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Analytical Solution to Temporally Dependent One-Dimensional Advection-Diffusion Equation with Source Term

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Abstract
In advection-diffusion equation, the diffusion or dispersion coefficient and the velocity may occur as a constant, spatially or temporally dependent. The purpose of this research is to find the analytical solution for the case where both dispersion and velocity are temporally dependent. Specifically, to find analytical solutions for one-dimensional advection-diffusion equation with source term of uniform input and increasing nature input. A new time variable which is hyperbolically decreasing function of time is introduced and then solved by using Laplace transformation method. Apart from that, the results for the behavior of the solute concentration distribution in the domain at various parameters is illustrated graphically by using MATLAB.
Keywords: Advection; Diffusion; Dispersion; Laplace Transformation

1. Introduction
Diffusion refers to the movement of a substance from one place to another as a result of random fluctuations while advection is defined as the transport of particles by the bulk motion of the fluid [1]. Diffusive transport can be governed by $\bar{J}_d = D'_d \nabla C$ where $\bar{J}_d$ is the diffusive flux, D'_d is the effective diffusion equation and ∇C is the concentration gradient. The flux of a substance in a particular direction is defined as thequantity of that substance passing through a section perpendicular to that direction per unit areaand per unit time, flux =q = (amount that passes through cross-section)/(cross-sectional area × time duration) =$\frac{m}{A\Delta t}$. Depending on the nature of the transport process, the quantity of material crossing the cross-section over which the count is taken depends on the nature of the transport process [2]. In short, this process is called advection.

The advection-diffusion equation can be solved numerically [3,4,5,6]. In addition, several authors had also solve it analytically [7,8,9,10]. The basic approach was to reduce the diffusion-advection equation into a diffusion equation by eliminating the advection terms. It was done either by introducing moving coordinates [11] or by introducing another dependent variable [12]. The purpose of this study is to find the analytical solution of one-dimensional advective-diffusion for uniform and increasing nature input point source condition. The analytical solution produced will be imported into MATLAB to visualise the behaviour of the concentration at various parameter which includes the dispersion parameter, flow velocity and flow resistance. Furthermore, in this study, a scheme of source term for advection-diffusion equation is proposed where γ represents first order decay rate coefficient while μ represents as zero-order production which is the solute's internal/external production. The first-order decay and zero order production term are considered as time dependent [13].

The solutions to advection-diffusion equation, aid in understanding the contaminant or pollutant concentration distribution behaviour through an open medium such as air, rivers, lakes and a porous medium for example an aquifer, on the basis of which remedial processes to reduce or eliminate damages can be implemented [14].

2. Literature Review

2.1. Diffusion-Advection

In a study by Zoppou and Knight [15], an analytical solutions for advection and diffusion-advection equations with spatially variable coefficients was conducted. Source term are not included in this study. Analytical solutions are provided for one-dimensional transport of a pollutant in an open channel with steady unpolluted lateral inflow uniformly distributed over its whole length. Spatially variable coefficient advection and advection-diffusion equations with the velocity proportional to distance and the diffusion coefficient proportional to the square of the velocity.

Kumar et al. [16] has conducted a study on analytical solutions of one-dimensional advection-diffusion equation with variable coefficients in a finite domain and source term is not included in this study. The analytical solutions initially provide two dipersion problems. The first investigates temporally dependent solute dispersion in a homogeneous region along a uniform flow. Due to the inhomogeneity of the domain, the velocity is regarded spatially dependent and the dispersion is proportional to the square of the velocity is the second problem. The analytical solutions are solved by Laplace transformation technique. New independent space and time variables were introduced during this process.

A generalized analytical solution for one-dimensional solute transport in finite spatial domains with any time-dependent inlet boundary condition and spatial variable coefficients has obtained [17]. Also, source term is considered in this study. Advection, hydrodynamic dispersion, linear equilibrium sorption and first order decay processes are all incuded in the governing equation. The Laplace transform with respect to time and the generalised integral transform technique with respect to the spatial coordinate are used to construct the generalised analytical solution.

The fractional diffusion advection equations with temporally dependent coefficients are generated from a fractional power law for the atter flux [18]. Diffusion processes in several types of porous media with fractal geometry can be accurately simulated using these equations. However, source term is not included in this study. Non-local fractional derivatives appear to be difficult and lack several essential features that regular derivatives have. In addition, this study focuses on obtained a numerical solution for the combination of diffusive flux governed by Fick’s Law and advection flux connected with the velocity field.

Pudasaini et al. [19] has conducted a study in finding some exact and analytical solutions to a nonlinear diffusion-advection model with variable coefficients using four different method which includes boundary layer methods, travelling wave, generalized separation of variables and Lie symmetry. The model’s nonlinearity is related to the quadratic diffusion and advection fluxes in porous media, which are modelled by sub-diffusive and sub-advective fluid flow. The analytical solutions obtained is similar even using the different methods. In contrast to classical linear diffusion and advection, nonlinear diffusion and advection is characterised by a gradually thinning tail that reaches to the rear of fluid and the evolution of a forward advecting frontal bore head.

An analytical solutions have considered spatially, temporally or spatiotemporally dependent dispersion coefficients and velocity [20]. The analytical solutions in this study consider eight different cases of dispersion coefficients and velocity with temporally dependent,spatially dependent spatiotemporally dependent coefficients. The temporally dependence is considered to be exponential, asymptotical or sinusoidal while the the spatially dependence is linear. By using the relevant coordinate transform method, an advection-diffusion equation was reduced to a non-homogeneous diffusion equation and the solutions were obtained by using Green’s function. The analytical solution could be used for various of spatiotemporal patterns of both dispersion coefficient and velocity.

2.2. Laplace Transform

The Laplace transformation is one of the most useful instruments for solving ordinary and partial differential equations and it has had a lot of success. Laplace transformation can be said as more accuracy to justify the analytical solutions and this methods also has been commonly used because of the ability to solve complex problem. Overall, the wide range of practical applications for which the Laplace transform is so well suited addressed [21].

The Laplace transformation method is used to find analytical solutions to the one-dimensional advection-diffusion equation with temporally dependent coefficients which consider two cases, uniform input and input of increasing nature as well as two particular cases, uniform dispersion along unsteady flow and temporally dependent dispersion along uniform flow [22]. In this study, inverse Laplace transform also involves to find the analytical solution.

Buske et al. [23] used the Laplace transform to obtain the solution where the Navier-Stokes equation is coupled to the advection-diffusion equation which the model is determine. In addition , the pollutant concentration also the mean wind field , which is assumed to be the carrier of the pollutant substance. The generalized integral Laplace transform method also involves since this method solved many advection-diffusion problems for pollution dispersion simulation.

A Laplace transform technique was used to generate two alternative analytic solutions to a single diffusion-convection problem [24]. The purpose of this paper was to obtain an analytic solution valid at the end points of the domain of interest. By using Laplace transform method, the solution of diffusion-convection solute transport models can be successfully obtained as part of the end points of the domain of interest solutions. The Laplace transform method also can be applied to more complex solute transport problem with successful solution.

The Laplace transformation approach is used to construct a simple analytical solution for the unsteady advection-dispersion equation describing the pollutant concentration in one dimension . Analytical unsteady solution obtained shows the variation of concentration of pollutant with time and space [25].

3. Mathematical Formulation and Analytical Solutions

3.1. Uniform Input Point Source Condition

Diffusion-advection equation in one dimension with source term is written as,

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D(x,t)\frac{\partial C}{\partial x} - u(x,t)C\right) - \gamma C + \mu \qquad (1)$$

where C is the solute concentration in liquid phase at position x and time t. D is the solute diffusion parameter, u is the velocity of the flow. γ and μ are the source term, which represents the first order and zero-order production of the solute respectively.

For temporally-dipendent dispersion of uniform input,

$$u(x,t) = \frac{u_0}{1+\sinh(mt)} \qquad (2)$$

$$D(x,t) = \frac{D_0}{1+\sinh\,(mt)} \qquad (3)$$

$$\gamma(x,t) = \frac{\gamma_0}{1+\sinh\,(mt)} \qquad (4)$$

$$\mu(x,t) = \frac{\mu_0}{1+\sinh\,(mt)} \qquad (5)$$

where m is the resistance of the flow u_0 is the initial constants of velocity of flow, D_0 is the initial constants of dispersion parameter. γ_0 and μ_0 are initial constants of first order decay and zero order production respectively.

The initial and boundary conditions are

$$C(x,t) = 0 \quad ; \; t = 0, x > 0 \qquad (6)$$

$$C(x,t) = C_0 \quad ; \; t \geq 0, x = 0 \qquad (7)$$

$$\frac{\partial C(x,t)}{\partial x} = 0 \;;\; t \geq 0, x \to \infty \qquad (8)$$

Substitute Equation (2) to (5) into Equation (1)

$$\frac{\partial C}{\partial t} = \frac{1}{1+\sinh(mt)}\left(D_0 \frac{\partial^2 C}{\partial x^2} - u_0 \frac{\partial C}{\partial x} - \gamma_0 C + \mu_0\right) \qquad (9)$$

Introducing a new time variable by a transformation

$$T = \int_0^t \frac{1}{1+\sinh\,(mt)}\, dt \qquad (10)$$

Differentiation on both LHS and RHS of Equation (10) to get

$$\frac{dT}{dt} = (1 + \sinh\,(mt))^{-1} \qquad (11)$$

Reducing Equation (9) by chain rule into the diffusion-advection equation with constant coefficients as

$$\frac{\partial C}{\partial T} = D_0 \frac{\partial^2 C}{\partial x^2} - u_0 \frac{\partial C}{\partial x} - \gamma_0 C + \mu_0 \qquad (12)$$

The initial and boundary condition in terms of new time variable T are

$$C(x,T) = 0 \quad ; \quad T = 0, x > 0 \qquad (13)$$

$$C(x,T) = C_0 \quad ; \quad T \geq 0, x = 0 \qquad (14)$$

$$\frac{\partial C(x,T)}{\partial x} = 0 \quad ; \quad T \geq 0, x \to \infty \qquad (15)$$

The basic approach for solving the diffusion-advection equation is to convert Equation (12) by eliminating the convection term. A new dependent variable $K(x,T)$ is introduced

$$C(x,T) = K(x,T)\exp\left[\frac{u_0}{2D_0}x - \left(\frac{{u_0}^2}{4D_0} + \gamma_0\right)T\right] + \frac{\mu_0}{\gamma_0} \qquad (16)$$

Now, differentiate Equation (16) with respect to T and x and will get

$$\frac{\partial K}{\partial T} = D_0 \frac{\partial^2 K}{\partial x^2} \qquad (17)$$

The initial and boundary value problem in terms of $K(x,T)$ are

$$K(x,T) = -\frac{\mu_0}{\gamma_0} exp\left[-\frac{u_0}{2D_0}x\right]; \quad T = 0, \quad x > 0 \qquad (18)$$

$$K(x,T) = \left(C_0 - \frac{\mu_0}{\gamma_0}\right)\exp[\alpha^2 T]; \quad T \geq 0, \quad x = 0 \qquad (19)$$

$$\frac{\partial K}{\partial x} + \frac{u_0}{2D_0}K = 0; \quad T \geq 0, \quad x \to \infty \qquad (20)$$

where $\alpha^2 = \frac{{u_0}^2}{4D_0} + \gamma_0$.

Method of Laplace transform will be used to solve the diffusion equation. Applying the Laplace transform into Equation (17), (19) and (20) and the general solution obtained as

$$\overline{K}(x,s) = C_1 \exp\left[-\sqrt{\frac{s}{D_0}}\,x\right] + C_2 \exp\left[\sqrt{\frac{s}{D_0}}\,x\right] - \frac{\mu_0}{\gamma_0}\frac{1}{(s-\beta^2)}\exp\left[-\frac{u_0}{2D_0}x\right] \qquad (21)$$

where $\beta^2 = \frac{{u_0}^2}{4D_0}$.

The particular solution of this in the Laplace domain may be obtained as

$$\bar{K}(x,s)=\left[\left(C_0-\frac{\mu_0}{\gamma_0}\right)\frac{1}{(s-\alpha^2)}+\frac{\mu_0}{\gamma_0}\frac{1}{(s-\beta^2)}\right]\exp\left[-\sqrt{\frac{s}{D_0}}\;x\right]-\frac{\mu_0}{\gamma_0}\frac{1}{(s-\beta^2)}\exp\left[-\frac{u_0}{2D_0}x\right] \tag{22}$$

Now, applying the inverse Laplace transformation on Equation (22), will give $K(x,T)$ which hence $C(x,T)$ may be obtained. Based on Kumar (2017), the final solution of temporally dependent dispersion of uniform input is

$$C(x,T)=\frac{\mu_0}{\gamma_0}+\left(C_0-\frac{\mu_0}{\gamma_0}\right)F(x,T)-\frac{\mu_0}{\gamma_0}G(x,T) \tag{23}$$

where

$$F(x,T)=\frac{1}{2}\exp\left[\frac{\left\{u_0-\left(u_0{}^2+4D_0\gamma_0\right)^{1/2}\right\}x}{2D_0}\right]\operatorname{erfc}\left(\frac{x-\left(u_0{}^2+4D_0\gamma_0\right)^{\frac{1}{2}}T}{2\sqrt{D_0}\,T}\right)$$
$$+\frac{1}{2}\exp\left[\frac{\left\{u_0+\left(u_0{}^2+4D_0\gamma_0\right)^{\frac{1}{2}}\right\}x}{2D_0}\right]\operatorname{erfc}\left(\frac{x+\left(u_0{}^2+4D_0\gamma_0\right)^{\frac{1}{2}}T}{2\sqrt{D_0}\,T}\right)$$

and

$$G(x,T)=\exp(-\gamma_0T)\left[1-\frac{1}{2}\operatorname{erfc}\left(\frac{x-u_0T}{2\sqrt{D_0}\,T}\right)-\frac{1}{2}\exp\left(\frac{u_0}{D_0}x\right)\operatorname{erfc}\left(\frac{x+u_0T}{2\sqrt{D_0}\,T}\right)\right]$$

3.2. Increasing Nature Input Point Source Condition

The mathematical formulation is the same as in (1) to (8) but the increasing nature input introduced at the origin of the domain will make the boundary condition becomes

$$-D(x,t)\frac{\partial C}{\partial x}+u(x,t)C=u_0C_0;\quad x=0,\quad t>0 \tag{24}$$

Using Equation (2), (3) and (10) the above condition obtained as

$$-D_0\frac{\partial C}{\partial x}+u_0C=u_0C_0;\quad x=0,\quad T>0 \tag{25}$$

Now, by reducing Equation (25) becomes

$$-D_0\frac{\partial K}{\partial x}+K\left(\frac{u_0}{2}\right)=u_0\left(C_0-\frac{\mu_0}{\gamma_0}\right)\exp[\alpha^2T];\quad x=0,\quad T>0 \tag{26}$$

Applying the Laplace transform on Equation (26) will get

$$-D_0\frac{\partial\bar{K}}{\partial x}+\bar{K}\left(\frac{u_0}{2}\right)=u_0\left(C_0-\frac{\mu_0}{\gamma_0}\right)\frac{1}{s-\alpha^2};\quad x=0 \tag{27}$$

Now, differentiate Equation (21) then substitute into Equation (27) and will get the particular solution in the Laplace domain as

$$\bar{K}(x,s)=\left(\frac{u_0}{\sqrt{D_0}}\left(C_0-\frac{\mu_0}{\gamma_0}\right)\frac{1}{(s-\alpha^2)}\frac{1}{(\sqrt{s}+\beta)}+\frac{u_0}{\sqrt{D_0}}\frac{\mu_0}{\gamma_0}\frac{1}{(s-\beta^2)}\frac{1}{(\sqrt{s}+\beta)}\right)\exp\left[-\sqrt{\frac{s}{D_0}}\;x\right]-\frac{\mu_0}{\gamma_0}\frac{1}{(s-\beta^2)}\exp\left[-\frac{u_0}{2D_0}x\right] \tag{28}$$

Applying inverse Laplace transform into the Equation (28) will give $K(x,T)$ which hence $C(x,T)$ may be obtained as

$$C(x,T)=\frac{\mu_0}{\gamma_0}+\left(C_0-\frac{\mu_0}{\gamma_0}\right)F(x,T)-\frac{\mu_0}{\gamma_0}G(x,T) \tag{29}$$

where

$$F(x,T)=\frac{u_0}{\left\{u_0+\left(u_0{}^2+4\gamma_0D_0\right)^{\frac{1}{2}}\right\}}\exp\left[\left(\frac{\left\{u_0+\left(u_0{}^2+4\gamma_0D_0\right)^{\frac{1}{2}}\right\}}{2D_0}\right)x\right]\operatorname{erfc}\left[\frac{\left\{x-\left(u_0{}^2+4\gamma_0D_0\right)^{\frac{1}{2}}T\right\}}{2\sqrt{D_0T}}\right]$$

$$+\frac{u_0}{\left\{u_0+(u_0{}^2+4\gamma_0 D_0)^{\frac{1}{2}}\right\}}\exp\left[\left(\frac{\left\{u_0-(u_0{}^2+4\gamma_0 D_0)^{\frac{1}{2}}\right\}}{2D_0}\right)x\right]\mathrm{erfc}\left[\frac{\left\{x-(u_0{}^2+4\gamma_0 D_0)^{\frac{1}{2}}T\right\}}{2\sqrt{D_0 T}}\right]+\frac{u_0{}^2}{2\gamma_0 D_0}\exp\left(\frac{u_0}{D_0}x-\gamma_0 T\right)\mathrm{erfc}\left[\frac{\{x+u_0 T\}}{2\sqrt{D_0 T}}\right]$$

and

$$G(x,T)=\exp\{-\gamma_0 T\}1-\frac{1}{2}\mathrm{erfc}\left\{\frac{\{x-u_0 T\}}{2\sqrt{D_0 T}}\right\}-\left(\frac{u_0{}^2}{\pi D_0}T\right)^{\frac{1}{2}}\exp\left\{\frac{-(x-u_0 T)^2}{4D_0 T}\right\}$$
$$+\frac{1}{2}\left[1+\frac{u_0 x}{D_0}+\frac{u_0{}^2}{D_0}T\right]\exp\left\{\frac{u_0 x}{D_0}\right\}\mathrm{erfc}\left\{\frac{\{x+u_0 T\}}{2\sqrt{D_0 T}}\right\}$$

4. Results and discussion

The results of the behaviour of the solute concentration distribution in the domain at various parameter for both cases are illustrated graphically by using MATLAB.

4.1. Uniform Input

The curves in Figure 1 until Figure 4 represent the concentration value in a finite domain of $0 \le x \le 10$ with $\gamma_0 = 0.02\ (\mathrm{day})^{-1}$, $\mu_0 = 0.001\ (\mathrm{kg/meter^3 day})$ and $C_0 = 1.0$ are chosen while m, u_0 and D_0 are change depending on the investigation.

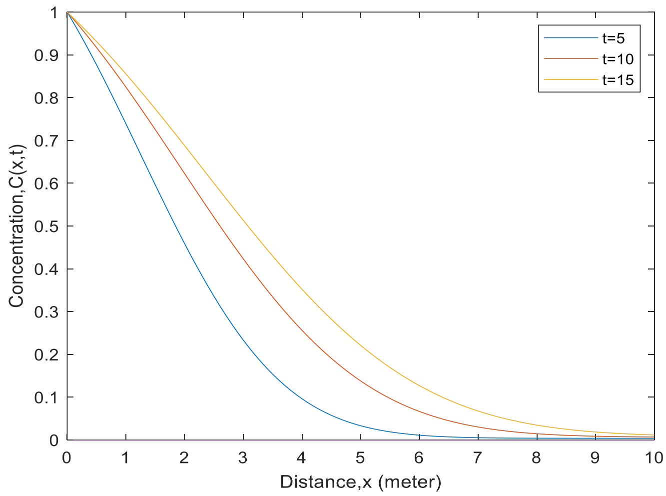

Figure 1 Comparison of solute concentration distributions of Equation (3.23) at $t\ (\mathrm{day}) = 5, 10$ and 15.

Figure 1 depicts the concentration values for temporally dependent dispersion of uniform input at different times $t\ (\mathrm{day}) = 5, 10$ and 15. The input value of $m(\mathrm{day})^{-1} = 0.1$, $u_0(\mathrm{meter/day}) = 0.25$ and $D_0\ (\mathrm{meter^2/day}) = 0.45$ are chosen. It is observed that, as t increases, the concentration value at respective position will also increases (become higher). Alternatively it might be claimed that the concentration value is lower at slower time and higher for a longer time.

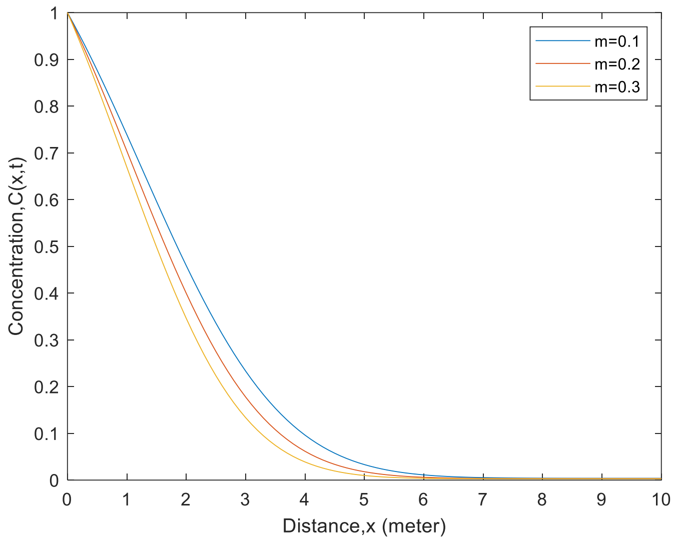

Figure 2 Comparison of solute concentration distributions for different flow resistance $m(\text{day})^{-1} = 0.1$, 0.2 and 0.3 at t $(\text{day}) = 5$.

Figure 2 illustrates the concentration profile on different flow resistance for $m(\text{day})^{-1} = 0.1$, 0.2 and 0.3 at t $(\text{day}) = 5$. For this investigation, D_0 $(\text{meter}^2/\text{day}) = 0.45$, and $u_0(\text{meter}/\text{day}) = 0.25$ are chosen. It can be seen that the concentration level lower for higher flow resistance. Alternatively it can be claimed that the concentration level is higher for lower flow resistance.

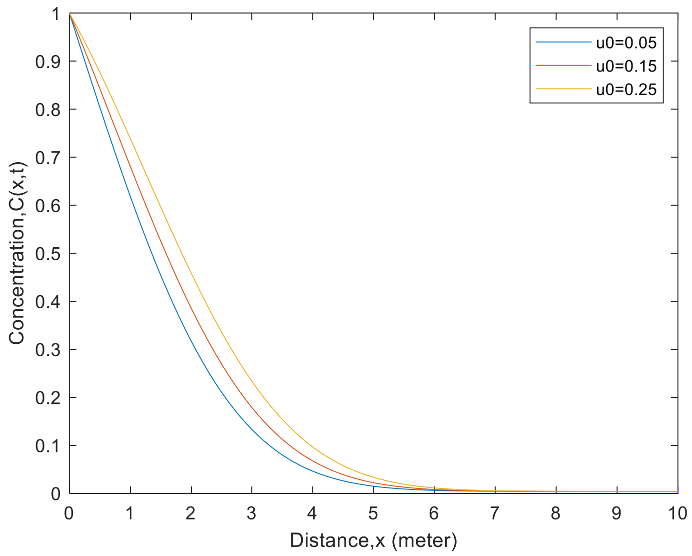

Figure 3 Comparison of solute concentration distributions for different flow velocity, $u_0(\text{meter}/\text{day}) = 0.05, 0.15$ and 0.25 at t $(\text{day}) = 5$.

Figure 3 shows the concentration distribution in the domain for varied flow velocity, $u_0(\text{meter}/\text{day}) = 0.05, 0.15$ and 0.25 at t $(\text{day}) = 5$. For this investigation, D_0 $(\text{meter}^2/\text{day}) = 0.45$ and $m(\text{day})^{-1} = 0.1$ are chosen. It is found that the concentration level is decreases faster for lower velocity as x increases or it can also be said that concentration level is higher for higher velocity at particular distance, x.

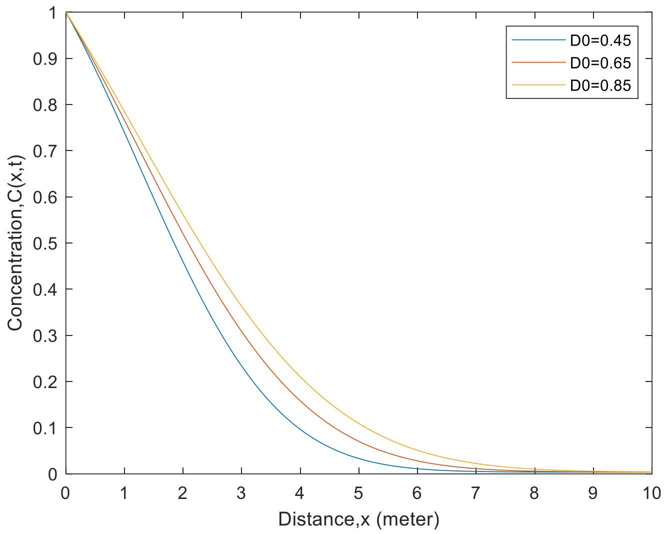

Figure 4 Comparison of solute concentration distributions for various dispersion parameter, D_0 $(\text{meter}^2/\text{day}) = 0.45$, 0.65 and 0.85 at t $(\text{day}) = 5$.

Besides, Figure 4 represents the effects on concentration distribution of various dispersion parameter, D_0 $(\text{meter}^2/\text{day}) = 0.45$, 0.65 and 0.85 at particular time, velocity and resistance where t $(\text{day}) = 5$, $u_0(\text{meter/day}) = 0.25$ and $m(\text{day})^{-1} = 0.1$ respectively. From the figure, it is noticed that the concentration level is lower for lower velocity as x increases alternatively it can be said that the concentration value is higher for higher flow velocity.

4.2. Increasing Nature Input

The curves in Figure 5 until Figure 8 represent the concentration value in a finite domain of $0 \leq x \leq 10$ with $\gamma_0 = 0.02$ $(\text{day})^{-1}$, $\mu_0 = 0.001$ $(\text{kg/meter}^3\text{day})$ and $C_0 = 1.0$ are chosen while m, u_0 and D_0 are changes depends on the investigation.

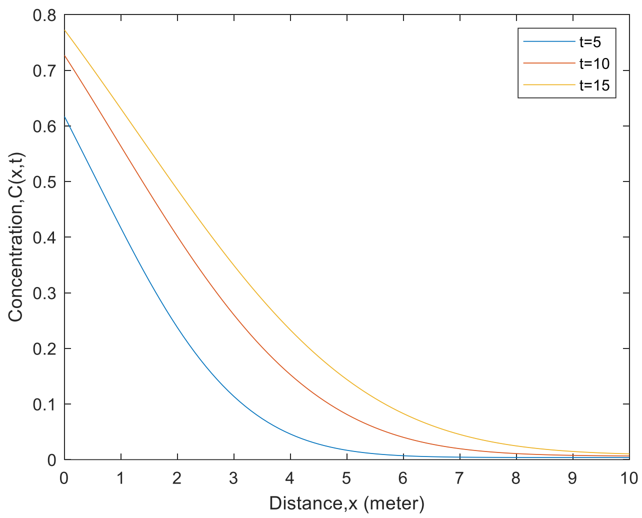

Figure 5 Comparison of solute concentration distributions of Equation (3.9) at t $(\text{day}) = 5, 10$ and 15.

Figure 5 depicts the results of final solution as in Equation (3.48) which is the concentration values for temporally dependent dispersion of increasing nature input at different times t $(\text{day}) = 5, 10$ and 15. The input values of $u_0 = 0.25$, $D_0 = 0.45$, $m = 0.1$ are chosen. It is observed that the longer the time taken is, the higher the concentration as the curve move along $x = 0$ to the $x = 10$ or it also can be said the concentration value is lower at slower time.

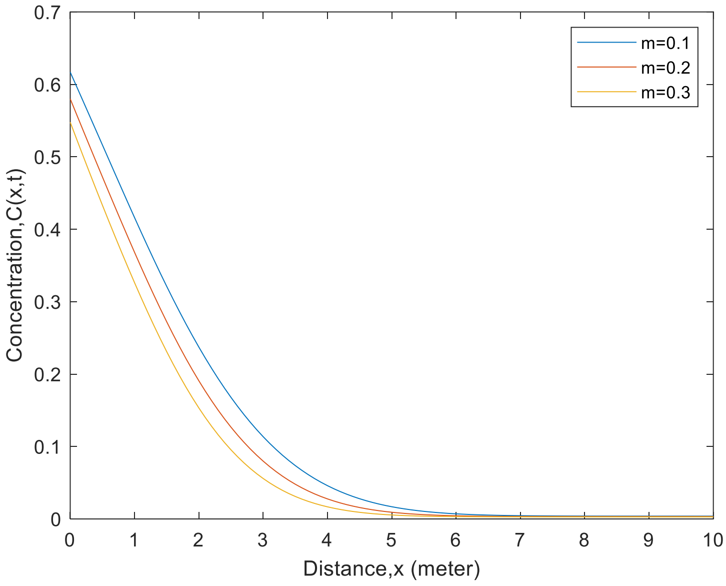

Figure 6 Comparison of solute concentration distributions for different flow resistance $m(\text{day})^{-1} = 0.1$, 0.2 and 0.3 at t $(\text{day}) = 5$.

Figure 6 illustrates the concentration profile on different flow resistance for $m(\text{day})^{-1} = 0.1$, 0.2 and 0.3 at t $(\text{day}) = 5$. It can be seen that the concentration level is lower for higher flow resistance and alternatively it can be claimed that the concentration higher for lower flow resistance. For this investigation, D_0 $(\text{meter}^2/\text{day}) = 0.45$ and $u_0(\text{meter}/\text{day}) = 0.25$ are used.

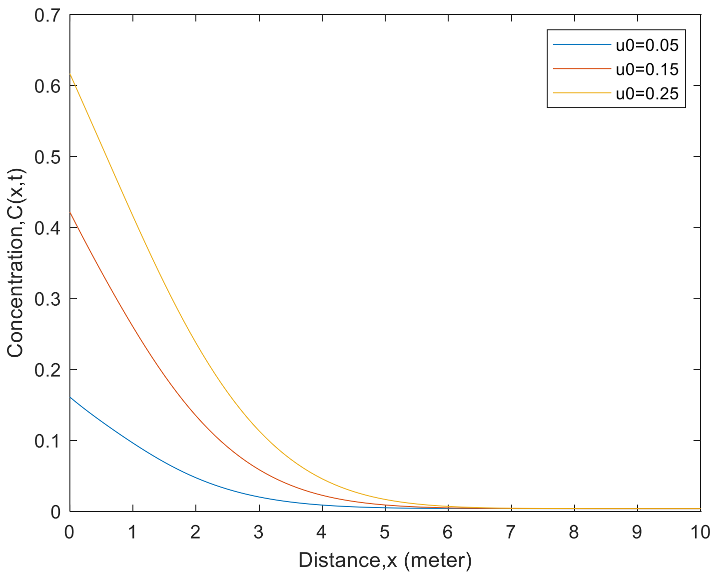

Figure 7 Comparison of solute concentration distributions for different flow velocity, $u_0(\text{meter}/\text{day}) = 0.05, 0.15$ and 0.25 at t $(\text{day}) = 5$.

Figure 7 shows the concentration distribution in the domain for varied flow velocity, $u_0(\text{meter}/\text{day}) = 0.05, 0.15$ and 0.25 at t $(\text{day}) = 5$. For this investigation, D_0 $(\text{meter}^2/\text{day}) = 0.45$ and $m(\text{day})^{-1} = 0.1$ are used. It is found that the concentration is decreases faster for lower flow velocity or it can also be said that concentration is higher for higher flow velocity.

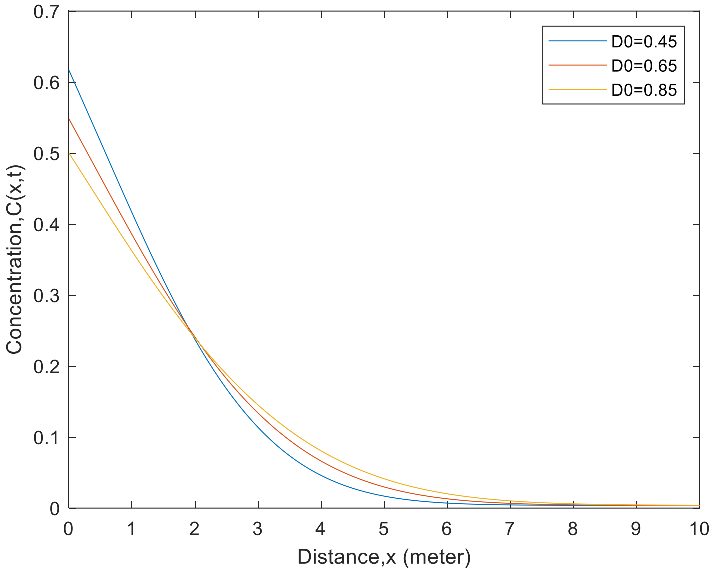

Figure 8 Comparison of solute concentration distributions for various dispersion parameter, $D_0\ (\text{meter}^2/\text{day}) = 0.45, 0.65$ and 0.85 at $t\ (\text{day}) = 5$.

Based on Figure 8, it represents the effects on concentration distribution of various dispersion parameter, $D_0\ (\text{meter}^2/\text{day}) = 0.45$, 0.65 and 0.85 at particular time, $t\ (\text{day}) = 5$. For this investigation, flow resistance and flow velocity which $m(\text{day})^{-1} = 0.1$ and $u_0(\text{meter}/\text{day}) = 0.25$ are used respectively. From the figure, it is noticed that the concentration level is lower for lower dispersion coefficient while at higher dispersion coefficient, the concentration level is higher as the distance increases.

Conclusion

The analytical solutions to a one-dimensional advection-diffusion equation with source term have been obtained by considering two cases, uniform input and increasing nature input condition where both of solute dispersion and velocity are temporally dependent. In addition, the solute dispersion is considered proportional to the velocity. The application of new transformation to the diffusion-advection equation, which introduces spatially dependent variables allows using the Laplace transformation method to obtain the analytical solution. It can be conclude that such analytical solution can be used in realistic dispersion problem in order to manage the pollutant distributions along ground water, surface water and air flow domains.

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