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Analytical Solutions to Advection-Diffusion Equation with Time-Fractional Caputo-Fabrizio Derivative

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Abstract
An analytical solution to advection-diffusion in half plane is obtained with considering Dirichlet problem. The purpose of this research is to solve analytically a two-dimensional advection-diffusion equation with the time-fractional derivative with different fractional parameter and velocity for Dirichlet problem. The solution is compared and interpreted into graph for fractional advection-diffusion equation $(\alpha \in (0,1), v \neq 0)$ with ordinary advection-diffusion equation $(\alpha = 1, v \neq 0)$, and for ordinary diffusion equation $(\alpha = 1, v = 0)$ with fractional diffusion equation $(\alpha \in (0,1), v = 0)$**.** Specifically, to study the dependence of fundamental solutions by spatial position, time and fractional parameter. The solution obtained by using Laplace transform, Sine-Fourier transform and exponential Fourier transform and also expressed by elementary and Bessel functions. The numerical calculation is carried out and the results obtain is illustrated using MATLAB. The presence of fractional parameter gives an adequate model for given problem.
Keywords: Caputo-Fabrizio; advection-diffusion; singular kernel; MATLAB.

1. Introduction

The concept of diffusion has been widely used especially in biology, chemistry and physics. Diffusion can be described as the changes of molecules from high concentration to low concentration because of the driving force [1]. Advection is defined as a transmission of a matter from one place to another inside a moving fluid [2]. The combination of advection and diffusion describes the physical phenomena where the physical quantities are transmitted inside a physical system due to processes of diffusion and advection(convection).

Equation of advection-diffusion has attracted many authors since the equation frequently used in chemistry and engineering field such as transferring mass, heat, energy and vorticity [3]. Not only that, advection-diffusion model is used in water-pollution accident in middle and lower reach of Hanjiang river [4], to study the measure of concentration pollutant and dissolved oxygen concentrations in river water [5], fumigation [6], movement of substances in biological tissues and many more.

On the other hand, fractional order derivative is more accurate and realistic that make researchers focused to investigate the solutions of nonlinear differential equations to find the approximate solutions. Example of fractional order derivative is Riemann-Liouville, Caputo and Hilfer. However, these operators have some difficulties and limitations in modelling physical problems. Therefore, Caputo and Fabrizio has introduced alternative fractional differential operator that having a kernel with exponential decay. This fractional derivative is known as Caputo-Fabrizio operator which attracted researcher because of having a non-singular kernel [7].

In this paper, the purpose of this study is (1) to solve analytically the fractional advection diffusion equation (ADE) with the time-fractional derivative without singular kernel in the half-plane $(x, y) \in (-\infty, \infty) \times [0, \infty)$ by considering Dirichlet cases with different fractional parameter, $\propto$. (2) To compare and interpret the solution into graph for fractional advection-diffusion equation $(\alpha \in (0,1), v \neq 0)$ with ordinary advection-diffusion equation $(\alpha = 1, v \neq 0)$ and (3) compare and interpret the solution into graph for ordinary diffusion equation $(\alpha = 1, v = 0)$ with fractional diffusion equation $(\alpha \in (0,1), v = 0)$. The technique use for this study is Laplace transform with respect to temporal variable t and Fourier transform with respect to space coordinate x and *y*. From the analytical solution, MATLAB will be used to make comparisons between the solution.

2. Literature Review

The application of advection-diffusion equation (ADE) has been used widely in real-world to solve transport process. For example, research about heterogeneity of hydro-geological media like river bed or aquifer [8]. In this paper, the authors extended the research from previous paper by considering velocity dependence on independent variable and dispersion parameter on the time-variable [9]. By using Green's function method, the analytical solutions of one-dimensional ADE are obtained in an infinite domain subjected to the instantaneous and continuous injected sources. By this, it helps to get the solution of the ADE in general form for different particular cases. The solution for both research is matched through particular solutions and figures for the instantons and continuous sources.

Furthermore, research on the advection-diffusion equation in two-dimensional and its initial and boundary value problems with using Atangana-Baleanu derivative. This Atangana-Baleanu (AB) derivative is proposed in sense of Riemann-Liouville and Caputo definitions with non-singular Mittag-Leffler function as a memory kernel. The AB derivative also has interpreted as a filter regulator. The Laplace transformation of AB derivative needs physically interpretable integer order initial conditions. From the results, it can be observed that the current results for concentration function is different from results of advection-diffusion model with Caputo fractional derivative with some parameter coefficients. Due to the Mittag-Leffler kernel, AB derivative has been more advantageous than Caputo fractional derivative for different types of diffusive transport [10]

Other research on one-dimensional advection-diffusion is about analytical solution of advection-diffusion equation using finite difference schemes. The solution is obtained by using explicit centered difference scheme and Crank-Nicolson scheme. The qualitative behaviour of the ADE for different option of advection and diffusion coefficient is obtained and the schemes are applied in the pollutant distribution in a river for different time and space coordinates [11].

Caputo-Fabrizio operator has been used to construct a dengue model [12]. The numerical results that using the fractal and fractional orders is found to be best fitting with the real statistical data. Application of the fractal-fractional operators in real world problem gives better outcome compared to ordinary order.

To redevelop the proposed Coronavirus transmission model with fractional order, the researchers applied the Caputo-Fabrizio derivative with non-singular exponential kernel. Using the Caputo-Fabrizio operator, it can help the researchers to earn more insights into the disease transmission dynamics.

Laplace transform is an integral transform method that solve linear ordinary differential equation. It has been used widely in physics, control and electrical engineering, optics, mathematics and signal processing. Laplace transform is very effective mathematical tool that can simplify the complex problem in the area of stability and control [13]. Laplace transform gives a powerful method of solving the differential and integral solutions. The benefit of using Laplace transform method is it solves the value problems straight without need to find the general solution first [14].

The analytical solutions of advection-diffusion equation for a point source with a linear pulse time pattern involving constant-parameters condition which is constant velocity and diffusion coefficient used Laplace transform and inverse Laplace transform is used in solving the advection-diffusion equation. The results show that the superposition principle was employed to extend the derived solution for few point sources in arbitrary patterns. The method used for different problems and it gives results that highly accurate [15].

Fourier series has been used widely in may field such as electronics, quantum mechanics and electrodynamics. Fourier series also can be used to analyse the square wave that occur in electric circuit and how to convert analog to digital systems [16]. The researcher discusses about the applications in electronics and digital multimedia visualization signal process communication system.

2.1. Fractional Derivatives

The concept of differentiation operator $D = \frac{d}{dx}$ is already well known and for function f with the nth derivative of function f is described as $D^n f(x) = \frac{d^n f(x)}{dx^n}$. The question arise in 1695 what would happen to $D^n f$ if n were a fraction and not a regular integer. Since that, the fractional calculus has attracted many famous mathematicians such as Euler, Laplace, Fourier, Abel, Liouville, Riemann and Laurent [14]. The following will provide the definition of some fractional derivatives.

2.1.1 Riemann-Liouville

The definition of Riemann-Liouville time-fractional derivative of order $\alpha \in [0,1)$ is

$$^{RL}D_t^{\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^t (t-\tau)^{-\alpha} f(\tau) d\tau \qquad (2.1)$$

where:

$\alpha \;=\; fractional\, parameter$

$\Gamma \;=\; gamma\, function$

The Laplace transform to equation (2.1) is given by

$$\mathcal{L}\{^{RL}D_t^{\alpha} f(t)\} = s^{\alpha} F(t) - {}^{RL}D_t^{\alpha-1} f(t)|_{t=0}\,.$$

2.1.2 Caputo

The definition of Caputo-time fractional derivative of order $\alpha \in [0,1)$ is

$$^{C}D_t^{\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} f'(\tau) d\tau. \qquad (2.2)$$

Laplace transform to equation (2.2) is

$$\mathcal{L}\{\, ^{C}D_t^{\alpha} f(t)\} = s^{\alpha} F(s) - s^{\alpha-1} f(0).$$

2.1.3 Caputo-Fabrizio

The definition of Caputo Fabrizio-time fractional derivative of order $\alpha \in [0,1)$ is

$$^{CF}D_t^{\alpha} = \frac{1}{(1-\alpha)} \int_0^t \exp\left(\frac{-\alpha(t-\tau)}{1-\alpha}\right) f'(\tau) d\tau \qquad (2.3)$$

Applying Laplace transform to equation (2.3)

$$\mathcal{L}\{^{CF}D_t^{\alpha}\} = \frac{sL[f(t)] - f(0)}{(1-\alpha)s + \alpha}\,.$$

3. Methodology

3.1. Mathematical Formulation

The partial differential equation of advection-diffusion is given by

$$\frac{\partial C}{\partial t} + \nabla.(\boldsymbol{u}C) = D\nabla^2 C \qquad (2.4)$$

The advection-diffusion in equation (2.4) is in (*x,y*) plane into an incompressible fluid, $(div(u) = 0)$, with velocity $\boldsymbol{u} = (v_0, v_0, 0)$ that is constant cross flow velocity. Then the partial differential equation of advection-diffusion become:

$$\frac{\partial C(x,y,t)}{\partial t} = D\left(\frac{\partial^2 C(x,y,t)}{\partial x^2} + \frac{\partial^2 C(x,y,t)}{\partial y^2}\right) - v_0 \frac{\partial C(x,y,t)}{\partial x} - v_0 \frac{\partial C(x,y,t)}{\partial y} \qquad (2.5)$$

with the initial and boundary equations:

$C(x,y,0)=0$ (2.6)
$C(x,0,t)=C_0\delta(x)\delta(t)$ (2.7)
$\lim_{x\to\pm\infty} C(x,y,t)=0$ (2.8)
$\lim_{y\to\infty} C(x,y,t)=0$ (2.9)

3.2. Analytical solution

3.2.1. Nondimensionalisation

Consider the non-dimensional variables,

$$C^*=\frac{C}{C_1},\qquad x^*=\frac{x}{L},\qquad y^*=\frac{y}{L},\qquad t^*=\frac{Dt}{L^2},\qquad v^*=\frac{v_0L}{D},$$

After substituting non-dimensional variable into equation (2.5), then the non-dimensional form of the two-dimensional advection-diffusion equation is

$$\frac{\partial C}{\partial t}=\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right)-v\left(\frac{\partial C}{\partial x}+\frac{\partial C}{\partial y}\right) \tag{2.10}$$

3.2.1. The Fundamental Solutions to the Fractional Dirichlet Problem

New unknown function $u(x,y,t)$ defined as,

$$C(x,y,t)=\exp\left(\frac{v}{2}(x+y)\right)u(x,y,t) \tag{2.11}$$

and taking into consideration $f(x)\delta(x)=f(0)\delta(x)$, it is known that u(*x*,*y*,*t*) the solution to the problem:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{v^2}{2}$$

with

$u(x,y,0)=0,$ (2.12)
$u(x,0,t)=\delta(x)\delta(t),$ (2.13)
$\lim_{x\to\pm\infty} u(x,y,t)=0,$ (2.14)
$\lim_{y\to\infty} u(x,y,t)=0.$ (2.15)

In this research the advection-diffusion is described by the time fractional advection-diffusion equation with Caputo-Fabrizio time-fractional derivative,

$$^{CF}D_t^{\alpha}u(x,y,t)=\frac{\partial^2 u(x,y,t)}{\partial x^2}+\frac{\partial^2 u(x,y,t)}{\partial y^2}-\frac{v^2}{2}u(x,y,t) \tag{2.16}$$

with $0<\alpha<1$

In order to solve equation (2.10) with initial and boundary equation in equation (2.12) to (2.15), Laplace transform is use with respect to time, $\bar{u}(x,y,x)=\int_0^\infty u(x,y,t)e^{-st}\,ds$, Sine-Fourier transform with respect to variable *y*, $\tilde{\bar{u}}(x,\eta,s)=\sqrt{\frac{2}{\pi}}\int_0^\infty \bar{u}(x,y,s)\sin(\eta y)dy$ and exponential Fourier transform with respect to variable *x*, $\tilde{\bar{u}}^*=\int_{-\infty}^{\infty}\tilde{\bar{u}}(x,\eta,s)\,e^{-i\xi x}dx.$ Then,

$$\tilde{\bar{u}}^*(\xi,\eta,s)=\sqrt{\frac{2}{\pi}}\frac{\eta}{\left[\frac{s\gamma}{(s+\alpha\gamma)}+\xi^2+\eta^2+\frac{v^2}{2}\right]} \tag{2.17}$$

where $\gamma=\frac{1}{(1-\alpha)}$

Equation (2.17) can be written in equivalent form which is:

$\tilde{\tilde{u}}^*(\xi,\eta,s) = \tilde{\tilde{u}}_1^*(\xi,\eta,s) + \tilde{\tilde{u}}_2^*(\xi,\eta,s)$
where

$$\tilde{\tilde{u}}_1^*(\xi,\eta,s) = \sqrt{\frac{2}{\pi}}\frac{\eta}{\left[\xi^2+\eta^2+\gamma+\frac{v^2}{2}\right]} \qquad (2.18)$$

and

$$\tilde{\tilde{u}}_2^*(\xi,\eta,s) = \sqrt{\frac{2}{\pi}}\frac{\alpha\gamma^2\eta}{(\xi^2+\eta^2+\varsigma^2)^2}\cdot\frac{1}{s+\frac{\alpha\gamma\left(\xi^2+\eta^2+\frac{v^2}{2}\right)}{\xi^2+\eta^2+\varsigma^2}} \qquad (2.19)$$

where $\varsigma^2 = \gamma + \frac{v^2}{2}$

Then, applying the inversion of integral transform to equation (2.18) and applying some properties of integrals,

$$u_1(x,y,t) = \frac{4}{\pi}\delta(t)\int_0^\infty\int_0^\infty \frac{\eta\, sin(\eta y) cos\,(\xi x)}{[\xi^2+\eta^2+\varsigma^2]}\,d\eta d\xi$$

Next, introduced the polar coordinates coordinates in the plane (ξ,η), that are $\xi = \rho cos\theta$, and $\eta = \rho sin\theta$, where $\rho \in [0,\infty), \theta \in \left[0,\frac{\pi}{2}\right]$.

$$u_1(x,y,t) = \frac{4}{\pi}\delta(t)\int_0^\infty\int_0^{\frac{\pi}{2}}\frac{\rho^2}{\rho^2+\varsigma^2} sin\theta \sin(\rho sin\theta y)\cos\,(\rho cos\theta x)\,d\rho d\theta.$$

By changing the variable $z = cos\theta$,

$$I_1 = \int_0^{\frac{\pi}{2}} sin\,\theta\, sin(y\rho sin\theta)\cos(x\rho\, cos\,\theta)\,d\theta$$
$$= \int_0^1 \sin(y\rho sin\theta)\cos(x\rho z)\,dz$$

and by referring to [18] it is known that

$$\int_0^1 cos(mz)\, sin\left(n\sqrt{1-z^2}\right)dz = \frac{\pi}{2}\left(\frac{n}{\sqrt{m^2+n^2}}\right)J_1\left(\sqrt{m^2+n^2}\right).$$

Then, equation (2.18) becomes,

$$u_1(x,y,t) = \frac{4}{\pi}\delta(t)\int_0^\infty \frac{\pi}{2}\left(\frac{n}{\sqrt{m^2+n^2}}\right)J_1\left(\sqrt{m^2+n^2}\right)\left(\frac{\rho^2}{\rho^2+\varsigma^2}\right)d\rho.$$

by referring to [19] where

$$\int_0^\infty \frac{z^2}{z^2+c^2}J_1(bz)dz = cK_1(bc).$$

Substitute the integrals,

$$u_1(x,y,t) = \frac{2y\delta(t)}{\sqrt{x^2+y^2}}\varsigma K_1\left(\varsigma\sqrt{x^2+y^2}\right).$$

From the above relations, $J_n(.)$ is the Bessel function of first kind of order *n* and $K_n(.)$ Is the modified Bessel function of second kind of order *n*.
Let assume that

$$f_1(x,y) = \frac{2y\delta(t)}{\sqrt{x^2+y^2}}\varsigma K_1\left(\varsigma\sqrt{x^2+y^2}\right).$$

Then it can be written as

$$f_1(x,y) = \frac{2\varsigma y}{\sqrt{1+z}}K_1\left(\varsigma|x|\sqrt{1+\frac{1}{z}}\right)$$

where $z = \left(\frac{x}{y}\right)^2$.

As a conclusion, the function $u_1(x,y,t)$ is undefined at the point $(x,y,t) = (0,0,0)$ and $u_1(x,y,t) = 0$ for $|x| > 0, y > 0, t > 0$.
Next, similarly for $u_2(\xi,\eta,s)$ given by equation (2.19), then,

$$u_2(x,y,t) = \alpha\gamma^2 y\rho e^{-\alpha\gamma t}\int_0^\infty \frac{\rho^2}{(\rho^2+\varsigma^2)^2}\mathrm{e}^{\frac{\alpha\gamma^2 t}{\rho^2+\varsigma^2}}\left[J_0\left(\rho\sqrt{x^2+y^2}\right)+J_2\left(\rho\sqrt{x^2+y^2}\right)\right]d\rho$$

Then, the solution of equation (2.11) with the initial and boundary condition in equation (2.12) to (2.15) given by [20] is,

$$u(x,y,t) = \alpha\gamma^2 y e^{-\alpha\gamma t}\int_0^\infty \frac{\rho^3}{(\rho^2+\varsigma^2)^2}\mathrm{e}^{\frac{\alpha\gamma^2 t}{\rho^2+\varsigma^2}}\left[J_0\left(\rho\sqrt{x^2+y^2}\right)+J_2\left(\rho\sqrt{x^2+y^2}\right)\right]d\rho \qquad (2.20)$$

Replacing equation (2.20) into equation (2.11), then the fundamental solution of the fractional Dirichlet problem for half-plane is

$$C(x,y,t) = \alpha\gamma^2 y\exp\left(\frac{v}{2}(x+y)-\alpha\gamma t\right)\int_0^\infty \frac{\rho^3}{(\rho^2+\varsigma^2)^2}\exp\left(\frac{\alpha\gamma^2 t}{\rho^2+\varsigma^2}\right)\left[J_0\left(\rho\sqrt{x^2+y^2}\right)+J_2\left(\rho\sqrt{x^2+y^2}\right)\right]d\rho. \qquad (2.21)$$

Case 1 : The Fundamental Solution to Dirichlet Problem for the Ordinary Advection-Diffusion Equation ($\alpha \to 1$)

$$C(x,y,t) = \frac{y}{2t^2}\exp\left(\frac{v}{2}(x+y)-\frac{v^2}{2}t-\frac{x^2+y^2}{4t}\right) \qquad (2.22)$$

Case 2 : The Fundamental Solution to Dirichlet Problem for the time-fractional diffusion equation $v = 0$ and $0 < \alpha < 1$

$$C(x,y,t) = \alpha\gamma^2 y\exp(-\alpha\gamma t)\int_0^\infty \frac{\rho^3}{(\rho^2+\gamma)^2}\exp\left(\frac{\alpha\gamma^2 t}{\rho^2+\gamma}\right)\left[J_0\left(\rho\sqrt{x^2+y^2}\right)+J_2\left(\rho\sqrt{x^2+y^2}\right)\right]d\rho. \qquad (2.23)$$

Case 3 : The Fundamental Solution to Dirichlet Problem for the normal diffusion equation $v = 0$ and $\alpha \to 1$

$$C(x,y,t) = \frac{y}{2t^2}\exp\left(-\frac{x^2+y^2}{4t}\right). \qquad (2.24)$$

4. Results and discussion

4.1. Graphical Results

In order to determine the influence of fractional parameter α on the advection-diffusion process, the numerical calculations has been carried out. The results then are illustrated graphically by using MATLAB. Comparison were made between fractional advection-diffusion equation $(\alpha \in (0,1), v \neq 0)$ and ordinary advection-diffusion equation $(\alpha = 1, v \neq 0)$. Other than that, comparison between fractional advection-diffusion equation $(\alpha \in (0,1), v = 0)$ and ordinary advection-diffusion equation $(\alpha = 1, v = 0)$ are made.

Figure 1 illustrates the results of fundamental solution (Equation (2.21)) and solution to Case 1 (Equation (2.22)) which is fractional advection-diffusion equation and ordinary advection-diffusion equation for Dirichlet problem respectively. The graph is plotted for x-variables versus concentration $C(x, y, t)$ by using $y = 0.4, v = 1.5$ for different values of parameter α.

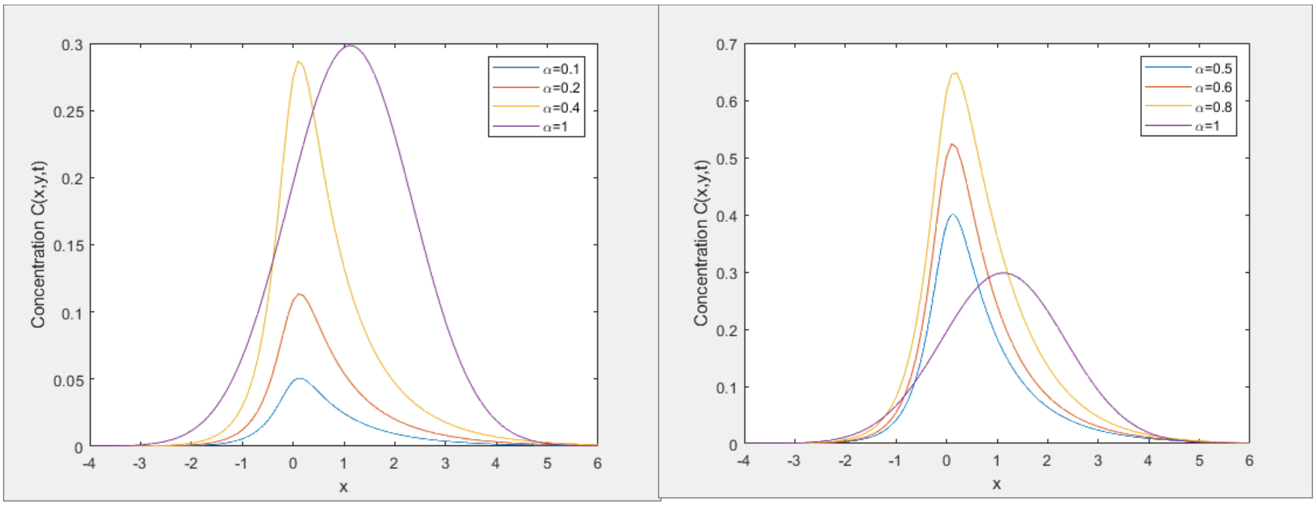

Figure 1 Fractional ADE and ordinary ADE $(\alpha = 1)$ when time t=0.75 for different fractional parameter α

Figure 1 shows the curves of fractional advection-diffusion equation when time t=0.75 for small values of $\alpha = 0.1, 0.2, 0.4$ on the left hand side and on the right hand side, large value of $\alpha = 0.5, 0.6, 0.8$ and ordinary advection-diffusion equation when $\alpha = 1$. When value $\alpha = 0.1, 0.2, 0.4$, the concentration for ordinary advection-diffusion equation is larger than concentration of fractional advection-diffusion equation but when value of $\alpha = 0.5, 0.6, 0.8$, then there is an area which concentration for fractional advection-diffusion equation is larger than concentration of ordinary advection-diffusion equation. Besides, when fractional parameter α is decreasing, the concentration is also decreasing.

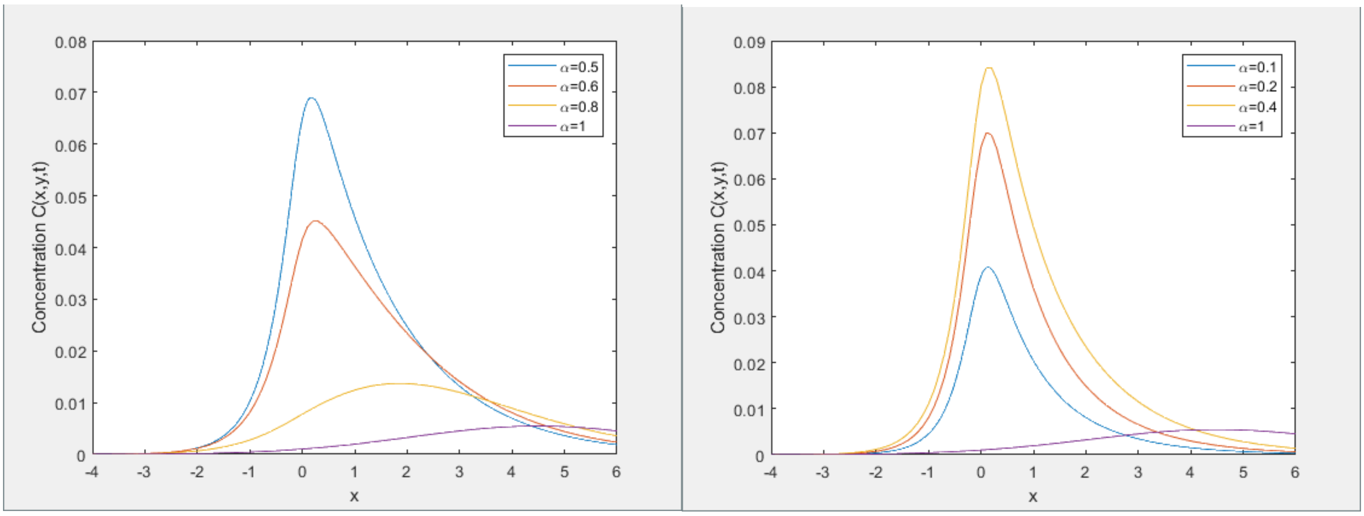

Figure 2 Fractional ADE and ordinary ADE $(\alpha = 1)$ when time t=0.75 for different fractional parameter α

Figure 2 shows the curves of fractional advection-diffusion equation when time t=3 for small values of $\alpha = 0.1,0.2,0.4$, on the left and large value of $\alpha = 0.5,0.6,0.8$ on the right and ordinary advection-diffusion equation when $\alpha = 1$. When value $\alpha = 0.1,0.2,0.4$, the concentration for fractional advection-diffusion equation is larger than concentration of ordinary advection-diffusion equation. This also similar when values of $\alpha = 0.5,0.6,0.8$. It can be concluded that, for larger value of time *t*, the concentration of ordinary advection-diffusion equation is lesser compared to smaller value of time

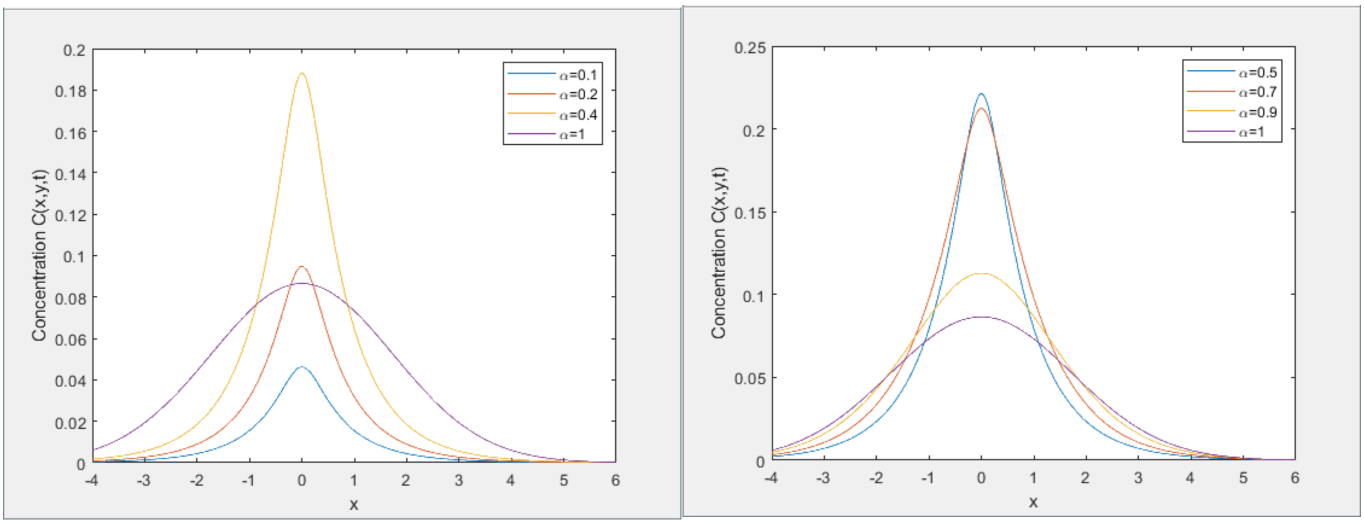

Figure 3 Fractional diffusion equation and ordinary diffusion equation $(\alpha = 1)$ when time *t*=1.5 and *v*=0 for different fractional parameter α

Figure 3 shows the curves of Equation (2.23) (Case 2) which is the fractional diffusion equation modeled by Caputo-Fabrizio time-fractional derivative and Equation (2.24) (Case 3) which is normal diffusion equation such that transport velocity, v=0. The curves is plotted versus x-variable with time t=1.5 for small value $\alpha = 0.1,0.2,0.4$, large value of $\alpha = 0.5,0.7,0.9$ and ordinary advection-diffusion equation when $\alpha = 1$. It can be seen that, there is values of fractional parameter α which the concentration corresponding to fractional advection-diffusion equation is smaller (or larger) than concentration corresponding to ordinary advection-diffusion equation.

Conclusion

The objectives of this study is to solve analytically the time-fractional derivative with different fractional parameter and velocity for Dirichlet problem and to compare and interpret the solution into graph for fractional advection-diffusion equation ($\alpha \in (0,1)$, $v \neq 0$) with ordinary advection-diffusion equation ($\alpha=1$, $v \neq 0$) and for ordinary diffusion equation ($\alpha=1$, $v=0$) with fractional diffusion equation ($\alpha \in (0,1)$, $v=0$). After getting the final analytical solution for Dirichlet conditions, the behavior of the graph is observed for various fractional parameter, α with different value of velocity by using MATLAB.

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