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Properties and Some Applications of Fuzzy Metric Space

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Abstract
The purpose of this project is to study in detail of metric and fuzzy metric spaces. All the main definitions and properties of fuzzy metric spaces that are available in literature are used in this project. The definition of fuzzy set and metric space are introduced in this project. The fundamental differences between metric and fuzzy metric spaces are studied and highlighted. Both structures can generate metrizable topological space. In addition, the properties of metric and fuzzy metric spaces are both similar except their concept distance and inner product. Some related theorems and their proofs are provided in this project.
Keywords: Fuzzy metric space; Metric space;

1. Introduction

Fuzzy metric space has been introduced in different ways by many authors. In this study, the possible relationship of fuzzy set, metric space and fuzzy metric space is investigated, through some samples of fuzzy metric space. The theory of fuzzy set was introduced as a set defined by membership functions that takes values in the interval [0,1] by Zadeh [1]. It is also defined as the generalization of crisp sets, whereas crisp sets are defined as to put/place the individuals in some given universe of discourse into two groups: members and nonmembers. A mathematical formulation, for a metric, of a distance is by abstracting the real line, plane and three-dimensional space. Metric space is a topological space which defined by a distance function, *d*, where *d* must be a non-negative, real valued mapping of $X \times X$ and all the points of x, y and z must be in X. Metric spaces are seen as fundamental concept in all areas of mathematics. Distance function is defined by a norm. In the first, any norm; in the second, by a special kind of norm that is derived from an inner product. While inner product is a generalization of inner product (or dot product) of vectors in finite dimensional spaces.

In this study, a comparison of metric and fuzzy metric spaces will be presented. In 1975, Kramosil and Michalek [2] introduced the concept of fuzzy metric space. However, it is still one of the fundamental problems in fuzzy mathematics even though it has been modified and studied by many researchers. There is no uniform measure that can be proven to be exist which can be used in all kinds of fuzzy environment.

This project aims to study in detail the definition of metric and fuzzy metric spaces, study some samples of fuzzy metric spaces, expose the fundamental differences between metric and fuzzy metric spaces and study the impact of the differences to other mathematical disciplines. The results is obtained by comparing metric space and fuzzy metric space and some applcations of fuzzy metric space in real life is presented.

2. Literature Review

*2.1.*Fuzzy Set

Definition 2.1 [1]. A fuzzy set is *empty* if and only if its membership function is identically zero on X.

$$A = B, \text{ if and only if } f_A(x) = f_B(x) \text{ for all } x \text{ in } X. \quad (1)$$

The *complement* of a fuzzy set A is denoted by A' and is defined by

$$f_{A'} = 1 - f_A \quad (2)$$

In fuzzy sets, the notion of containment and the notions of union and intersection are defined as follows;

For *containment*, A is said to be contained in B (or A is a *subset* of B or we can also say that A is smaller than or equal to B) if and only if $f_A \leqq f_B$,

$$A \subset B \Leftrightarrow f_A \leqq f_B \quad (3)$$

While for *union*, the union of two fuzzy sets A and B with respective membership functions $f_A(x)$ and $f_B(x)$ is a fuzzy set C, written as $C = A \cup B$, whose membership function is related to A and B by

$$f_c(x) = \text{Max}\ [f_A(x), f_B(x)], x \in X \quad (4)$$

or, in abbreviated form

$$f_C = f_A \vee f_B \quad (5)$$

Note that the associative property for ∪ is $A \cup (B \cup C) = (A \cup B) \cup C$.

The union of A and B is the smallest fuzzy set containing both A and B. This means if D is any fuzzy set which contains both A and B, then it also contains the union of A and B.

To prove this, first let Max $[f_A, f_B] \geqq f_A$ and Max $[f_A, f_B] \geqq f_B$. Since D is any fuzzy set containing A and B, then

$$f_D \geqq f_A \quad (6)$$

$$f_D \geqq f_B \quad (7)$$

Hence

$$f_D \geqq \text{Max}\ [f_A, f_B] = f_C \quad (8)$$

which implies that $C \subset D$.

For *intersection*, the intersection of two fuzzy sets A and B with respective membership functions $f_A(x)$ and $f_B(x)$ is a fuzzy set C, written as $C = A \cap B$, whose membership function is related to A and B by

$$f_c(x) = \text{Min}\ [f_A(x), f_B(x)], x \in X \quad (9)$$

or, in abbreviated form

$$f_C = f_A \wedge f_B \quad (10)$$

The intersection of A and B is the largest fuzzy set which is contained in both A and B. In ordinary sets, A and B are *disjoint* if $A \cap B$ is empty.

The membership function of a classical set can only take two values – zero or one whereas the membership function of a fuzzy set is a continuous function with the range [0,1]. It is also known as the generalization of crisp sets

2.2 Metric Space

Definition 2.2 [3]. Let X is a set. *A* metric on X is a function, $d : X \times X \to \mathbb{R}^{+} \ni d(x,y) > 0$

i. $d(x,y) = 0$ if and only if $x = y$
ii. $d(x,y) = d(y,x), \forall x,y \in X$
iii. $d(x,y) + d(y,z) \geq d(x,z), \forall x,y,z$

The third condition is called **triangle inequality.** Generally, metric space is a pair (X,d) of a set X and a metric d on X. For a metric space (X,d), elements of X are called points and the value $d(x,y)$ is called distance between x and y.

2.3 Fuzzy Metric Space

The definition of fuzzy metric space is given by;

Definition 2.3 [4]. Suppose X is a nonempty set and

$$d_F : P_F(X) \times P_F(X) \to S_F^{+}(R)$$

is a mapping. $(P_F(X), d_F)$ is said to be a *fuzzy metric space* if for any $\{(x,\lambda),(y,\gamma),(z,\rho)\} \subset P_F(X)$, d_F satisfies the following three conditions,

Nonnegative: $d_F\left((x,\lambda),(y,\gamma)\right) = 0$ iff $x = y$ and $\lambda = \gamma = 1$

Symmetric: $d_F\left((x,\lambda),(y,\gamma)\right) = d_F\left((y,\gamma),(x,\lambda)\right)$

Triangle inequality: $d_F\left((x,\lambda),(z,\rho)\right) \prec d_F\left((x,\lambda),(y,\gamma)\right) + d_F\left((y,\gamma),(z,\rho)\right)$

where d_F is called a *fuzzy metric* defined in $P_F(X)$ and $d_F\left((x,\lambda),(y,\gamma)\right)$ is called a *fuzzy distance* between the two fuzzy points.

In 1975, Kramosil and Michalek [2] introduced the definition of 3-tuple fuzzy metric space which is given by the following definition.

Definition 2.4 [2]. The 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions:

i. $M(x,y,0) = 0$
ii. $M(x,y,t) = 1, \forall t > 0$ iff $x = y$
iii. $M(x,y,t) = M(y,x,t)$
iv. $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$
v. $M(x,y,\cdot) : [0,\infty) \to [0,1]$ is left continuous

$x,y,z \in X$ and $t,s > 0$.

However, the above definition was modified by George and Veeramani to Definition 2.7 [4] in order to introduce Hausdorff topology on the fuzzy metric space.

Definition 2.5 [4]. The 3-tuple $(X,M,*)$ is said to be a fuzzy metric space if X is an arbitary set, $*$ is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions:

i. $M(x,y,t) > 0$
ii. $M(x,y,t) = 1$ iff $x = y$
iii. $M(x,y,t) = M(y,x,t)$
iv. $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$
v. $M(x,y,\cdot): (0,\infty) \to [0,1]$ is continuous

$x,y,z \in X$ and $t,s > 0$.

3. Methodology

This research begins by investigating the definitions of fuzzy set and metric space. Other than that, the definitions and theorems of fuzzy metric spaces are also studied by using multiple examples. Next, the properties of fuzzy metric spaces are reviewed. The comparison on metric and fuzzy metric spaces is made. Some applications of fuzzy metric space is studied The design and procedure of the research are summarized in Table 3.**1** as follow:

Table 3.1: Research Design and Procedure

Phase	**Design and Procedure**
1.	Literature review on fuzzy sets. • To understand on how fuzzy sets was developed including its background and the application.
2.	Literature review on metric spaces. • To gain some insight about the development and history of metric spaces.
3.	Literature review on fuzzy metric spaces. • To have a better understanding of how fuzzy sets were created including their history and applications.
4.	Properties of fuzzy metric spaces • To gain some knowledge about the properties of fuzzy metric spaces.
5.	Comparison and relationship • To discuss and compare the properties of metric and fuzzy metric spaces.
6.	Applications of fuzzy metric space • To gain knowledge about the applications of fuzzy metric space in real life
7.	Submission of report writing

4. Results and discussion

4.1 Comparison between Metric Space and Fuzzy Metric Space

The comparison of both metric spaces is given in Table 4.**2**.

Table 4.2: Comparison between metric space and fuzzy metric space

	Metric space	Fuzzy metric space
Similarities	Both metric spaces can induce topological space which is also said to be continuous.	
	Both metric spaces are metrizable space.	
	Since both metric spaces are metrizable, so both spaces are second-countable.	
	Both metric spaces are Lindelöf space as they are second-countable.	
	Metric space and fuzzy metric space are also said to be separable.	
	Metric space and fuzzy metric space are Hausdorff space	
	As both metric spaces are Hausdorff space, then metric space and fuzzy metric space are countably compact.	
Differences	$d(x,y)$ is referred as distance function between x and y	$d_F(x,y,t)$ is denoted as the degree of nearness between x and y with respect to t.

4.2 Application of Fuzzy Metric Space

4.2.1 Color Image Filtering

The fuzzy metrics embodies a multicode of concepts that are influenced by image pixels which is given as (i, J_i) (position, colour). Commonly used color are *red*, *green* and *blue* (RGB) that are combined in different amounts to produce different colors [5]. It can be characterized by spatial coordinates of pixel (i_1, i_2) and by vector $J_i = (J_i^1, J_i^2, J_i^3)$. $J_i^1$is denoted as the quantity of red colour, J_i^2 as the quantity of green colour and J_i^3 is the quantity of blue colour. Since 2005, there are several successful works in color image processing using fuzzy metric space have already been published. There are six classical vector filtering techniques.

i. Vector median filter (VMF)

The distance between two colour vectors is derived from the generalized Minkowski metric which defined as [6]

$$d(I_i, I_j) = \left(\sum_{k=1}^{m} |I_i(k) - I_j(k)|^{\gamma} \right)^{\frac{1}{\gamma}} \tag{11}$$

where *I* denoted as a multichannel image and γ is the used metric

ii. Extended vector median filter (EVMF)

This filter is the combination of VMF and the arithmetic mean filter (AMF) which used to suppress gaussian noise in a better way. This filter selects output according to the following rule [6].

$$I_{EVMF} = \left\{ \begin{array}{c} \bar{I} \text{ if } \sum_{j=1}^{n} d(\bar{I}, I_j) \leq \sum j = 1^n d(I_{VMF}, I_j) \\ I_{VMF} \text{ otherwise} \end{array} \right\} \tag{12}$$

where I_{VMF} denoted as the output of VMF and $\bar{I}$ denoted as AMF output.

iii. Basic vector directional filter (BVDF)

Another filter which can parallelizes VMF is the basic vector directional filter (BVDF). This filter is suitable for designing multichannel image filters. The equation of this filter is given by [6]

$$\alpha_i = \sum_{j=1}^{N} A(I_i, I_j) \tag{13}$$

where $A(I_i, I_j) = arc\cos\left(\frac{I_i \cdot I_j}{|I_i| \cdot |I_j|}\right)$ is the angle between two vectors. BVDF may outperform VMF in terms of chromaticity preservation.

iv. Generalized directional distance filter (GVDF)

Normally, GVDF is used as a second level filter so that its output become an input for other filtes to compute the final output. The output of this filter is a set of *r* lowest ranked samples which defined as [6]

$$I_{GVDF} = \{I_{(1)}, I_{(2)}, \ldots, I_{(r)}\} \tag{14}$$

v. Directional distance filter (DDF)

To overcome the disadvantages of BVDF, researchers use directional distance filter (DDF). This filter uses both magnitude and direction in the distance criterion. This filter also works simultaneously to generate distance criteria by VMF and BVDF. The distance criterion is given as follow [6].

$$\Omega_i = \left(\sum_{j=1}^{N}\left(\sum_{k=1}^{m}|I_i(k) - I_j(k)|^{\gamma}\right)^{\frac{1}{\gamma}}\right)^{1-p} \cdot \left(\sum_{j=1}^{n} A(I_i, I_j)\right)^{p} \tag{15}$$

where p is the parameters which gives effect on the magnitude criterion in front of the angular criterion

vi. Hybrid filters

If DDF works simultaneously, directional hybrid filters HVF operate independently on direction and magnitude of vectors then apply a combination to generate output. This filter makes a nonlinear combination of VMF and BVDF according to the following rule [6].

$$I_{HVF} = \left\{\begin{matrix} I_{VMF} \text{ if } I_{VMF} = I_{BVDF} \\ \left(\frac{|I_{VMF}|}{|I_{BVDF}|}\right) I_{BVDF} \text{ otherwise} \end{matrix}\right\} \tag{16}$$

where I_{VMF} denotes as the VMF output, I_{BVDF} denotes as the BVDF output and $|\cdot|$ denotes as vector magnitude.

4.2.2 Building composite indicators

Fuzzy metric space can be used for building composite indicators. This year, Jiménez-Fernández et al. published a work on building composite indicators through an unsupervised machine learning technique based on fuzzy metrics. They stated in their research that the use of a metric is necessary due to the fact that it is a natural way to establish the proximity between the analysed observations and also to perform benchmarking. This is to prevent an ordering of the units studied is obtained that lacks structure and thus, the notion of distance [7]

There are a few steps required to construct a composite indicator. Firstly, researchers have to construct a composite indicator formula. The definiton of fuzzy metric space introduced by George and Veeramani [4] can be used to construct the formula. As an example, the composite fuzzy indicator formula used by Jiménez-Fernández et al. is defined as [7]

$$CFI_i = CFI(x_*, x_i) = \prod_{j=i}^{m} M_j(x_{*j}, x_{ij}) \tag{17}$$

After the formula was constructed, the sensitivity constant of fuzzy metric have to be computed in order to check the existence of statistical relationships between all single indicators. Then, the best functional relationship among single indicators and CFI is chosen to select an efficient approach to the data set. In Jiménez-Fernández et al. recent work, the MARS model used is defined as [7].

$$CFI_i = \beta_0 + \sum_{j=1}^{m} \beta_i B(x_{ij}) + \varepsilon_i \tag{18}$$

where CFI_i is the composite indicator, x_{ij} is the observation of the j-normalized single indicator $j \in \{1, \dots, m\}, \beta_0$ is the intercept, $B(x_{ij})$ is a basis of disjoint functions and ε is the error term.

Then, the sensitivity scores (k_j) need to be determined to quantify the strength of the relationship between single indicators and composite indicator. Finally, the algorithm for building composite indicator can be constructed to examine the iterative method of calculation and statistical properties required to stop the iterative process. The process will stop when a similar rank correlation between two composite indicators met.

4.2.3 Medical image processing

Fuzzy metric space have been applied to brain image diagnosis and prediction. The most popular approach used on this application is the Fuzzy C-Means (FCM). As medical brain image have loads of uncertainty during diagnosis and prediction due to the layer of tissues in brain, so FCM is one of the best approach to deal with uncertainty and inaccuracy of images. The objective function of classification is given as follow [8].

$$J_{min}(U,V) = \sum_{i=1}^{c} J_i = \sum_{i=1}^{c}\sum_{j}^{n} u_{ij}^{m} d_{ij}^{2} \tag{19}$$

where *U* denotes as the classification matrix, *C* denotes the number of sample classes which is determined by u_{ij}, the corresponding sample membersip function, *V* (classification center vector) refers as the distance from *i*-th classification center to the *j*-th sample point and *m* is a constant. The objective function can be optimized iteratively by solving its minimum value.

Besides brain image diagnosis and prediction, recent work by Zararsiz and Riaz [9] has shown fuzzy metric space also have been used for vaccine selection in COVID-19. the fuzzy metric space used in this work is bipolar fuzzy metric space which is defined as,

Definition 4.3 [9]. The tetra set $(A,\mathcal{B},\Delta,\nabla)$ is called bipolar fuzzy metric space (BFMS) when the following conditions are satisfied for all $u,v,z \in A$, where A be arbitrary set, $\{< u,\mu_A^r(u),\mu_A^\ell(u) > : u \in A\}$ be BFS such that μ_A^r,μ_A^ℓ are defined as fuzzy sets on $A \times A \times (0,\infty)$, $\mathcal{B} : A \times A \times (0,\infty) \to [-1,1]$, $\alpha,\gamma > 0$ and Δ,∇ show the continuous t-norm and continuous symmetry t_s-conorm, respectively:

i. $0 \le \mu_A^r(u,v,\gamma) \le 1, -1 \le \mu_A^\ell(u,v,\gamma) \le 0$
ii. $\mu_A^r(u,v,\gamma) + \mu_A^\ell(u,v,\gamma) \le 1$
iii. $\mu_A^r(u,v,\gamma) = 1 \Leftrightarrow u = v$
iv. $\mu_A^r(u,v,\gamma) = \mu_A^r(v,u,\gamma)$
v. $\mu_A^r(u,v,\gamma)\Delta\mu_A^r(v,z,\alpha) \le \mu_A^r(u,z,\gamma+\alpha)$
vi. $\mu_A^r(u,v,.) : [0,\infty) \to [-1,1]$ is continuous
vii. $\lim_{k} \mu_A^r(u,v,\gamma) = 1, (\forall \gamma > 0)$
viii. $\mu_A^\ell(u,v,\gamma) = -1 \Leftrightarrow u = v$
ix. $\mu_A^\ell(u,v,\gamma) = \mu_A^\ell(v,u,\gamma)$
x. $\mu_A^\ell(u,v,\gamma)\nabla\mu_A^\ell(v,z,\alpha) \ge \mu_A^\ell(u,z,\gamma+\alpha)$
xi. $\mu_A^\ell(u,v,.) : [0,\infty) \to [-1,1]$ is continuous
xii. $\lim_{k} \mu_A^\ell(u,v,\gamma) = -1, (\forall \gamma > 0)$.

The functions $\mu_A^r(u,v,\gamma)$ and $\mu_A^\ell(u,v,\gamma)$ represent the value of closeness and the value of distance between u and v with respect to γ , respectively.

5. Conclusion

It can be concluded that metric space and fuzzy metric space have many similar properties except the relationship between distances and inner products of fuzzy points. Both metric spaces also can generate topological space and Hausdorff space. In addition, both metric spaces can be induced by each other which means we can induce fuzzy metric space from metric space and vice versa. The comparison of both metric spaces is given. Some applications of fuzzy metric space are also provided.

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