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Solving Wave Equation In a Circular Membrane of a Drum

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Abstract
Drum is one of the most popular instruments and exists since 5500 BC until today. The drum can be related to mathematics to help the musicians in finding the perfect steady beat. Therefore, this paper proposed a visualization system of the amplitude of a drum head by calculating their eigen-frequencies and eigen-wavenumbers using different modes of transverse standing waves on a circular membrane. The separation of variables method is used to derive the equation of the transverse standing waves on a circular membrane. The graph of Bessel's function is plotted using Mathematica and is used to visualize the system of the amplitude of a drum head. It is showed that the frequency and form of distinct modes are unaffected at beginning velocities and displacements. The first six modes' amplitudes are determined, and the displacements of the first three modes are graphically showed. The mode shape is observed to be invariant regardless of the applied initial displacement and velocity
Keywords: drum; separation of variables; amplitude; wave equation; circular membrane

1. Introduction

People have been making music using sound for thousands of years. They have come up with a vast range of musical instruments. Vibrating materials make sound in all musical instruments. As a result of the vibrations, sound waves travel through the air. Most musical instruments use resonance to amplify and increase the volume of sound waves. Resonance occurs when an object vibrates in response to sound waves of a given frequency. When a musical instrument's head, such as a drum, is struck, the entire instrument, as well as the air within it, may vibrate. Most musical instruments have the ability to adjust the frequency of their sound waves. This changes the pitch of the sound, or how high or low it seems to the listener.

Drumming has a long and rich history, with origins in a wide range of civilizations around the globe. Drumming has been used for a number of purposes throughout history, including religious rites, psychological well-being, and healing [1].

Derivation of the solutions of the two-dimensional wave equation with circular boundary conditions, which reflect the frame constraint and describe the vibrations of the membrane.Measuring on a drum shows that the shape of the drum has a significant impact on its acoustic qualities which can be seen through its visualization in Mathematica.

This research aims to (1) derive one-dimensional wave equation for a circular domain, (2) analytically solve the wave equation for a circular domain by using the method of separation of variables and (3) plot the solution by using Mathematica and analyze its physical interpretation.

2. Literature Review

2.1 Introduction

In the DCT (discrete cosine transform) transform domain, drum modes are represented as chirplike signals, which is a significantly simpler representation than the time domain signal itself. The Hilbert transform can convert these chirps into an amplitude and phase function representation that is much simpler. Both of these signals showed a distinct progression with strike velocity, reflecting changes in strength, pitch glide, and decay of the modal oscillation [2].

2.2 Finite Difference

With tension T, area density $(x; y = m(x; y)) = A$, and damping constant D with displacement u, the drum membrane is handled as a Finite Difference Time Domain (FDTD) model. The density of the area is determined by the mass *m(x,y)*, which has a geographical distribution determined by the extra paste. When it came to modes, just one membrane model was used. To account for the influence of the back membrane and enclosed air on the fundamental frequency amplitudes, the back membrane and enclosed air are still included in the model. The back membrane is simulated similarly to the front membrane, with the exception that viscoelastic damping was removed when viscoelastic damping was added, as described above [3].

2.3 Separation of Variable

The separation of variable method is one of the general approaches for addressing the boundary value problem of many forms of linear partial differential equations (also known as Fourier method). The probabilistic solution of the variable is substituted into a partial differential equation, which is then divided into many ordinary differential equations in the separation of variable approach. The separation variable approach can only tackle linear issues because it is based on the linear superposition concept. In the Cartesian coordinate system, spherical coordinate system, cylindrical coordinate system, and other coordinate systems, the separation variable approach can be used [4].

Suppose a differential equation has the form:

$$\frac{d}{dx}f(x) = j(x)k(f(x)) \tag{1}$$

Then let $y = f(x)$, the equation will become

$$\frac{dy}{dx} = j(x)k(y) \tag{2}$$

It must $k(y) \neq 0$, so we can separate x with dx and y with dy then rearrange the equation into

$$\frac{dy}{k(y)} = j(x)dx \tag{3}$$

Integrating both sides of the equation with respect to y and x.

$$\int \frac{1}{k(y)}dy \quad = \int j\,(x)dx \tag{4}$$

$$\int \frac{1}{k(y)}dy \ + C_1 = \int j(x)dx \ + C_2 \tag{5}$$

Where C_1 and C_2 are constants of integration.

3. Methodology

3.1 Stage of research methodology

Stage 1: Transform the 2-dimensions of wave equation (in rectangular cartesian coordinates) into polar coordinates.

Stage 2: Use separation variables method to solve the partial differential equation in two independent variables.

Stage 3: We will get the special case of Bessel equation then will use *Mathematica* to solve and plot the graph of the function

Stage 4: Every information from the Bessel function graph such as eigen- wavenumbers and eigenfrequencies will be plotted in *Mathematica* to get the visualization

3.2 Mathematical Formulation

3.2.1 Transvers Standing Waves on a Circular Membrane

The general equation that describes the wave equation is given by

$$\nabla^2 u(\vec{r},t) - \left(\frac{2}{v^2}\right)\left(\frac{\partial^2 \Psi(\vec{r},t)}{\partial t^2}\right) = 0 \tag{6}$$

where □ ($\vec{r},t$)is the displacement amplitude of the wave at space position ($\vec{r}$) with time *t*, from its equilibrium position, *v* represents the longitudinal speed of propagation of the wave and $\nabla^2$is the Laplacian operator.

As the drumhead has circular geometry, the wave equation is written in polar cylindrical coordinates. This means the Cartesian coordinate (*x,y*) is transformed to the (*r,φ*) for the displacement amplitude, *u(r,φ,t)* is given by:

$$\nabla^2 u\,(r,\varphi,t) - \frac{1}{v^2}\;\frac{\partial^2 u(r,\varphi,t)}{\partial t^2} = 0 \;. \tag{7}$$

In cylindrical polar coordinates, the first term in Eq. (3.2) is

$$\nabla^2 u\,(r,\theta,t) = \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\,, \tag{8}$$

$$\Rightarrow \;= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\,. \tag{9}$$

Laplacian operator∇^2,is given by

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial a}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

The two-dimensional wave equation describing the behaviour of waves on a cylindrical membrane is rewrite as:

$$\frac{\partial^2 u(r,\theta,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u\,(r,\theta,t)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 (r,\theta,t)}{\partial \theta^2} - \frac{1}{v^2}\frac{\partial^2 u(r,\theta,t)}{\partial t^2} = 0. \tag{10}$$

The circularly symmetric solutions of the wave equation are only dependent on *r* and *t*, not on θ. In other words, the elimination of a coordinate from the field variable indicates symmetry, then wave equation is reduce to

$$\frac{\partial^2 u(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,t)}{\partial r} - \frac{1}{v^2}\frac{\partial^2 u(r,t)}{\partial t^2} = 0.$$

By using separation of variable method, assume

$$u(r,t) = R(r)T(t). \tag{11}$$

$$\frac{d^2R}{dr^2}T + \frac{1}{r}\frac{dR}{dr} - \frac{1}{v^2}R\frac{d^2T}{dt^2} = 0. \tag{12}$$

Then dividing through *u(r,t)=R(r)T(t)* both sides give

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} = \frac{1}{v^2T}\frac{d^2T}{dt^2}. \tag{13}$$

The left and right sides are independent in this case, which can only be the case if they are both equal to a constant. Let the constant as $-k^2$for simplicity, to satisfy the boundary conditions.
Let the left hand side as constant$-k^2$,

$$\frac{d^2T}{dt^2}\frac{1}{v^2T} = -k^2 \;\Rightarrow\; \frac{d^2T}{dt^2} + k^2v^2T = 0 \tag{14}$$

Equation (3.3) is a second order ordinary differential equation. It also can be used to represent a simple harmonic oscillator.
Let right hand side as constant$-k^2$:
Then,

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{rR}\frac{dR}{dr} = -k^2 \Rightarrow \frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + k^2R = 0 \tag{15}$$

The first equation, for the time function, is nothing new where it is the simple harmonic oscillator equation.
This has solutions

$$T_k(t) = A\cos(kvt) + B\sin(kvt) \tag{16}$$

The second equation, for the radial function is new. It's a linear ordinary differential equation, but the coefficients are not a constant. The answers get a little more difficult as a result of this. The substitution $s = Rr$ can be used to simplify the problem. The variable *R* will be removed from the equation as a result of this. The substitution yields the following *R* as a function of s equation:

$$\frac{d^2R}{ds^2} + \frac{1}{s}\frac{dR}{ds} + R = 0 \tag{17}$$

This is a special case of Bessel’s equation

$$R = A\,Bessel\,J\,[O,S] + BBesel\,Y[O,S] \tag{18}$$

Here *A* and *B* are constants and they multiply Bessel functions

$$R = AJ_0(s) + B\,Y_0(s) \tag{19}$$

Therefore the boundary condition is at the edge of the drum head is fixed so that it cannot be displaced

$$J_0(s) = J_0(kR) = 0. \tag{20}$$

Since *kR* must correspond to a zero of the Bessel function, this specifies the possible values of *k*. To put it another way, if α_m are the values of *s* for which $J_0(s)$is zero, then the possible values of *k* are

$$k_m = \frac{\alpha_m}{R} \Longrightarrow R_m(r) = J_0(\frac{\alpha_m}{R}r) \tag{21}$$

Finally combining equations 3.13 and 3.17, the solution for the wave equation (3.6) is

$$u_m(r,t) = \{a_m \cos(k_m vt) + b_m \sin(k_m vt)\}J_0\left(\frac{a_m}{R}r\right) \tag{22}$$

The general solution is a linear combination all these

$$u(r,t) = \sum_{m=1}^{\infty}\{a_m \cos(k_m vt) + b_m \sin(k_m vt)\}\,J_0\left(\frac{a_m}{R}r\right). \tag{23}$$

The coefficients $\alpha_m$and $b_m$are determined from the initial conditions. It is specified as

$$u(r,o) = f(r), \tag{24}$$

$$\frac{\partial u}{\partial t}|_{t=0} = g(r). \tag{25}$$

Substitute the general solution into these expressions, it gives the initial functions $f(r)$ and $g(r)$ in terms of the coefficients α_m and b_m.

$$\sum_{m=1}^{\infty} a_m J_0\left(\frac{\alpha_m}{R} r\right) = f(r), \tag{26}$$

$$\sum_{m=1}^{\infty} b_m k_m v J_0\left(\frac{\alpha_m}{R} r\right) = g(r). \tag{27}$$

In terms of the coefficients a_m and b_m, these formulas give the initial functions *f(r)* and *g(r)* by and therefore, inverting this result, it shows the coefficients a_m and b_m in terms of the starting functions *f(r)* and *g(r).* Then complete solution for the drum head (circularly symmetric) is given by:

$$u(r,t) = \sum_{m=1}^{\infty}\{a_m \cos(k_m v t) + b_m \sin(k_m v t)\} J_0\left(\frac{a_m}{R} r\right) \tag{28}$$

4. Results and discussion

4.2 Frequency

Striking in the ring creates a large increase in amplitudes within the ring when compared to striking at the ring rim or outside the ring. Above 400 Hz, this rise is still discernible. We can deduce that low frequencies are unaffected by the ring because the drum's fundamental frequency is 34 Hz, but higher frequencies are [3].

4.3 Normal Mode

If the ratio of the radius is roughly equal to the ratio of two zero-crossings of the same Bessel function, an approximate solution can be determined. The nodal circles for those modes can be considered to coincide with the inner and outer radii of the annular membrane if output and input have the same ratio of two zero-crossings of some Bessel function [5].

4.4 Bessel Function Graph

The number of nodal diameters (m) and nodal circles are used to classify modes (*n*). This is written as (*m*, *n*), where m denotes the fundamental frequency and *n* is the interest frequency (0,1) [6].

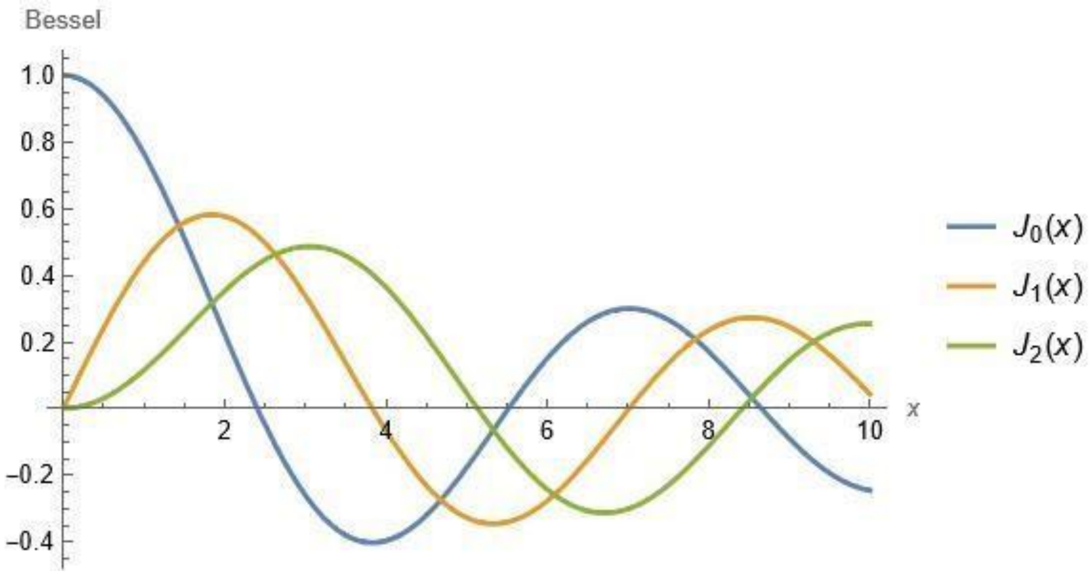


Figure 4.1

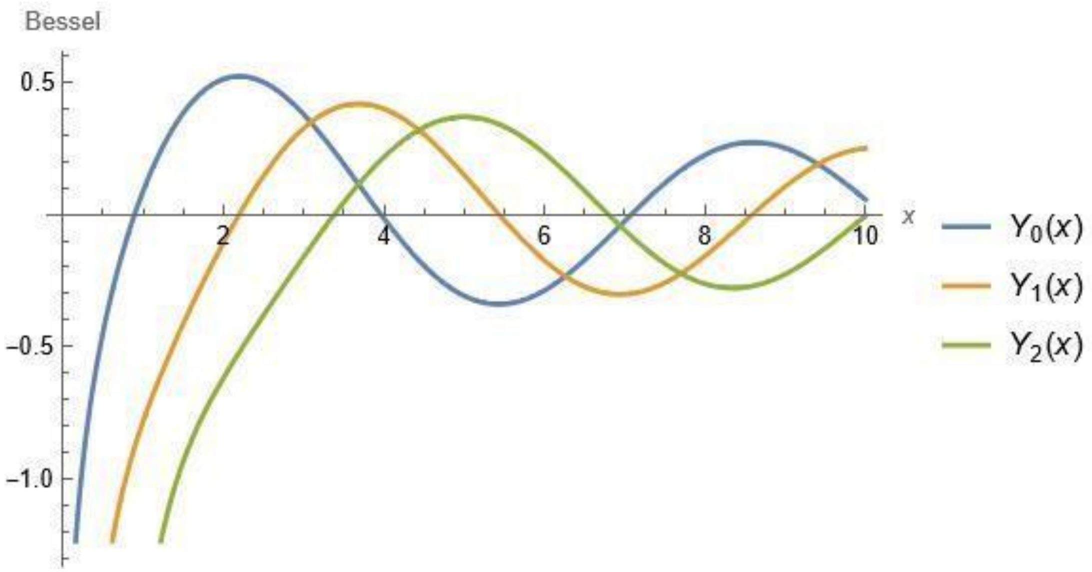


Figure 4.2

The first few zeroes of the low-order $J_m(x)$:

		$n = 1$	$n = 2$	$n = 3$
m	$J_m(x)$	x_1	x_2	x_3
0	0	2.40	5.52	8.65
1	0	3.83	7.02	10.17
2	0	5.14	8.42	11.62

Table 4.1

m=0, n=1: $k_{0,1} \approx 2.40/R \quad \omega_{0,1} \approx 2.40v/R$

m=1, n=1 $k_{1,1} \approx 3.83/R$: $\omega_{1,1} \approx 3.83v/\mathrm{R}$

m=2, n=1: $k_{2,1} \approx 5.14/R \quad \omega_{2,1} \approx 5.14v/R$

m=0, n=2: $k_{0,2} \approx 5.52/R \quad \omega_{0,2} \approx 5.52v/R$

4.5 Visualization by Mathematica

The normal mode frequencies, are related to the speed of transverse waves, and the radius, of the membrane through the wavenumber.

$$k_{m,n}R = \frac{\omega_{m,n}R}{c} = \frac{2\pi f_{m,n}R}{c} = j_{m,n} \Rightarrow f_{m,n} = \frac{j_{m,n}c}{2\pi R} \tag{29}$$

The mode shapes corresponding to these normal mode frequencies for a few low-order modes are visualized below along with the corresponding values of $j_{m,n} = k_{m,n}a$ and the ratio of the modal frequency, $f_{m,n}$, to the frequency of the lowest pure radial mode, $f_{0,1} \cong \left(\frac{2.40483}{2x}\right)\left(\frac{c}{R}\right) = 0.38274\left(\frac{c}{R}\right)$.

Lower order eigenmodes of vibration for transverse standing wave on a circular membrane on two dimensional visualization:

Two-Dimensional Visualization

The normal mode frequencies, are related to the speed of transverse waves, and the radius, of the membrane through the wave number.

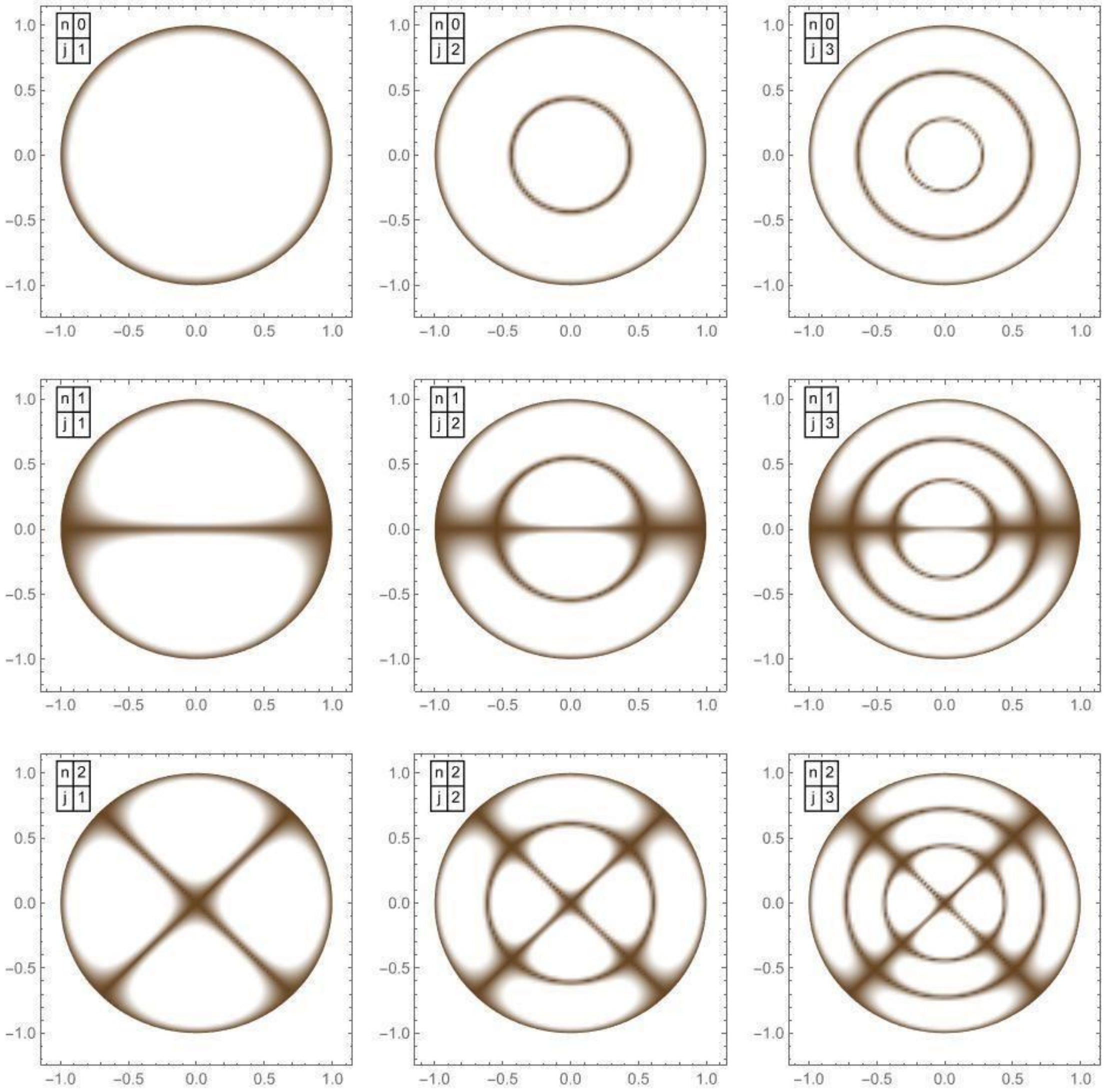
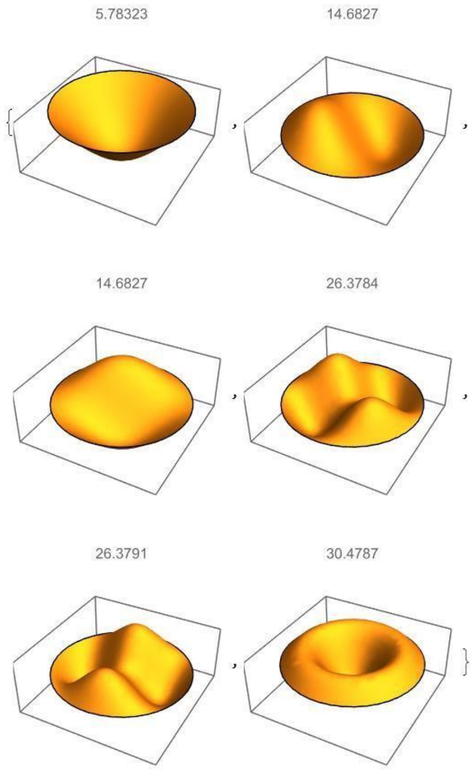


Figure 4.3

Three-Dimensional Visualization



The mode shapes for a circular membrane with radius R that are greatly exaggerated. The top row contains the first "pure radial" J_0 modes: (0, 1) and (0, 2). The middle row shows the first two J_1 modes, each with a single nodal diameter: (1, 1) and (1, 2). The bottom row has the first two J_2 modes with two perpendicular nodal diameters: (2, 1) and (2, 2). Above each mode shape is their corresponding value of $J_{m,n} = k_{m,n}R$.

The vibration of a circular membrane is investigated for various initial velocity and initial displacements. As can be seen, the mode form is unaffected by the initial displacement and speed that are applied. The amplitude of vibration of modes is the parameter that alters with the application of initial displacement and velocity. The relationship between the wave's amplitude and frequency is such that it is inversely proportional to the frequency. The amplitude decreases as the frequency increases. The significance of modes in any vibration is once again demonstrated by this study.

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