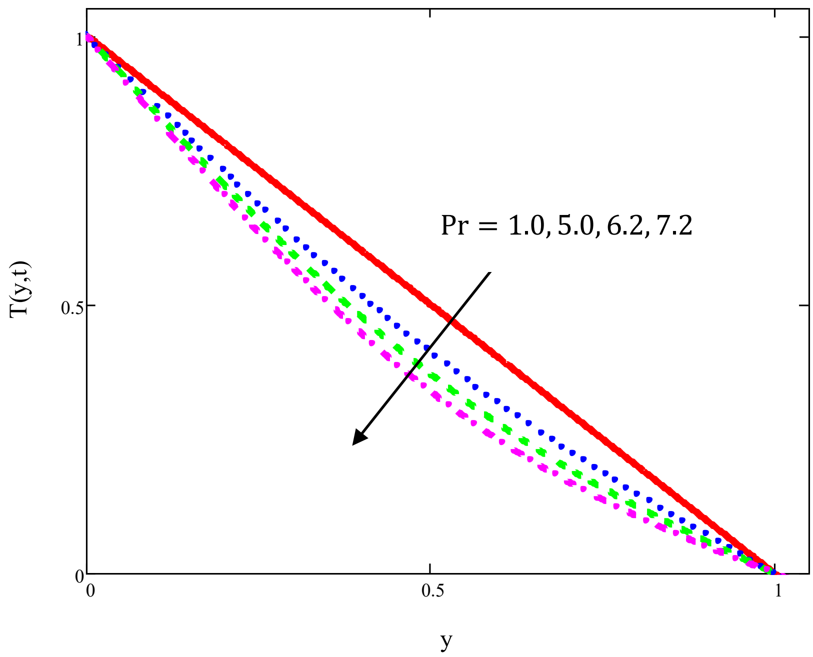

Figure 7 Temperature profiles for different values of Prandtl number, Pr with $t = 1.0$.

Conclusion

The free convection flow through two vertical channels is studied using the generalised Brinkman fluid model. The Laplace transform technique is used to obtain analytical solutions, and the effects of different parameters on velocity and temperature profiles are discussed. The obtained results are illustrated graphically for different values of parameters. This paper concludes the following main points:

- Velocity increased when the Grashof number, *Gr* increases.
- Velocity increased when the time, *t* increases.
- Velocity decreased with increasing values of Prandtl number, Pr.
- Velocity decreases with increasing values of Brinkman type fluid parameter, β_1.
- Temperature decreased when the Prandtl number, Pr increases.
- Temperature increased with increasing values of time, *t*.

Acknowledgement

The researcher would like to thank all people who have supported the research, Dr. Ahmad Qushairi Mohamad as supervisor of Final Year Project and Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia.

References

[1] Lissant, KJ 1977, 'Non-Newtonian Pharmaceutical Compositions', U.S. Patent, no. 4, 040, 857.

[2] Varma, S. V., & Babu, M. S. (1985). A Brinkman model for MHD viscous incompressible flow through a porous channel. Indian Journal of Pure and Applied Mathematics,16(7), 796-806.

[3] Islam, S., Asif, M., & Haq, S. (2018). Exact Solutions of Brinkman Type Fluid Between Side Walls Over an Infinite Plate.

[4] Zulkiflee, F., Mohamad, A. Q., Shafie, S., & Khan, A. (2019). Unsteady Free Convection Flow between Two Vertical Parallel Plates with Newtonian Heating. *Matematika*, *35*.

[5] Zakaria, M. N., Hussanan, A., Khan, I., & Shafie, S. (2013). The effects of radiation on free convection flow with ramped wall temperature in Brinkman type fluid. Jurnal Teknologi, 62(3).

[6] Gorla, R. S. R., Mansour, M. A., & Sahar, M. G. (1999). Natural convection from a vertical plate in a porous medium using Brinkman's model. *Transport in porous media*, *36*(3), 357-371.

[7] Rajagopal, K. R. (2007). On a hierarchy of approximate models for flows of incompressible fluids through porous solids. *Mathematical Models and Methods in Applied Sciences*, *17*(02), 215-252.

[8] Fetecau, C., Fetecau, C., & Imran, M. A. (2011). On Stokes problem for fluids of Brinkman type. *Mathematical Reports*, *13*(1), 15-26.

[9] Ali, F., Khan, I., & Shafie, S. (2012). A note on new exact solutions for some unsteady flows of Brinkman-type fluids over a plane wall. Zeitschrift für Naturforschung A, 67(6-7), 377-380.Ali,

[10] Khan, Z. A., Haq, S. U., Khan, T. S., Khan, I., & Tlili, I. (2018). Unsteady MHD flow of a Brinkman type fluid between two side walls perpendicular to an infinite plate. Results in Physics, 9, 1602-1608.

[11] Narahari, M., & Ishak, A. (2011). Radiation effects on free convection flow near a moving vertical plate with Newtonian heating. Journal of Applied Sciences, 11(7), 1096-1104.

[12] Ali, F., Khan, I., Ul Haq, S., & Shafie, S. (2013). Influence of thermal radiation on unsteady free convection MHD flow of Brinkman type fluid in a porous medium with Newtonian heating. Mathematical Problems in Engineering, 2013.

[13] ATHAL, I., & SIVASANKARI, M. (2021). Approximate Solutions Of Transient Free-Convective Flow In A Vertical Channel Due To Symmetric Heating Using Laplace Transform Technique. International Journal of Modern Agriculture, 10(1), 163-167.

[14] Qushairi, M. A., Yeou, J. L., Shafie, S., Khan, I., & Ismail, Z. (2018). Exact solution for unsteady free convection flow of Casson fluid in vertical channel. In MATEC Web of Conferences (Vol. 189, p. 01007). EDP Sciences.

[15] Karwe, M. V., & Deo, I. (2003). Grashof number. Encyclopedia of agricultural, food, and biological engineering, 1(1), 454-456.

[16] Shah, N. A., Animasaun, I. L., Ibraheem, R. O., Babatunde, H. A., Sandeep, N., & Pop, I. (2018). Scrutinization of the effects of Grashof number on the flow of different fluids driven by convection over various surfaces. Journal of Molecular liquids, 249, 980-990.

[17] Chandran, P., Sacheti, N. C., & Singh, A. K. (2005). Natural convection near a vertical plate with ramped wall temperature. *Heat and Mass Transfer*, *41*(5), 459-464.