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Moment by Integration for a few 2D and 3D Objects with Symmetries

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Abstract
Recently, the concept of symmetry is widely used in numerous areas of science and technology with the outcomes of its application. A symmetry known as rotational invariance which is a universal property shared by many physical systems. Numerous researchers have quantitatively investigated symmetry with variety types of moments. However, there is no study has been undertaken on the moment of mass by integration using the idea of symmetry in 2D and 3D objects. The purpose of this study is to investigate symmetry of 2D and 3D objects, to determine moment of a few symmetric objects and to relate the symmetry with moments approach in integration. In this report, integration is used to present a new method for analysing and detecting the reflectional symmetry about axes and planes that works on both completely and approximately symmetrical objects. The moment obtained describes the tendency of an object to rotates about the axis or about the plane. The result was proposed to demonstrate how the symmetry object can affect the result of moment. The technique is also expected to benefit in the continuing of the symmetry's study object in future analysation
Keywords: Symmetry Objects; Moments; Integration

1. Introduction

Recently, the concept of symmetry is widely used in numerous areas of science and technology with the outcomes of its application. Besides, many real-world objects exhibit some kind of symmetry. Symmetry is a fundamental principle of perceptual organisation that influences perceptual grouping, object recognition, and figure-ground segregation [1]. Furthermore, several researchers are particularly interested in the study of the symmetry objects due to its widespread prevalence in the object recognition and interpretation. The manifestation of symmetry is determined by the way the pieces are organized, or by the sort of transformation used, such as rotation, reflection, and translation [2].

A symmetry known as rotational invariance which is a universal property shared by many physical systems [3]. Numerous researchers have quantitatively investigated symmetry with moments such as moment invariants, complex moment, and orthogonal moments. However, there is no study has been undertaken on the moment of mass by integration using the idea of symmetry in 2D and 3D objects. In this report project, the reflectional symmetry will mainly address of a 2D, and 3D object. A 3D of object is reflectionally symmetrical with respect to plane if the object stays the same when it reflected over the plane.

As a result, this study focuses on the symmetry of 2D and 3D object, the tendency of symmetry objects to rotate, as well as the condition that match the symmetry requirement. The moment technique was utilised as the core for analysing symmetry and how symmetry in 2D and 3D influences the moment by integration.

The coverage of this study of 2D and 3D objects includes the formulation of moments in integration. Using a mathematical equation, this research proves and examines the difference of the symmetrical properties between various objects and moments, numerically and descriptively. Dimensionless mathematical equations are considered to prevent the mathematical difficulty of dimensions that provide complication in the proving. The shapes and regions of the symmetry objects are simulated using computer programming to aid in the study of symmetries' effect on moments. Discussion of the relationship between symmetry properties is conducted to understand the effect towards moment with integration.

This research aims to (1) investigate symmetry of 2D and 3D objects, (2) determine moment of a few symmetric objects and (3) relate the symmetry with moments approach in integration. The moment value is obtained.

2. Literature Review

2.1. Historical Development of Symmetry

The concept of symmetry originated in Italy at the beginning of the Renaissance. From the outset it was associated with the belief that Nature's forms are symmetric and that nothing can be beautiful unless it has a symmetric shape. Symmetry is called as a norm and was soon adopted through western Europe. Symmetry also describes as a fundamental concept pervading both science and culture [4]. In popular terms, symmetry is often viewed as a kind of balance. But in Mathematics, symmetry has been given more precise meaning. A symmetry of some mathematical objects is a transformation that preserves the object's structure [5]. Therefore, symmetrical structure looks the same before and after when something done on it.

Study of symmetry is one of the most important concepts unifying many areas of modern mathematics. Many researchers have an intuitive idea of symmetry, and often think about certain shapes or patterns as being more or less symmetric than others. Group theory is one of the mathematical studies of symmetry and explores general ways of studying it in many distinct settings. In addition, the perspective of symmetry is wide due to its widespread prevalence in their application. Therefore, there are also varies study conducted about this topic in other fields such as in calculus, linear algebra, abstract algebra, and geometry.

2.2. Axial and Central Symmetry

Balance is implied by the idea of symmetry [6]. The axis of symmetry is an imaginary straight line that splits a geometry into two identical parts, thereby creating one part as the mirror image of the other part [7]. This line can be horizontal, vertical, or slanted. Axial symmetries are inverse isometries since the distance between its point and its homologous is preserved, but the orientation is inverse [8]. The axial symmetry does not present just between an object and its reflection, because many figures that can divided into two halves by a line are symmetrical with respect to the line. It can be claimed that distinct forms have different lines of symmetry.

The central symmetry is referred to as a special point since it has a significant distinguishing characteristic. It said to be remained unchanged regardless of the type of symmetry operation is performed. For instance, a circle region, has a centre with a unique point that results in the pattern being unchanged if the circular pattern is rotated about its centre, independent of its position.

2.3. Application of Symmetry

Symmetry is one of the fundamental properties of shapes and objects that has received a great deal of attention in the field of computer vision. Symmetrical shape descriptions and the identification of symmetrical properties of objects are highly beneficial in guiding shape matching, model-based object matching, and object recognition. In robotics, symmetrical information is also useful for recognition, inspection, grasping, and reasoning [9]. Symmetries are suitable choices for characterising form. It is a powerful concept that can help with object detection and recognition in a variety of situations [10]. Furthermore, the fact that an image has symmetry allows it to be economically characterised. For instance, if one half of an object is the mirror image of the other half, only one half has to be described.

2.4. Application of Symmetry

Moments are scalar quantities that are used to characterise and capture the significant aspects of a function [11]. For hundreds of years, they have been frequently employed in statistics to describe the shape of a probability density function and in traditional rigid-body mechanics to estimate the mass distribution of a body. Besides, moments are mathematically defined as projections of a function onto a polynomial base, and similarly, the Fourier transform is a projection onto a basis of harmonic functions.

When it comes to different applications, the concept of moment varies. In physics, the moment is an expression involving the product of a distance and physical quantity and accounts for how the physical quantity is positioned or arranged. For example, the moment of force is the product of a force on an object and the distance from the object's reference point where it measures the potential to cause a body to spin around a given point or axis. It is also related to the focus of the study on the moment of mass, also known as lamina mass, which analyse the tendency of lamina to rotate around the x −axis and y −axis [12].

3. Methodology

3.1. Research Data

Data of 2D and 3D symmetrical objects were retrieved from the Wolfram MathWorld through the portal that known as *solid geometry* [13]. The information taken from the web includes the general equation of different 2D and 3D objects and their general equations. The estimated value from the general equations for this research was based on different type of symmetries. The representation of mathematical equation of the object and axis symmetry were recorded.

The dataset of symmetry object was gathered to evaluate the object analysis by the moment technique. Table 1 displays the equation of 2D object region based on standard features of different symmetry meanwhile Table 2 presents the equation of 3D objects.

Table 1: Equation of 2D objects

Object	Equation of Region
A	$x + y = 2, y - x = 2, y = 0$
B	$x = 0, x = 3, y = -1, y = 1$
C	$x^2 + y^2 = 1$

Table 2: Equation of 3D objects

Object	Equation of Region
D	$-2 \leq x \leq 2, -1 \leq y \leq 5,$ $-1 \leq z \leq 1$
E	$x^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = 1$
F	$x^2 + y^2 + z^2 = 1$
G	$z = \sqrt{1 - x^2 - y^2}$
H	$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 + z^2 = 1$

3.2. The Simulation of Symmetry Object

The simulations involve more flexible software to solve a wide range of problems [14]. Therefore, the representation of mathematical equations is required for the development of the object region. Graphical interphase for constructing and editing mathematical expression is also taken into account while developing symmetric objects.

3.2. The Symmetrical Condition of 2D Object

To test the symmetries, it is required to test either one of the conditions. The condition of symmetry is divided into two categories which are in Cartesian and Polar coordinates. There are three types of symmetry that need to consider which are symmetry about x −axis, y −axis, and the origin.

3.2.1 In Cartesian Coordinate

Table 3: Symmetrical Condition in Cartesian Coordinate

Object	Equation of Region
About the x-axis	If (x, y) satisfy the equation, so does $(x, -y)$
About the y-axis	If (x, y) satisfy the equation, so does $(-x, y)$
About the origin	If (x, y) satisfy the equation, so does $(-x, -y)$

3.2.1 In Polar Coordinate

Table 4: Symmetrical Condition in Polar Coordinate

Object	Equation of Region
About the x-axis	If (r, θ) satisfy the equation, so does $(r, -\theta)$ or $(-r, \pi - \theta)$
About the y-axis	If (r, θ) satisfy the equation, so does $(r, \pi - \theta)$ or $(-r, -\theta)$
About the origin	If (r, θ) satisfy the equation, so does $(r, \pi + \theta)$ or $(-r, \theta)$

3.3. The Development of Moment Equation 2D Object

3.3.1 Definition 1
Let $\rho(x, y)$ be a continuous density function on a lamina corresponding to a plane region D, then the mass m of the lamina is given by

$$m = \iint_D \rho(x, y)\ dA \tag{1}$$

3.3.2 Definition 2
The measure of the tendencies of a lamina to rotate about the x –axis and y –axis is called moments of mass with respect to the x and y –axis.

If a region D break up into grid of equivalent rectangles and select the sample point (x_i, y_i) from the upper right corner of the inside the ijth rectangle, it can use to approximate the moment of mass of the rectangle with respect to x –axis and y –axis using the formula:

$$(mass)(directed\ distance\ from\ x - axis) \approx (\rho(x_i, y_i)\Delta A)y_j. \tag{2}$$

Likewise, the moment of mass estimation with respect to the *y*-axis:

$$(mass)(directed\ distance\ from\ y - axis) \approx (\rho(x_i, y_i)\Delta A)x_i. \tag{3}$$

3.3.3 Definition 3
Using Riemann sums involving equation (3.1) and (3.2), the definition for moments of a lamina R with respect to the x –axis and y –axis is given as

$$M_x = \iint_R y\rho(x, y)\ dA, \tag{4}$$

and

$$M_y = \iint_R x\rho(x,y)\, dA \tag{5}$$

3.3.4 Definition 4
The density $\rho(x,y)$ is considered to be constant number when the lamina or the object is homogenous. That is, the object has uniform density.

3.4. The Development of Moment Equation 3D Object

3.4.1 Definition 5
Let $\rho(x,y,z)$ be the density of a solid R at the point (x,y,z). Then the total mass of the solid is the triple integral of

$$m = \iiint_D \rho(x,y,z)\ dV. \tag{6}$$

3.4.2 Definition 6
The first moment of a 3D solid region D about coordinate plane is defined as the triple integral over D of the distance from a point (x,y,z) in D to the plane multiplied by the density of the solid at the point. First moments of the coordinate planes are as follows,

$$M_{yz} = \iiint_a^b x\rho(x,y,z)\ dV, \tag{7}$$

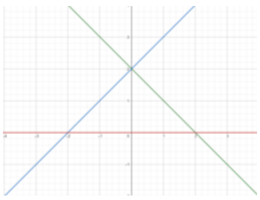
$$M_{xz} = \iiint_a^b y\rho(x,y,z)\ dV, \tag{8}$$

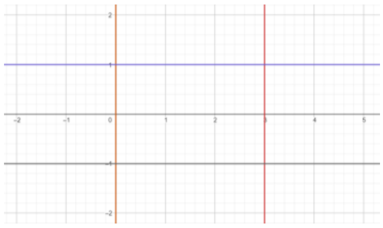
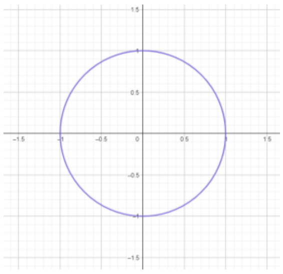
$$M_{xy} = \iiint_a^b z\rho(x,y,z)\ dV. \tag{9}$$

4. Results and discussion

4.1. Simulation of 2D Objects with Different Symmetrical Axis
Three regions of 2D objects with different axis symmetry were simulated for this study in Table 5 that referred from the mathematical equation of object to calculate and analyse the relation with moment approach.

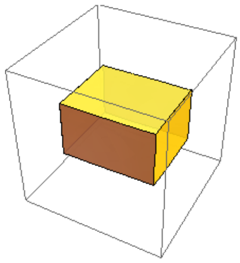
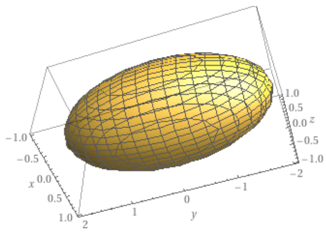
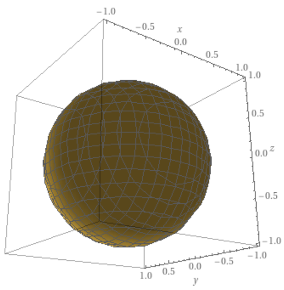
Table 5: Simulation of 2D objects

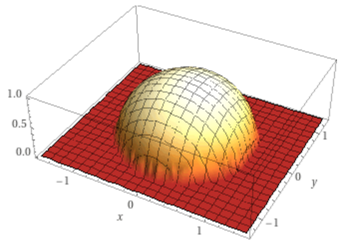
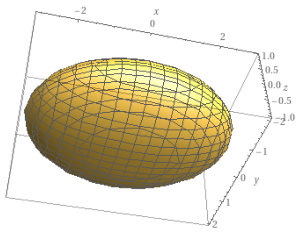
Object	Name	Region
A	Triangular	

B	Rectangular	
C	Circle	

4.2. Simulation of 3D Objects with Different Symmetrical Axis

Table 6: Simulation of 3D objects

Object	Name	Region
D	Cuboid	
E	Spheroid	
F	Sphere	

G	Hemisphere	
H	Ellipsoid	

4.3. Application of Moment by Integration Technique

The proven of moment based on the symmetry conditions in polar coordinate for 2D region will be discussed in this part. As a result, the proof focuses on the relation of symmetry object about x −axis and y −axis with moment in integration. Next, calculation of moment of symmetrical 2D and 3D objects will be analysed from different type of coordinates which used in solving the integrals.

4.3.1. Symmetry about different axes in 2D

The value of moment with integration can be calculated by using Equation (4) and (5). As a preliminary result, this research will exclusively focus on the relation on the moment of symmetry objects only for three objects which are Object A, Object B and Object C respectively.

4.3.1.1. Moments in Cartesian Coordinates

The mathematical representation for Object A in the formulation of moment by integration in cartesian coordinates are as follows:

$$M_x = \iint_R y\rho(x,y)\ dA = \int_{y=0}^{y=2} \int_{x=y-2}^{x=2-y} y\ dxdy = \frac{8}{3},$$

$$M_y = \iint_R x\rho(x,y)\ dA = \int_{y=0}^{y=2} \int_{x=y-2}^{x=2-y} x\ dxdy = 0.$$

Meanwhile, the mathematical representation for Object B in the formulation of moment by integration in cartesian coordinates is as follows:

$$M_x = \iint_R y\rho(x,y)\ dA = \int_{y=1}^{y=-1} \int_{x=0}^{x=3} y\ dxdy = 0\,,$$

$$M_y = \iint_R x\rho(x,y)\ dA = \int_{y=1}^{y=-1} \int_{x=0}^{x=3} x\ dxdy = 9.$$

4.3.1.2. Moments in Polar Coordinates

The mathematical representation for Object C in the formulation of moment by integration in polar coordinates is as follows:

$$M_x = \iint_R y\rho(x,y)\ dA = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} r^2 sin\theta\ drd\theta = 0,$$
$$M_y = \iint_R x\rho(x,y)\ dA = \int_{\theta=0}^{\theta=2\pi}\int_{r=0}^{r=1} r^2 cos\theta\ drd\theta = 0.$$

4.3.2. Symmetry about different axes in 3D

The value of moment with integration can be calculated by using Equation (7), (8) and (9). As a preliminary result, this research will exclusively focus on the relation on the moment of symmetry objects only for five objects which are Object D, Object E, Object F, Object G and Object H respectively.

4.3.2.1. Moments in Cylindrical Coordinates

The mathematical representation for Object D in the formulation of moment by integration in cylindrical coordinates is as follows:

$$M_{xy} = \iiint_a^b z\rho(x,y,z)\ dV = \int_{y=-2}^{y=2}\int_{z=-\sqrt{4-y^2}}^{z=\sqrt{4-y^2}}\int_{x=0}^{x=\sqrt{1-\frac{z^2}{4}-\frac{y^2}{4}}} zdxdzdy = 0,$$
$$M_{xz} = \iiint_a^b y\rho(x,y,z)\ dV = \int_{y=-2}^{y=2}\int_{z=-\sqrt{4-y^2}}^{z=\sqrt{4-y^2}}\int_{x=0}^{x=\sqrt{1-\frac{z^2}{4}-\frac{y^2}{4}}} ydxdzdy = 0,$$
$$M_{yz} = \iiint_a^b x\rho(x,y,z)\ dV = \int_{y=-2}^{y=2}\int_{z=-\sqrt{4-y^2}}^{z=\sqrt{4-y^2}}\int_{x=0}^{x=\sqrt{1-\frac{z^2}{4}-\frac{y^2}{4}}} xdxdzdy = \pi.$$

The mathematical representation for Object E in the formulation of moment by integration in cylindrical coordinates is as follows:

$$M_{xy} = \iiint_a^b z\rho(x,y,z)\ dV = \int_{y=-1}^{y=1}\int_{z=-\sqrt{1-y^2}}^{z=\sqrt{1-y^2}}\int_{x=0}^{x=\sqrt{1-z^2-y^2}} zdxdzdy = 0,$$
$$M_{xz} = \iiint_a^b y\rho(x,y,z)\ dV = \int_{y=-1}^{y=1}\int_{z=-\sqrt{1-y^2}}^{z=\sqrt{1-y^2}}\int_{x=0}^{x=\sqrt{1-z^2-y^2}} ydxdzdy = 0,$$
$$M_{yz} = \iiint_a^b x\rho(x,y,z)\ dV = \int_{y=-1}^{y=1}\int_{z=-\sqrt{1-y^2}}^{z=\sqrt{1-y^2}}\int_{x=0}^{x=\sqrt{1-z^2-y^2}} xdxdzdy = 0.$$

4.3.2.2. Moments in Cartesian Coordinates

The mathematical representation for Object F in the formulation of moment by integration in cartesian coordinates is as follows:

$$M_{xy} = \iiint_a^b z\rho(x,y,z)\ dV = \int_{x=-2}^{x=2}\int_{y=1}^{y=5}\int_{z=-1}^{z=1} zdxdzdy = 0,$$
$$M_{xz} = \iiint_a^b y\rho(x,y,z)\ dV = \int_{x=-2}^{x=2}\int_{y=1}^{y=5}\int_{z=-1}^{z=1} ydxdzdy = 96,$$
$$M_{yz} = \iiint_a^b x\rho(x,y,z)\ dV = \int_{x=-2}^{x=2}\int_{y=1}^{y=5}\int_{z=-1}^{z=1} xdxdzdy = 0.$$

4.3.2.3. Moments in Spherical Coordinates

The mathematical representation for Object G in the formulation of moment by integration in spherical coordinates is as follows:

$$M_{xy} = \iiint_a^b z\rho(x,y,z)\ dV = \iiint_a^b \varphi cos\theta\ \varphi^2 sin\theta d\varphi d\phi d\theta = \int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\varphi=0}^{\varphi=1}\int_{\theta=0}^{\theta=\frac{\pi}{2}} \varphi^3 cos\theta sin\theta d\varphi d\phi d\theta = 0,$$

$$M_{xz}=\iiint_a^b y\rho(x,y,z)\ dV=\iiint_a^b \varphi sin\phi sin\theta\ \varphi^2 sin\theta d\varphi d\phi d\theta=\int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\varphi=0}^{\varphi=1}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\varphi^3 sin\phi sin^2\theta d\varphi d\phi d\theta$$
$$=\frac{\pi}{8},$$
$$M_{yz}=\iiint_a^b x\rho(x,y,z)\ dV=\iiint_a^b \varphi cos\phi sin\theta\ \varphi^2 sin\theta d\varphi d\phi d\theta=\int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\varphi=0}^{\varphi=1}\int_{\theta=0}^{\theta=2\pi}\varphi^3 cos\phi sin^2\theta d\varphi d\phi d\theta.$$
$$=\frac{\pi}{8}.$$

The mathematical representation for Object H in the formulation of moment by integration in spherical coordinates is as follows:

$$M_{xy}=\iiint_a^b z\rho(x,y,z)\ dV=8\iiint_a^b 3(2)(1)\varphi cos\theta\ \varphi^2 sin\theta d\varphi d\phi d\theta$$
$$=48\int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\varphi=0}^{\varphi=1}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\varphi^3 cos\theta sin\theta d\varphi d\phi d\theta=3\pi,$$
$$M_{xz}=\iiint_a^b y\rho(x,y,z)\ dV=8\iiint_a^b 2(6)\varphi sin\phi sin\theta\ \varphi^2 sin\theta d\varphi d\phi d\theta$$
$$=96\int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\varphi=0}^{\varphi=1}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\varphi^3 sin\phi sin^2\theta d\varphi d\phi d\theta=6\pi,$$
$$M_{yz}=\iiint_a^b x\rho(x,y,z)\ dV=8\iiint_a^b 3(6)\varphi cos\phi sin\theta\ \varphi^2 sin\theta d\varphi d\phi d\theta$$
$$=144\int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\varphi=0}^{\varphi=1}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\varphi^3 cos\phi sin^2\theta d\varphi d\phi d\theta=9\pi.$$

4.4. Result of Moments

The moment results for both dimensions were compiled into Table 7 so that further analysis could be performed to determine the relation of the moment with symmetry objects.

Table 7: Moment value of Symmetry 2D Objects

Object	Moment value	
	M_x	M_y
A	8/3	0
B	0	9
C	0	0

Table 8: Moment value of Symmetry 3D Objects

Object	Moment value		
	M_{xy}	M_{xz}	M_{yz}
D	0	96	0
E	0	0	π
F	0	0	0
G	0	$\pi/8$	$\pi/8$
H	3π	6π	9π

Moment in general seem to be very good for detecting the rotational occurred on the 2D and 3D models representing solid geometries where the symmetry exists about the axis and about the plane respectively. Therefore, the moment proof was analysed at the first part which uses the symmetry condition in both cartesian and polar coordinates that can be used to analyse the moment of symmetry of 2D objects. Table 5 and Table 6 show the result of the simulations of symmetry 2D, and 3D objects based on the mathematical equation of geometry from the data collected. Each of geometry is simulated by online graph generator, Desmos and Wolfram Alpha [15][16].

The moment obtained describes the tendency of an object to rotates about $x-$axis, $y-$axis or both axes. Based on the result in Table 7, the moment value with respect to $y-$axis for Object A is zero and the moment value of Object B is zero about $x-$axis. However, the moment value for both axes of Object C is zero which satisfies the symmetrical conditions that were discussed in Table 3 and Table 4. Therefore, it can be said that zero moments obtained indicates there is no rotation occur about the axis.

In 3D region, the value of moment shows the tendencies of an object to rotate about the plane. Based on the result in Table 8, Object D has zero moment about $xy-$ and $yz-$planes, object E obtains zero moment about $xy-$ and $xz-$planes, and the moment value of Object G is zero about $xy-$plane only. However, the moment value about $xz-$ and $yz-$planes is the same which indicates that both regions are mirroring each other results to existence of rotation symmetry. In other cases, Object F has zero moment of all planes, but Object H is in contrast. Therefore, it can be concluded the nonzero moments obtained describes there is a rotation occur about the plane in 3D region.

This result was proposed to demonstrate how the symmetry objects can affect the result of moment. This approach is designed to give as a reference for investigating the symmetry of 2D and 3D objects. The technique is expected to benefit in the continuing of the symmetry's study object in future analysation.

Conclusion

In this research a new method for analysing the symmetry of 2D and 3D objects represented by moment approach was described and its results were presented. The method seems to work well on perfectly as well as approximately symmetrical objects and exhibits good results even when used on objects with different type of symmetries with respect to the axes and the planes.

The results of moment value were also compared to another symmetry objects, especially in 2D and 3D regions. It was shown that the moment value obtained for each of the geometries explains the tendency of an object to rotate about an axis or a plane. Hence, the zero moment can be described as there is no rotation occurred about symmetry axis or symmetry plane. The proof of moment based on the symmetrical condition of 2D objects in polar coordinates has been shown to be zero which met the symmetrical criteria and showed that there is no tendency of the object to rotate about the axis.

Even though the result of the moment that have been calculated did not having any problems, however in certain shapes of objects the moment approach cannot met the conclusions. It can be said that this method available for solid geometry only. In some respects, when estimating the complex geometry, the tendencies of the object to rotate cannot be achieved, meanwhile the simple solid object can be quickly and easily calculated.

Since the focus of the study of symmetry is only measures the tendencies of object to rotate, the approach needs to be advanced to make the method detect more than one plane of a complex object such as human face and make the method to be generalized for detection of symmetries of different types as well. In the future, the deeper study would be the best to examine the possibilities to further extend or generalize the method.

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