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Energy of the Composite Order Cayley Graphs of the Quaternion Groups of Order at Most 32

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Abstract
Energy of a graph is the sum of the absolute value of the eigenvalues of the graphs' adjacency matrix. The energy of graphs has various applications in mathematics and other areas of science, for example, in understanding the partition problem of the adjacency matrix associated to the optimization problem. Thus, many researches have been conducted in investigating the energy of graphs including the energy of Cayley graphs of symmetric groups and energy of connected and undirected graph of order less than ten. In this research, the main objective is to compute the energy of the composite order Cayley graphs of the quaternion groups of order at most 32. A composite order Cayley graph of a group is a pair of set of vertices and set of edges, where the set of vertices contains the elements of the group and two distinct vertices x and y are adjacent when $xy^{-1} \in S$, where S is a subset of composite order elements of the group. In order to compute the energy of the composite order Cayley graphs of the quaternion groups of order at most 32, the structures of the graphs are first examined by reconstructing the graphs by using GeoGebra software based on the definition of the composite order Cayley graph and the presentation of each of the groups. Then, by referring to the adjacency of the graph's vertices, the adjacency matrix of the graph is determined, and the energy of the graph is computed based on the eigenvalues of the adjacency matrix. Maple software is used to assist in the computation of the eigenvalues of the graph's adjacency matrix. As a result, the energy of the composite order Cayley graphs of quaternion group of order eight, 16 and 32 are found to be 12, 28 and 60, respectively.
Keywords: Energy of graph; Cayley graph; quaternion groups; adjacency matrix; eigenvalues.

1. Introduction

A graph is a structure consisting of vertices and edges such that there exists a connection on these vertices with the edges. In the construction of the graph, the vertices and edges are usually represented by dots and lines respectively. The theory of graph has been introduced by Euler, who was a great mathematician. Euler introduced this theory after he solved the Seven Bridge of Königsberg problem [1]. After graph theory was introduced and studied in more detail, it has become essential knowledge in modern mathematics and being utilised in many applications, such as linguistic, computer science, chemistry and many more [2].

The energy of a graph is the sum of the absolute values of the eigenvalues of its adjacency matrix where the adjacency matrix of a graph is the square matrix with the entries in matrix depend on the adjacency of the graph's vertices. Moreover, the energy of a graph has been defined by Gutman in 1978 [3]. Gutman became motivated to introduce the energy of a graph after finding out that Hückel proposed the Hückel Molecular Theory in 1930s. This theory is an approximate method that has been used by chemist in approximating energies correlated with π −electron orbitals in conjugated hydrocarbons.

The energy of various graph of groups became well known and has been extensively investigated since 1978. For example, Shalini and Joseph [4], investigated the energy of connected and undirected graphs of order less than 10. Since then, there are many researchers who investigate the energy of certain graph of groups such as Gaidhani [5] and Prasad [6]. In addition, Ahmad Fadzil *et. al* also investigated the energy of Cayley graphs for symmetric groups of order 24 [7].

Although many researches have been conducted on energy of graphs, there is no research done yet on the energy of the composite order Cayley graph of quaternion groups. Therefore, the energy of composite order Cayley graph of quaternion groups of order at most 32 is determined in this research.

A Cayley graph (also known as Cayley color graph or Cayley diagram) is one of the graph that can be associated to group and is denoted by $Cay(G,S)$, where S is a non-empty subset of a finite group G. Tolue in [8] has introduced a new type of Cayley graph namely composite order Cayley graph. A composite order Cayley graph denoted as $Cay_c(G,S)$, is a graph with a set of vertices consisting of the elements of G and two distinct vertices x and y are adjacent if $xy^{-1} \in S$, where S is a subset of G containing elements of composite order.

Cayley graphs has been introduced by Cayley in 1878 to explain the concept of abstract groups [9]. From the Cayley diagram, the elements of the group were represented by points and the operation on the elements was represented by a directed line connecting the two points. This diagram concept was developed and leads to the formation of a new type of Cayley graph [10].

In addition, the quaternion group of order 2^n is a nonabelian group with presentation

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = e, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle,$$

where $n \geq 3$ [11]. This research will be focus for $n = 3, 4$ and 5 which are the quaternion group of order eight, Q_8, order 16, Q_{16} and order 32, Q_{32}.

Therefore, the energy of composite order Cayley graph of quaternion groups of order at most 32 is determined in this research.

2. Basic Concepts in Graph Theory and Group Theory

In this section, the definition of graph and some related definitions which are referred in computing the energy of the graph are presented. To begin with, the definition of a graph is given as follows:

Definition 2.1 [12] Graph

A graph is a pair $\Gamma = (V(\Gamma), E(\Gamma))$ of sets such that $E \subseteq [V]^2$. Each element of $V(\Gamma)$ is called a vertex while the element of E is called an edges. An edge that connect two distinct vertices $u, v \in V(\Gamma)$ can be denoted as $\{u, v\}$.

The main idea to determine the energy of graph is by calculating the sum of the absolute value of the eigenvalues of the graphs' adjacency matrix. The definition of the adjacency matrix of a graph is given as follows:

Definition 2.2 [13] Adjacency Matrix

Let Γ be a graph with $V(\Gamma) = \{v_1, v_2, v_3, \dots, v_n\}$ and $E(\Gamma) = \{e_1, e_2, e_3, \dots, e_m\}$. The adjacency matrix of Γ denoted by $A(\Gamma)$ is the $n \times n$ matrix defined as follows.

$$A(\Gamma) = \begin{cases} x_{ij} = 1 \text{ if } v_i \sim v_j \\ x_{ij} = 0 \text{ if otherwise} \end{cases}$$

where $v_i \sim v_j$ represents the adjacency of v_i and v_j. Usually, $A(\Gamma)$ is simply denoted by A.

In order to get the eigenvalues of the adjacency matrix, the characteristic polynomial need to be obtained first.

Definition 2.3 [6] The Characteristic Polynomial

Let A be an $n \times n$ matrix. The characteristic polynomial of A is the function $f(\lambda)$ given by

$$f(\lambda) = \det(A - \lambda I_n)$$

Definition 2.4 [13] Energy of Graph
For any graph Γ, the energy of the graph, $\varepsilon(\Gamma)$, is the sum of the absolute value of the eigenvalues of the graph's adjacency matrix. The energy of the graph can be written as

$$\varepsilon(\Gamma) = \sum_{i=1}^{n} |\lambda_i|$$

where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the adjacency matrix of Γ.

The following is the basic concept needed in order to determine the energy of the graph for composite order Cayley graph of quaternion groups, starting with the definition of generalized quaternion group.

Definition 2.5 [11] Generalized Quaternion Group
A generalized quaternion group, also known as a generalized quaternion 2-group is a group of order 2^n. For $n \geq 3$, the group has the presentation:

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = e, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1} \rangle.$$

The definition of graph associated to groups also provided in this section and the following are some important definitions of a graph related with group.

Definition 2.6 [14] Graph of Group
A graph of group G denoted as Γ_G, is an object consisting of a collection of a pair of vertices, V and edges, E labelled as $\Gamma_G = (V, E)$. The elements of G are the vertices of Γ_G and the element of $E(G)$ are the lines that combine two elements of $V(G)$.

Definition 2.7 [15] Cayley Graph of a Group
A graph Γ_G is a Cayley graph on a group G, if there is a subset $S \subseteq G \backslash e$ with $S = S^{-1} = \{s^{-1} | s \in S\}$, such that $V(\Gamma_G) = G$, and two vertices g and h are adjacent if and only if $hg^{-1} \in S$. In other words, $hg^{-1} \in S$ implies that $\exists s \in S$ with $hg^{-1} = s$ or $h = sg$. A Cayley graph can be denoted as $Cay(G, S)$.

In 2019, Tolue defined a new type of Cayley graph namely composite order Cayley graph [8]. The definition of composite order Cayley graph are stated as follows:

Definition 2.8 [8] Composite Order Cayley graph
Let G be a group and S be a set of composite order elements of G. A composite order Cayley graph is a graph containing a set of vertex for the entire element of G and two distinct vertices x and y are adjacent when $xy^{-1} \in S$. A composite order Cayley graph can be denoted as $Cay_c(G, S)$.

Theorem 2.1 [16]
Let Q_{2^n} be a quaternion group of order 2^n, where $n \geq 3$. Let S be a subset of the composite order elements of Q_{2^n}. Then, $S = Q_{2^n} \setminus Z(Q_{2^n}) = Q_{2^n} \setminus \{e, a^{2^{n-2}}\}$.

In this research, the energy of composite order Cayley graph of the quaternion groups are computed.

3. The Energy of Composite Order Cayley Graph of the Quaternion Group of Order Eight

3.1. The Composite Order Cayley Graph of Quaternion Group of Order Eight

The group presentation of Q_8 is given by:

$$Q_8 = \langle a, b \mid a^4 = e, b^2 = a^2, b^{-1}ab = a^{-1} \rangle.$$

Based on the group presentation, eight elements of Q_8 is listed as follows:

$$Q_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$

The subset of the composite order Cayley graph, S_1, is identified based on Theorem 2.1, that is every elements of Q_{2^n} is of composite order except e and $a^{2^{n-2}}$. Since $Q_8 = Q_{2^3}$, $n = 3$, so, it implies that $e, a^2 \notin S_1$. Therefore, the subset S_1 of elements of Q_8 with composite order is

$$S_1 = \{a, a^3, b, ab, a^2b, a^3b\}.$$

From Definition 2.8, the set of vertices of $Cay_c(Q_8, S_1)$ is

$$V\big(Cay_c(Q_8, S_1)\big) = Q_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$

Let $s = a$, a is adjacent to e since $x(e^{-1}) = a \Rightarrow x = a \in S_1$ implies that there exist $s \in S_1$ such that $xy^{-1} = s$ or $x = sy$. Meanwhile, a^3 is not adjacent to a because $a^3(a^{-1}) \notin S_1$ which implies that there is not exist $s \in S_1$. Therefore, to find the set of edges, the adjacency of pair of element must be examined first. Hence, the set of edges of $Cay_c(Q_8, S_1)$ is:

$$\begin{aligned} E\big(Cay_c(Q_8, S_1)\big) = \{ &\{e, a^3\}, \{e, a^2b\}, \{e, a^3b\}, \{a, e\}, \{a, ab\}, \{a, a^2b\}, \{a^2, a\}, \{a^2, b\}, \{a^2, ab\}, \{a^3, b\}, \\ &\{a^3, a^2\}, \{a^3, a^3b\}, \{b, e\}, \{b, a\}, \{b, a^3b\}, \{ab, e\}, \{ab, a^3\}, \{ab, b\}, \\ &\{a^2b, a^2\}, \{a^2b, a^3\}, \{a^2b, ab\}, \{a^3b, a\}, \{a^3b, a^2\}, \{a^3b, a^2b\} \}. \end{aligned}$$

After that, the composite order Cayley graph of Q_8 can be constructed by using GeoGebra software by inserting the results for $V\big(Cay_c(Q_8, S_1)\big)$ and $E\big(Cay_c(Q_8, S_1)\big)$.

Next, based on the set of edges, the adjacency matrix of composite order Cayley graph of quaternion group of order eight is determined in order to compute the total energy of the graph.

3.2. The Adjacency Matrix of Composite Order Cayley Graph of Quaternion Group of Order Eight

The adjacency matrix of $Cay_c(Q_8, S_1)$ is determined based on the structure of $Cay_c(Q_8, S_1)$ observed from the set of edges of the graph. By Definition 2.2, the entry for the adjacency matrix is 1 when the pair of elements are adjacent by an edge and 0 if the elements are not adjacent by an edge. For an undirected graph, the value $x_{ij} = x_{ji}$ for all i, j.

In other words, the entry x_{11} is equal to 0 due to set $\{e, e\}$ is not included in $E\big(Cay_c(Q_8, S_1)\big)$ which shows that e is not adjacent to itself. Furthermore, the entry $x_{12} = x_{21}$ is equal to 1 since $\{e, a\} = \{a, e\}$ is included in $E\big(Cay_c(Q_8, S_1)\big)$. Therefore, the element e is adjacent to a and vice versa. So, the other entries are also determined in the same way. Therefore, the adjacency matrix of the composite order Cayley graph of Q_8 is

$$A\big(Cay_c(Q_8, S_1)\big) = \begin{bmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

The next subsection is on the computation of the energy of prime order Cayley graph of the quaternion group of order eight.

3.3. Computation of the Energy of Composite Order Cayley Graph of the Quaternion Group of Order Eight

Based on $A(Cay_c(Q_8, S_1))$, by using Definition 2.3, the characteristic polynomial, $f(\lambda)$ is obtained by determining $det\left(\lambda I - A(Cay_c(Q_8, S_1))\right)$ and the eigenvalues of $A(Cay_c(Q_8, S_1))$ is determined by $det\left(\lambda I - A(Cay_c(Q_8, S_1))\right) = 0$.

Maple software is used in order to assist the computation of the characteristic polynomial of $A(Cay_c(Q_8, S_1))$. Therefore, the characteristic polynomial of A is

$$f(\lambda) = \lambda^8 - 24\lambda^6 - 64\lambda^5 - 48\lambda^4.$$

Next, to find the eigenvalues of the graph, the characteristic polynomial is set to be equal to zero, i.e., $f(\lambda) = 0$. The characteristic equation is simplified and implies $\lambda^4(\lambda - 6)(\lambda + 2)^3 = 0$. Thus, the eigenvalues of the characteristic polynomial of the composite order Cayley graph of Q_8 are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$, $\lambda_5 = 6$ and $\lambda_6 = \lambda_7 = \lambda_8 = -2$.

Then, the total energy of the graph can be calculated as $\varepsilon(\Gamma) = \sum_{i=1}^{n}|\lambda_i|$ as stated in Definition 2.4. Hence,

$$\varepsilon(Cay_c(Q_8, S_1)) = 4|0| + 6 + 3|-2| = 12.$$

4. The Energy of Composite Order Cayley Graph of the Quaternion Group of Order 16

4.1. The Composite Order Cayley Graph of Quaternion Group of Order 16

The group presentation of Q_{16} is given by:

$$Q_{16} = \langle a, b \mid a^8 = e, b^2 = a^4, b^{-1}ab = a^{-1} \rangle.$$

Hence, based on the group presentation, sixteen elements of Q_{16} is listed as follows:

$$Q_{16} = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}.$$

The subset of the composite order Cayley graph, S_2, is selected with reference on Theorem 2.1, that is every elements of Q_{2^n} is of composite order except e and $a^{2^{n-2}}$. So, since $Q_{16} = Q_{2^4}$, $n = 4$ and this implies that $e, a^{2^{4-2}} \notin S_2$. Therefore,

$$S_2 = Q_{16} \setminus \{e, a^4\} = \{a, a^2, a^3, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}.$$

Based on Definition 2.8, the set of vertices of $Cay_c(Q_{16}, S_2)$ is

$$V(Cay_c(Q_{16}, S_2)) = Q_{16} = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}.$$

Let $s = a^2$, e is adjacent to a^2 since $x \cdot (a^2)^{-1} = a^2 \Rightarrow x = e \in S_2$ such that $xy^{-1} = s$ or $x = sy$. Meanwhile, a^2 is not adjacent to a^6 because $x \cdot (a^6)^{-1} = a^2 \Rightarrow x = a^8 \notin S_2$ which implies that there is not exist $s \in S_2$. Therefore, to find the set of edges, the adjacency of pair of element must be examined first. Hence, the set of edges of $Cay_c(Q_{16}, S_2)$ is:

$$\begin{aligned} E(Cay_c(Q_{16}, S_2)) = \{&\{e,a\},\{e,a^2\},\{e,a^3\},\{e,a^5\},\{e,a^6\},\{e,a^7\},\{e,b\},\{e,ab\},\{e,a^2b\},\{e,a^3b\},\\ &\{e,a^4b\},\{e,a^5b\},\{e,a^6b\},\{e,a^7b\},\{a,b\},\{a,ab\},\{a,a^2\},\{a,a^3\},\{a,a^4\},\\ &\{a,a^6\},\{a,a^7\},\{a,a^2b\},\{a,a^3b\},\{a,a^4b\},\{a,a^5b\},\{a,a^4b\},\{a,a^5b\},\\ &\{a,a^6b\},\{a,a^7b\},\{a^2,b\},\{a^2,ab\},\{a^2,a^3\},\{a^2,a^4\},\{a^2,a^5\},\{a^2,a^7\},\\ &\{a^2,a^2b\},\{a^2,a^3b\},\{a^2,a^4b\},\{a^2,a^5b\},\{a^2,a^6b\},\{a^2,a^7b\},\{a^3,b\},\{a^3,ab\},\\ &\{a^3,a^4\},\{a^3,a^5\},\{a^3,a^6\},\{a^3,a^2b\},\{a^3,a^3b\},\{a^3,a^4b\},\{a^3,a^5b\},\{a^3,a^6b\},\\ &\{a^3,a^7b\},\{a^4,a^5\},\{a^4,a^6\},\{a^4,a^5\},\{a^4,a^6\},\{a^4,a^7\},\{a^4,b\},\{a^4,ab\},\\ &\{a^4,a^2b\},\{a^4,a^3b\},\{a^4,a^4b\},\{a^4,a^5b\},\{a^4,a^6b\},\{a^4,a^7b\},\{a^5,b\},\{a^5,ab\},\\ &\{a^5,a^6\},\{a^5,a^7\},\{a^5,a^2b\},\{a^5,a^3b\},\{a^5,a^4b\},\{a^5,a^5b\},\{a^5,a^6b\},\\ &\{a^5,a^7b\},\{a^6,a^7\},\{a^6,b\},\{a^6,ab\},\{a^6,a^2b\},\{a^6,a^3b\},\{a^6,a^4b\},\{a^6,a^5b\},\\ &\{a^6,a^6b\},\{a^6,a^7b\},\{a^7,b\},\{a^7,ab\},\{a^7,a^2b\},\{a^7,a^3b\},\{a^7,a^4b\}, \end{aligned}$$

$\{a^7, a^5b\}, \{a^7, a^6b\}, \{a^7, a^7b\}\{b, ab\}, \{b, a^2b\}, \{b, a^3b\}, \{b, a^5b\}, \{b, a^6b\},$
$\{b, a^7b\}, \{ab, a^2b\}, \{ab, a^3b\}, \{ab, a^4b\}, \{ab, a^6b\}, \{ab, a^7b\}, \{a^2b, a^3b\},$
$\{a^2b, a^4b\}, \{a^2b, a^5b\}, \{a^2b, a^7b\}, \{a^3b, a^4b\}, \{a^3b, a^5b\}, \{a^3b, a^6b\},$
$\{a^4b, a^5b\}, \{a^4b, a^6b\}, \{a^4b, a^7b\}, \{a^5b, a^6b\}, \{a^5b, a^7b\}, \{a^6b, a^7b\}\}.$

Then, GeoGebra software is used to construct the composite order Cayley graph of Q_{16} by inserting the results for $V(Cay_c(Q_{16}, S_2))$ and $E(Cay_c(Q_{16}, S_2))$.

In the following subsection, based on the set of edges, the adjacency matrix of the composite order Cayley graph of quaternion group of order 16 is determined in order to compute the total energy of the graph.

4.2. The Adjacency Matrix of Composite Order Cayley Graph of Quaternion Group of Order 16

The adjacency matrix of $Cay_c(Q_{16}, S_2)$ is determined by using the group presentation of Q_{16} and the definition of composite order Cayley graph. By Definition 2.2, the entry for the adjacency matrix is 1 when the pair of elements are adjacent by an edge and 0 if the elements are not adjacent by an edge. For an undirected graph, the value $x_{ij} = x_{ji}$ for all i, j.

Therefore, the adjacency matrix of the composite order Cayley graph of Q_{16} is

$$A(Cay_c(Q_{16}, S_2)) = \begin{bmatrix} 0&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1 \\ 1&0&1&1&1&0&1&1&1&1&1&1&1&1&1&1 \\ 1&1&0&1&1&1&0&1&1&1&1&1&1&1&1&1 \\ 1&1&1&0&1&1&1&0&1&1&1&1&1&1&1&1 \\ 0&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1 \\ 1&0&1&1&1&0&1&1&1&1&1&1&1&1&1&1 \\ 1&1&0&1&1&1&0&1&1&1&1&1&1&1&1&1 \\ 1&1&1&0&1&1&1&0&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&0&1&1&1&0&1&1&1 \\ 1&1&1&1&1&1&1&1&1&0&1&1&1&0&1&1 \\ 1&1&1&1&1&1&1&1&1&1&0&1&1&1&0&1 \\ 1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&0 \\ 1&1&1&1&1&1&1&1&0&1&1&1&0&1&1&1 \\ 1&1&1&1&1&1&1&1&1&0&1&1&1&0&1&1 \\ 1&1&1&1&1&1&1&1&1&1&0&1&1&1&0&1 \\ 1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&0 \end{bmatrix}$$

4.3. Computation of the Energy of the Composite Order Cayley Graph of Quaternion group of the Order 16

The eigenvalue of adjacency matrix of graph must be determined first in order to compute the energy of the graph. By using Maple software, the characteristic polynomial of A is

$$f(\lambda) = \lambda^{16} - 112\lambda^{14} - 896\lambda^{13} - 3360\lambda^{12} - 7168\lambda^{11} - 8960\lambda^{10} - 6144\lambda^{9} - 1792\lambda^{8},$$

and the eigenvalues of the characteristic polynomial of the composite order Cayley graph of Q_{16} are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 0$, $\lambda_9 = 14$ and $\lambda_{10} = \lambda_{11} = \lambda_{12} = \lambda_{13} = \lambda_{14} = \lambda_{15} = \lambda_{16} = -2$.

Therefore, the energy of the composite order Cayley graph of Q_{16} is shown below:

$$\varepsilon(Cay_c(Q_{16}, S_2)) = 8|0| + 14 + 7|-2| = 28.$$

5. The Energy of Composite Order Cayley Graph of Quaternion Group of Order 32

5.1. The Composite Order Cayley Graph of Quaternion Group of Order 32

The group presentation of Q_{32} is given by:

$$Q_{32} = \langle a, b \mid a^{16} = e, b^2 = a^8, b^{-1}ab = a^{-1} \rangle.$$

Based on the group presentation, the set of the elements of Q_{32} is listed as follows:

$$\begin{aligned} Q_{32} = \{&e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}, b, ab, a^2b, a^3b, a^4b, \\ &a^5b, a^6b, a^7b, a^8b, a^9b, a^{10}b, a^{11}b, a^{12}b, a^{13}b, a^{14}b, a^{15}b\}. \end{aligned}$$

Next, for quaternion group of order 32, the subset of composite order Cayley graph is chosen to satisfy the condition of subset S_3, that is

$$\begin{aligned} S_3 &= Q_{32} \setminus \{e, a^8\} \\ &= \{a, a^2, a^3, a^4, a^5, a^6, a^7, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, \\ &\quad a^7b, a^8b, a^9b, a^{10}b, a^{11}b, a^{12}b, a^{13}b, a^{14}b, a^{15}b\}. \end{aligned}$$

The set of vertices and edges of $Cay_c(Q_{32}, S_3)$ are listed as follows:

$$\begin{aligned} V\big(Cay_c(Q_{32}, S_3)\big) &= Q_{32} \\ &= \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}, b, ab, a^2b, a^3b, a^4b, \\ &\quad a^5b, a^6b, a^7b, a^8b, a^9b, a^{10}b, a^{11}b, a^{12}b, a^{13}b, a^{14}b, a^{15}b\}. \end{aligned}$$

To determine the set of edges, the adjacency of pair of element must be checked first. Therefore, the set of edges of $Cay_c(Q_{32}, S_3)$ is

$$\begin{aligned} E\big(Cay_c(Q_{32}, S_3)\big) = \{ &\{e,a\},\{e,a^2\},\{e,a^3\},\{e,a^4\},\{e,a^5\},\{e,a^6\},\{e,a^7\},\{e,a^9\},\{e,a^{10}\},\{e,a^{11}\},\{e,a^{12}\}, \\
&\{e,a^{13}\},\{e,a^{14}\},\{e,a^{15}\},\{e,b\},\{e,ab\},\{e,a^2b\},\{e,a^3b\},\{e,a^4b\},\{e,a^5b\},\{e,a^6b\}, \\
&\{e,a^7b\},\{e,a^8b\},\{e,a^9b\},\{e,a^{10}b\},\{e,a^{11}b\},\{e,a^{12}b\},\{e,a^{13}b\},\{e,a^{14}b\}, \\
&\{e,a^{15}b\},\{a,a^2\},\{a,a^3\},\{a,a^4\},\{a,a^5\},\{a,a^6\},\{a,a^7\},\{a,a^8\},\{a,a^{10}\}, \\
&\{a,a^{11}\},\{a,a^{12}\},\{a,a^{13}\},\{a,a^{14}\},\{a,a^{15}\},\{a,b\},\{a,ab\},\{a,a^2b\},\{a,a^3b\},\{a,a^4b\}, \\
&\{a,a^5b\},\{a,a^6b\},\{a,a^7b\},\{a,a^8b\},\{a,a^9b\},\{a,a^{10}b\},\{a,a^{11}b\},\{a,a^{12}b\}, \\
&\{a,a^{13}b\},\{a,a^{14}b\},\{a,a^{15}b\},\{a^2,a^3\},\{a^2,a^4\},\{a^2,a^5\},\{a^2,a^6\},\{a^2,a^7\},\{a^2,a^8\}, \\
&\{a^2,a^9\},\{a^2,a^{11}\},\{a^2,a^{12}\},\{a^2,a^{13}\},\{a^2,a^{14}\},\{a^2,a^{15}\},\{a^2,b\},\{a^2,ab\},\{a^2,a^2b\}, \\
&\{a^2,a^3b\},\{a^2,a^4b\},\{a^2,a^5b\},\{a^2,a^6b\},\{a^2,a^7b\},\{a^2,a^8b\},\{a^2,a^9b\},\{a^2,a^{10}b\}, \\
&\{a^2,a^{11}b\},\{a^2,a^{12}b\},\{a^2,a^{13}b\},\{a^2,a^{14}b\},\{a^2,a^{15}b\},\{a^3,a^4\},\{a^3,a^5\},\{a^3,a^6\}, \\
&\{a^3,a^7\},\{a^3,a^8\},\{a^3,a^9\},\{a^3,a^{10}\},\{a^3,a^{12}\},\{a^3,a^{13}\},\{a^3,a^{14}\},\{a^3,a^{15}\},\{a^3,b\}, \\
&\{a^3,ab\},\{a^3,a^2b\},\{a^3,a^3b\},\{a^3,a^4b\},\{a^3,a^5b\},\{a^3,a^6b\},\{a^3,a^7b\},\{a^3,a^8b\}, \\
&\{a^3,a^9b\},\{a^3,a^{10}b\},\{a^3,a^{11}b\},\{a^3,a^{12}b\},\{a^3,a^{13}b\},\{a^3,a^{14}b\},\{a^3,a^{15}b\}, \\
&\{a^4,a^5\},\{a^4,a^6\},\{a^4,a^7\},\{a^4,a^8\},\{a^4,a^9\},\{a^4,a^{10}\},\{a^4,a^{11}\},\{a^4,a^{13}\},\{a^4,a^{14}\}, \\
&\{a^4,a^{15}\},\{a^4,b\},\{a^4,ab\},\{a^4,a^2b\},\{a^4,a^3b\},\{a^4,a^4b\},\{a^4,a^5b\},\{a^4,a^6b\}, \\
&\{a^4,a^7b\},\{a^4,a^8b\},\{a^4,a^9b\},\{a^4,a^{10}b\},\{a^4,a^{11}b\},\{a^4,a^{12}b\},\{a^4,a^{13}b\}, \\
&\{a^4,a^{14}b\},\{a^4,a^{15}b\},\{a^5,a^6\},\{a^5,a^7\},\{a^5,a^8\},\{a^5,a^9\},\{a^5,a^{10}\},\{a^5,a^{11}\}, \\
&\{a^5,a^{12}\},\{a^5,a^{14}\},\{a^5,a^{15}\},\{a^5,b\}\{a^5,ab\},\{a^5,a^2b\},\{a^5,a^3b\},\{a^5,a^4b\}, \\
&\{a^5,a^5b\},\{a^5,a^6b\},\{a^5,a^7b\},\{a^5,a^8b\},\{a^5,a^9b\},\{a^5,a^{10}b\},\{a^5,a^{11}b\},\{a^5,a^{12}b\}, \\
&\{a^5,a^{13}b\},\{a^5,a^{14}b\},\{a^5,a^{15}b\},\{a^6,a^7\},\{a^6,a^8\},\{a^6,a^9\},\{a^6,a^{10}\},\{a^6,a^{11}\}, \\
&\{a^6,a^{12}\},\{a^6,a^{13}\},\{a^6,a^{15}\},\{a^6,b\}\{a^6,ab\},\{a^6,a^2b\},\{a^6,a^3b\},\{a^6,a^4b\}, \\
&\{a^6,a^5b\},\{a^6,a^6b\},\{a^6,a^7b\},\{a^6,a^8b\},\{a^6,a^9b\},\{a^6,a^{10}b\},\{a^6,a^{11}b\},\{a^6,a^{12}b\}, \\
&\{a^6,a^{13}b\},\{a^6,a^{14}b\},\{a^6,a^{15}b\},\{a^7,a^8\},\{a^7,a^9\},\{a^7,a^{10}\},\{a^7,a^{11}\},\{a^7,a^{12}\}, \\
&\{a^7,a^{13}\},\{a^7,a^{14}\},\{a^7,b\}\{a^7,ab\},\{a^7,a^2b\},\{a^7,a^3b\},\{a^7,a^4b\},\{a^7,a^5b\}, \\
&\{a^7,a^6b\},\{a^7,a^7b\},\{a^7,a^8b\},\{a^7,a^9b\},\{a^7,a^{10}b\},\{a^7,a^{11}b\},\{a^7,a^{12}b\},\{a^7,a^{13}b\}, \\
&\{a^7,a^{14}b\},\{a^7,a^{15}b\},\{a^8,a^9\},\{a^8,a^{10}\},\{a^8,a^{11}\},\{a^8,a^{12}\},\{a^8,a^{13}\},\{a^8,a^{14}\}, \\
&\{a^8,a^{15}\},\{a^8,b\}\{a^8,ab\},\{a^8,a^2b\},\{a^8,a^3b\},\{a^8,a^4b\},\{a^8,a^5b\},\{a^8,a^6b\}, \\
&\{a^8,a^7b\},\{a^8,a^8b\},\{a^8,a^9b\},\{a^8,a^{10}b\},\{a^8,a^{11}b\},\{a^8,a^{12}b\},\{a^8,a^{13}b\}, \end{aligned}$$

$\{a^8, a^{14}b\}, \{a^8, a^{15}b\}, \{a^9, a^{10}\}, \{a^9, a^{11}\}, \{a^9, a^{12}\}, \{a^9, a^{13}\}, \{a^9, a^{14}\},$
$\{a^9, a^{15}\}, \{a^9, b\}\{a^9, ab\}, \{a^9, a^2b\}, \{a^9, a^3b\}, \{a^9, a^4b\}, \{a^9, a^5b\}, \{a^9, a^6b\},$
$\{a^9, a^7b\}, \{a^9, a^8b\}, \{a^9, a^9b\}, \{a^9, a^{10}b\}, \{a^9, a^{11}b\}, \{a^9, a^{12}b\}, \{a^9, a^{13}b\},$
$\{a^9, a^{14}b\}, \{a^9, a^{15}b\}, \{a^{10}, a^{11}\}, \{a^{10}, a^{12}\}, \{a^{10}, a^{13}\}, \{a^{10}, a^{14}\}, \{a^{10}, a^{15}\},$
$\{a^{10}, b\}\{a^{10}, ab\}, \{a^{10}, a^2b\}, \{a^{10}, a^3b\}, \{a^{10}, a^4b\}, \{a^{10}, a^5b\}, \{a^{10}, a^6b\}, \{a^{10}, a^7b\},$
$\{a^{10}, a^8b\}, \{a^{10}, a^9b\}, \{a^{10}, a^{10}b\}, \{a^{10}, a^{11}b\}, \{a^{10}, a^{12}b\}, \{a^{10}, a^{13}b\}, \{a^{10}, a^{14}b\},$
$\{a^{10}, a^{15}b\}, \{a^{11}, a^{12}\}, \{a^{11}, a^{13}\}, \{a^{11}, a^{14}\}, \{a^{11}, a^{15}\}, \{a^{11}, b\}\{a^{11}, ab\}, \{a^{11}, a^2b\},$
$\{a^{11}, a^3b\}, \{a^{11}, a^4b\}, \{a^{11}, a^5b\}, \{a^{11}, a^6b\}, \{a^{11}, a^7b\}, \{a^{11}, a^8b\}, \{a^{11}, a^9b\},$
$\{a^{11}, a^{10}b\}, \{a^{11}, a^{11}b\}, \{a^{11}, a^{12}b\}, \{a^{11}, a^{13}b\}, \{a^{11}, a^{14}b\}, \{a^{11}, a^{15}b\}, \{a^{12}, a^{13}\},$
$\{a^{12}, a^{14}\}, \{a^{12}, a^{15}\}, \{a^{12}, b\}, \{a^{12}, ab\}, \{a^{12}, a^2b\}, \{a^{12}, a^3b\}, \{a^{12}, a^4b\}, \{a^{12}, a^5b\},$
$\{a^{12}, a^6b\}, \{a^{12}, a^7b\}, \{a^{12}, a^8b\}, \{a^{12}, a^9b\}, \{a^{12}, a^{10}b\}, \{a^{12}, a^{11}b\}, \{a^{12}, a^{12}b\},$
$\{a^{12}, a^{13}b\}, \{a^{12}, a^{14}b\}, \{a^{12}, a^{15}b\}, \{a^{13}, a^{14}\}, \{a^{13}, a^{15}\}, \{a^{13}, b\}, \{a^{13}, ab\},$
$\{a^{13}, a^2b\}, \{a^{13}, a^3b\}, \{a^{13}, a^4b\}, \{a^{13}, a^5b\}, \{a^{13}, a^6b\}, \{a^{13}, a^7b\}, \{a^{13}, a^8b\},$
$\{a^{13}, a^9b\}, \{a^{13}, a^{10}b\}, \{a^{13}, a^{11}b\}, \{a^{13}, a^{12}b\}, \{a^{13}, a^{13}b\}, \{a^{13}, a^{14}b\}, \{a^{13}, a^{15}b\},$
$\{a^{14}, a^{15}\}, \{a^{14}, b\}\{a^{14}, ab\}, \{a^{14}, a^2b\}, \{a^{14}, a^3b\}, \{a^{14}, a^4b\}, \{a^{14}, a^5b\}, \{a^{14}, a^6b\},$
$\{a^{14}, a^7b\}, \{a^{14}, a^8b\}, \{a^{14}, a^9b\}, \{a^{14}, a^{10}b\}, \{a^{14}, a^{11}b\}, \{a^{14}, a^{12}b\}, \{a^{14}, a^{13}b\},$
$\{a^{14}, a^{14}b\}, \{a^{14}, a^{15}b\}, \{a^{15}, b\}\{a^{15}, ab\}, \{a^{15}, a^2b\}, \{a^{15}, a^3b\}, \{a^{15}, a^4b\},$
$\{a^{15}, a^5b\}, \{a^{15}, a^6b\}, \{a^{15}, a^7b\}, \{a^{15}, a^8b\}, \{a^{15}, a^9b\}, \{a^{15}, a^{10}b\}, \{a^{15}, a^{11}b\},$
$\{a^{15}, a^{12}b\}, \{a^{15}, a^{13}b\}, \{a^{15}, a^{14}b\}, \{a^{15}, a^{15}b\}, \{b, ab\}, \{b, a^2b\}, \{b, a^3b\}, \{b, a^4b\},$
$\{b, a^5b\}, \{b, a^6b\}, \{b, a^7b\}, \{b, a^9b\}, \{b, a^{10}b\}, \{b, a^{11}b\}, \{b, a^{12}b\}, \{b, a^{13}b\},$
$\{b, a^{14}b\}, \{b, a^{15}b\}, \{ab, a^2b\}, \{ab, a^3b\}, \{ab, a^4b\}, \{ab, a^5b\}, \{ab, a^6b\}, \{ab, a^7b\},$
$\{ab, a^8b\}, \{ab, a^{10}b\}, \{ab, a^{11}b\}, \{ab, a^{12}b\}, \{ab, a^{13}b\}, \{ab, a^{14}b\}, \{ab, a^{15}b\},$
$\{a^2b, a^3b\}, \{a^2b, a^4b\}, \{a^2b, a^5b\}, \{a^2b, a^6b\}, \{a^2b, a^7b\}, \{a^2b, a^8b\}, \{a^2b, a^9b\},$
$\{a^2b, a^{11}b\}, \{a^2b, a^{12}b\}, \{a^2b, a^{13}b\}, \{a^2b, a^{14}b\}, \{a^2b, a^{15}b\}, \{a^3b, a^4b\},$
$\{a^3b, a^5b\}, \{a^3b, a^6b\}, \{a^3b, a^7b\}, \{a^3b, a^8b\}, \{a^3b, a^9b\}, \{a^3b, a^{10}b\}, \{a^3b, a^{12}b\},$
$\{a^3b, a^{13}b\}, \{a^3b, a^{14}b\}, \{a^3b, a^{15}b\}, \{a^4b, a^5b\}, \{a^4b, a^6b\}, \{a^4b, a^7b\}, \{a^4b, a^8b\},$
$\{a^4b, a^9b\}, \{a^4b, a^{10}b\}, \{a^4b, a^{11}b\}, \{a^4b, a^{13}b\}, \{a^4b, a^{14}b\}, \{a^4b, a^{15}b\},$
$\{a^5b, a^6b\}, \{a^5b, a^7b\}, \{a^5b, a^8b\}, \{a^5b, a^9b\}, \{a^5b, a^{10}b\}, \{a^5b, a^{11}b\}, \{a^5b, a^{12}b\},$
$\{a^5b, a^{14}b\}, \{a^5b, a^{15}b\}, \{a^6b, a^7b\}, \{a^6b, a^8b\}, \{a^6b, a^9b\}, \{a^6b, a^{10}b\}, \{a^6b, a^{11}b\},$
$\{a^6b, a^{12}b\}, \{a^6b, a^{13}b\}, \{a^6b, a^{15}b\}, \{a^7b, a^8b\}, \{a^7b, a^9b\}, \{a^7b, a^{10}b\},$
$\{a^7b, a^{11}b\}, \{a^7b, a^{12}b\}, \{a^7b, a^{13}b\}, \{a^7b, a^{14}b\}, \{a^8b, a^9b\}, \{a^8b, a^{10}b\},$
$\{a^8b, a^{11}b\}, \{a^8b, a^{12}b\}, \{a^8b, a^{13}b\}, \{a^8b, a^{14}b\}, \{a^8b, a^{15}b\}, \{a^9b, a^{10}b\},$
$\{a^9b, a^{11}b\}, \{a^9b, a^{12}b\}, \{a^9b, a^{13}b\}, \{a^9b, a^{14}b\}, \{a^9b, a^{15}b\}, \{a^{10}b, a^{11}b\},$
$\{a^{10}b, a^{12}b\}, \{a^{10}b, a^{13}b\}, \{a^{10}b, a^{14}b\}, \{a^{10}b, a^{15}b\}, \{a^{11}b, a^{12}b\}, \{a^{11}b, a^{13}b\},$
$\{a^{11}b, a^{14}b\}, \{a^{11}b, a^{15}b\}, \{a^{12}b, a^{13}b\}, \{a^{12}b, a^{14}b\}, \{a^{12}b, a^{15}b\}, \{a^{13}b, a^{14}b\},$
$\{a^{13}b, a^{15}b\}, \{a^{14}b, a^{15}b\}\ \}$.

Then, the composite order Cayley graph of Q_{32} can be constructed by using GeoGebra software. Next, based on the set of edges, the adjacency matrix of composite order Cayley graph of quaternion group of order 32 is determined in order to compute the total energy of the graph.

5.2. The Adjacency Matrix of Composite Order Cayley Graph of Quaternion Group of Order 32

The adjacency matrix of $Cay_c(Q_{16}, S_2)$ is determined by using the group presentation of Q_{16} and the definition of composite order Cayley graph. Definition 2.2 says that the entry for the adjacency matrix is 1 when the pair of elements are adjacent by an edge. Meanwhile, the entry will be 0 if the elements are not adjacent by an edge. For an undirected graph, the value $x_{ij} = x_{ji}$ for all i, j.

Therefore, the adjacency matrix of the composite order Cayley graph of Q_{32} is
$A(Cay_c(Q_{32}, S_3))$

$$
= \begin{bmatrix}
0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0&1\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&1&1&1&1&1&1&1&0
\end{bmatrix}
$$

5.3. Computation of the Energy of the Composite Order Cayley Graph of the Quaternion Group of Order 32

Based on $A\big(Cay_c(Q_{32}, S_3)\big)$, by using Definition 2.3, the characteristic polynomial, $f(\lambda)$ is obtained. By using Maple software, the characteristic polynomial of A is

$$
\begin{aligned}
f(\lambda) = {} & \lambda^{32} - 480\lambda^{30} - 8960\lambda^{29} - 87360\lambda^{28} - 559104\lambda^{27} - 2562560\lambda^{26} - 8785920\lambda^{25} \\
& -23063040\lambda^{24} - 46858240\lambda^{23} - 73801728\lambda^{22} - 89456640\lambda^{21} - 82001920\lambda^{20} \\
& -55050240\lambda^{19} - 25559040\lambda^{18} - 7340032\lambda^{17} - 983040\lambda^{16}.
\end{aligned}
$$

Next, the eigenvalues of the characteristic polynomial of the composite order Cayley graph of Q_{32} are $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = \lambda_{10} = \lambda_{11} = \lambda_{12} = \lambda_{13} = \lambda_{14} = \lambda_{15} = \lambda_{16} = 0$, $\lambda_{17} = 30$ and $\lambda_{18} = \lambda_{19} = \lambda_{20} = \lambda_{21} = \lambda_{22} = \lambda_{23} = \lambda_{24} = \lambda_{25} = \lambda_{26} = \lambda_{27} = \lambda_{28} = \lambda_{29} = \lambda_{30} = \lambda_{31} = \lambda_{32} = -2$.

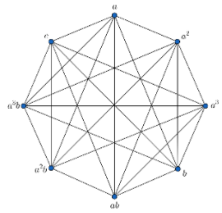
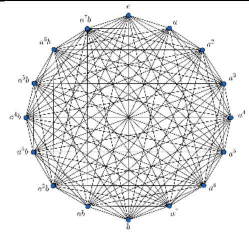
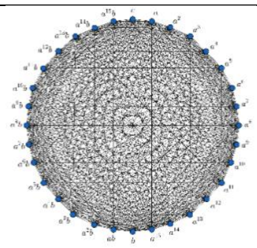
Then, the total energy of the composite order Cayley graph of Q_{32} is shown as below:

$$\varepsilon\big(Cay_c(Q_{32}, S_3)\big) = 16|0| + 30 + 15|-2| = 60.$$

Conclusion

As the conclusion, the energy of the composite order Cayley graphs of quaternion groups of order at most 32 are presented as in Table 1 below.

Table 1: The generalized quaternion groups and the composite order Cayley graphs of quaternion groups of order 8, 16, 32 with their energy of the graph

Group	Order	n	$Cay_c(Q_{2^n}, S_i)$	$\varepsilon\left(Cay_c(Q_{2^n}, S_i)\right)$
$Q_8 = \langle a, b \mid a^4 = e, b^2 = a^2, b^{-1}ab = a^{-1} \rangle$	8	3		12
$Q_{16} = \langle a, b \mid a^8 = e, b^2 = a^4, b^{-1}ab = a^{-1} \rangle$	16	4		28
$Q_{32} = \langle a, b \mid a^{16} = e, b^2 = a^8, b^{-1}ab = a^{-1} \rangle$	32	5		60

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