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Genetic Algebra Generated by Quadratic Stochastic Operators On Two-Dimensional Simplex

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Abstract In general, genetic algebra is known as commutative but non-associative. Moreover, all derivation of associative and commutative algebras are trivial as stated by Kadisons theorem. In the current paper, we induce a genetic algebra from Quadratic Stochastic Operators (QSO)s. Then, by considering general case of the algebra defined on two dimension, the description of associativity and derivation is obtained. We know that associative and commutative algebra only have trivial derivation, hence, the existence of non-trivial derivation on such algebra is studied.

Keywords: Genetic Algebra; Quadratic Stochastic Operator; Associative Algebra; Derivative Algebra

1. Introduction

Based on previous research, the researchers are presenting the classes of QSOs, for example, Volterra-QSOs, *b*-bistochastic QSOs, doubly QSOs, separable QSOs and etc. Therefore, we would like to study genetic algebra in general setting. The genetic Volterra algebras and some of their algebraic properties were studied by Ganikhodzhaev et al. [8]. Lately, Usmonov and Kodirova [26] made the relation for the case of Volterra QSOs between the evaluation algebras and the associated dynamical system. Motivated from those ideas, we are going to consider genetic algebra generated by QSOs which is simply called genetic algebra.

Usually, genetic algebra generated by QSOs are commutative but non-associative. Therefore, we are going to interpret the condition for associativity of such algebra. Furthermore, some of algebraic properties of these formation have genetic connotation. Kadisons Theorem states that all derivations of associative and commutative algebras are trivial. Thus, we are going to describe the derivation of genetic algebra in general setting.

Bernstein [6] state that the distribution evolution of individuals in a population was described using Quadratic Stochastic Operators (QSOs). Moreover, QSOs assumed a significant part of examination for the investigation of elements properties and models in various fields like biology, physics, economic and mathematics.

The most common method for modelling inheritance in genetics is to use genetic algebras (possibly non-associative) in mathematical genetics. Further, the derivation of the operators from the class of ξ^s-QSO is trivial (mean zero) because of it is non-associative [19]. However, the presence of the nontrivial derivations of genetic algebra were established in dimension two by Mukhamedov and Qaralleh [16,17].

In Mathematics, the interesting structure is the algebras that occurs in genetic either via gametic or zygotic. According to Reed [20], these algebras are generally commutative but non-associative. They also not compulsorily Lie or Jordan or any alternative algebra.

2. Preliminaries

Let us recall the definition of Quadratic Stochastic Operators (QSOs). Denote *V* be a mapping on the $(n-1)$ dimensional simplex S^{n-1},

$$S^{n-1} = \left\{ x = (x_1, x_2, ..., x_n) \in \mathbf{R}^n \mid x_i \geq 0, \sum_{i=1}^{n} x_i = 1 \right\}, \tag{1}$$

such that *V* takes the following form:

$$V(x)_k = \sum_{i,j=1}^{n} P_{ij,k} x_i y_j, \quad k = 1,2,3,...,n, \tag{2}$$

where $P_{ij,k}$ are coefficients of heredity that satisfy the coditions

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{i,j=1}^{n} P_{ij,k} = 1, \quad i,j,k = 1,2,3,...,n. \tag{3}$$

Furthermore, an algebra induces will be introduced by the operator *V*. Assume that $X = (x_1, x_2, \dots, x_n)$ and $Y = (y_1, y_2, \dots, y_n)$ are the arbitrary vectors in $\mathbb{R}^n$. Thus, we introduce the multiplication rule on $\mathbb{R}^n$ by

$$(\mathbf{x} \circ \mathbf{y}) = \sum_{i.j=1}^{n} P_{ij,k} x_i y_j \tag{4}$$

The pair $(\mathbb{R}^n, \circ)$ is called genetic algebra and note that, the algebra is commutative i.e., $(\mathbf{x} \circ \mathbf{y}) = (\mathbf{y} \circ \mathbf{x})$. In what follows, $\mathbf{e}_i$ is the standard *basis* in $\mathbb{R}^n$, i.e. $\boldsymbol{e}_i = (\delta_{i1}, \delta_{i2}, \dots, \delta_{in})$, where δ_{ij} is the Kroneker delta for any $i = 1,2, \dots, n$. Recall that, the genetic algebra *A* is associative if and only it satisfies the condition

$$((\mathbf{x} \circ \mathbf{y}) \circ \mathbf{z}) = (\mathbf{x} \circ (\mathbf{y} \circ \mathbf{z}))$$

Well-known Kadison's Theorem states that the derivation of associative and commutative algebras are trivial. On the other hand the reverse of the last statement may not be true, therefore it is interesting to describe the derivation of genetic algebras. We note that, a linear functional *D* is called derivation of the genetic algebra if *D* satisfying

$$D(\mathbf{x}, \mathbf{y}) = D(\mathbf{x}) \circ \mathbf{y} + \mathbf{x} \circ D(\mathbf{y}), \quad \text{for all } \mathbf{x}, \mathbf{y} \in A$$

$$(\mathbf{x} \circ \mathbf{y}) = x_1 y_1 (e_1 \circ e_1) + x_1 y_2 (e_1 \circ e_2) + x_2 y_1 (e_2 \circ e_1) + x_2 y_2 (e_2 \circ e_2)$$

By using the definition of the associativity, one can show that the description of associativity of the genetic algebra on $\mathbb{R}^n$ by considering the associativity on the standard basis only. This fact is formally formulated in the following remarks:

Remark 1 Suppose $(A, \circ)$ be a genetic algebra on $\mathbb{R}^2$ and e_i be the standard basis. *A* is associative if and only if

$$(e_i \circ e_j) \circ e_k = e_i \circ (e_j \circ e_k) \quad \text{for all } i,j,k = 1,2,...,n$$

Analogously, we have similar property to study the derivation of the genetic algebra as stated in the next remark:

Remark 2 The linear operator *D* is derivative if and only if

$$D(e_i \circ e_j) = D(e_i) \circ e_j + e_i \circ D(e_j) \text{ for all } i,j = 1,2,..,n.$$

3. Associativity on Two-Dimensional Genetic Algebra

In this section, a condition for associativity of genetic algebra defined on two dimensional is obtained. Due to Remark 1, the following multiplication table has created:

$\circ$	$\boldsymbol{e_1}$	$\boldsymbol{e_2}$
$\boldsymbol{e_1}$	$(a, 1-a)$	$(b, 1-b)$
$\boldsymbol{e_2}$	$(b, 1-b)$	$(c, 1-c)$

Table 4.1 Multiplication table $(\mathbb{R}^n, \circ)$

One can produce this table by the definition of genetic algebra.

The theorem has been published in [8], but the proving was not given. Thus, the theorem fully describes the associativity of a genetic algebra defined on $\mathbb{R}^2$.

Theorem 1 Let *A* be a genetic algebra and $\boldsymbol{e}_i$ be the standard basis in $\mathbb{R}^2$. *A* is associative if and only if $b(1-b)=c(1-a)$

Proof: Let the proof start by assuming *A* is associative and show that the equality $b(1-b)=c(1-a)$ is satisfied. By Remark 3.1, *A* is associative if and only if $\left(e_i \circ e_j\right)\circ e_k = e_i \circ \left(e_j \circ e_k\right)$. It is easy to show that $\left(e_i \circ e_j\right)\circ e_i = e_i \circ \left(e_j \circ e_i\right)$ and $\left(e_i \circ e_i\right)\circ e_i = e_i \circ \left(e_i \circ e_i\right)$ are satisfied.

Therefore, consider $\left(e_1 \circ e_1\right)\circ e_2 = e_1 \circ \left(e_1 \circ e_2\right)$

By expanding the LHS,

$$\begin{aligned}(e_1 \circ e_1)\circ e_2 &= \left(a,1-a\right)\circ\left(0,1\right)\\ &= (ab+c(1-a), a(1-b)+(1-c)(1-a))\\ &= (ab+c-ac, 1-ab-c+ac)\end{aligned}$$

Then, RHS were expand,

$$\begin{aligned}e_1 \circ (e_1 \circ e_2) &= \left(1,0\right)\circ\left(b,1-b\right)\\ &= (ab+b(1-b), b(1-a)+(1-b)^2)\\ &= (ab+b-b^2, 1-ab-b+b^2)\end{aligned}$$

This implies that $(ab+c-ac,1-ab-c+ac)=(ab+b-b^2,1-ab-b+b^2)$. So the corresponding components were equalized as below,

$$\begin{aligned}ab+c-ac &= ab+b-b^2\\ c(1-a) &= b(1-b),\end{aligned}$$

and

$$\begin{aligned}1-ab-c+ac &= 1-ab+b^2-b\\ c(1-a) &= b(1-b)\end{aligned}$$

Hence, one gets $b(1-b)=c(1-a)$.

The other cases can be done similarly.

Next, let us consider the reverse of the above theorem. Assume $b(1-b)=c(1-a)$ is true, then one needs to proof that *A* is associative. Let add *ab* both sides of the equality $b(1-b)=c(1-a),$ then,

$$\begin{aligned}b(1-b)+ab &= c(1-a)+ab\\ b-b^2+ab &= c-ac+ab\end{aligned} \tag{5}$$

Next, by multiplying -1 both sides followed by adding $1-ab$ to the equality $b(1-b)=c(1-a)$, one gets

$$\begin{aligned}c(a-1)+1-ab &= b(b-1)+1-ab\\ ac-c+1-ab &= b^2-b+1-ab\end{aligned} \tag{6}$$

Combining (5) and (6) yield

$$\begin{aligned}(b-b^2+ab, b^2-b+1-ab) &= (c-ac+ab, ac-c+1-ab)\\ (ab+b(1-b), b(1-a)+(1-b)^2) &= (ab+c(1-a), a(1-b)+(1-c)(1-a))\\ (1,0)\circ(b,1-b) &= (a,1-a)\circ(0,1)\\ e_1\circ(e_1\circ e_2) &= (e_1\circ e_1)\circ e_2\end{aligned}$$

Note that, the others cases can be proved in similar manner, thus show that *A* is associative. This ends the proof.

Remark 3: In Afini, they induce a genetic algebra from a class of QSOs, namely b-bistochastic QSOs, and the algebra is called b-bistochastic genetic algebra. In that case, the heredity coefficient satisfy $c=0,\ b\le\frac{1}{2}$ and they conclude that the algebra is associative if and only if *b = 0*. Hence, Theorem 1 generalized the result obtained by Afini [4].

4. Derivation of Genetic Algebra on Two-Dimensional Simplex

This section will describe the conditions on the genetic algebra that has trivial derivation only. However, the description of definition on derivation of genetic algebra has been proved by Afini [4].

Theorem 2 Let D be a derivative for genetic algebra A. By assuming the following cases,

(i) $a = b = 0,$

(ii) $c = 0$ and $\frac{1}{2} < b < 1,$

(iii) $c = 0$ and $b = 1$

Then, the associated genetic algebra is trivial derivation.

Proof Let D be a derivative for genetic algebra A. By Remarks 2, D is derivative if and only if $D(e_i \circ e_j) = D(e_i) \circ e_j + e_i \circ D(e_j)$ for all $i, j = 1, 2.$ Therefore, a system of equations is obtained as following:

$$
\begin{aligned}
& ad_{11} + 2bd_{12} - (1-a)d_{21} &&= 0, && (7)\\
& 2(1-a)d_{11} + (2-2b-a)d_{12} - (1-a)d_{22} &&= 0, && (8)\\
& cd_{12} + (a+b-1)d_{21} + bd_{22} &&= 0, && (9)\\
& (1-b)d_{11} + (1-c-b)d_{12} + (1-a)d_{21} &&= 0, && (10)\\
& cd_{11} + (1-2b-c)d_{21} - 2cbd_{12} &&= 0, && (11)\\
& cd_{12} - 2(1-b)d_{21} + (1-c)d_{22} &&= 0. && (12)
\end{aligned}
$$

i) Assume that $a = b = 0,$ then, the Equation (7) until (12) reduces to the following system:

$$
\begin{aligned}
& -d_{21} &&= 0, && (13)\\
& 2d_{11} + 2d_{12} - d_{22} &&= 0, && (14)\\
& cd_{12} - d_{21} &&= 0, && (15)\\
& d_{11} + (1-c)d_{12} + d_{21} &&= 0, && (16)\\
& cd_{11} + (1-c)d_{21} &&= 0, && (17)\\
& cd_{12} - 2d_{21} + (1-c)d_{22} &&= 0. && (18)
\end{aligned}
$$

From Equation (13), then $d_{21} = 0$ and substitute the value into Equation (15) and Equation (17) to obtain that $d_{11} = d_{12} = 0.$ Hence, it is implies $d_{22} = 0.$ Therefore, the derivation is trivial.

ii) Consider that $c = 0$ and $\frac{1}{2} < b < 1,$ then the system of Equation (7) until Equation (12) will reduce to the following system,

$$
\begin{aligned}
& ad_{11} + 2bd_{12} - (1-a)d_{21} &&= 0, && (19)\\
& 2(1-a)d_{11} + (2-2b-a)d_{12} - (1-a)d_{22} &&= 0, && (20)\\
& (a+b-1)d_{21} + bd_{22} &&= 0, && (21)\\
& (1-b)d_{11} + (1-b)d_{12} + (1-a)d_{21} &&= 0, && (22)\\
& (1-2b)d_{21} &&= 0, && (23)\\
& -2(1-b)d_{21} + d_{22} &&= 0. && (24)
\end{aligned}
$$

From Equation (23), the value of d_{21} is obtained which are $d_{21} = 0$. Then substitute the value into Equation (21) and (24). It becomes $d_{22} = 0.$ Thus, the system has been reduced as below,

$$
\begin{aligned}
& ad_{11} + 2bd_{12} &&= 0, && (25)\\
& (2-2a)d_{11} + (2-2b-a)d_{12} &&= 0, && (26)\\
& (1-b)d_{11} + (1-b)d_{12} &&= 0. && (27)
\end{aligned}
$$

Since $b < 1,$ then by Equation (27) will obtain that $d_{11} = -d_{12}$. Thus substitute the value obtained to the other system (Equation (26) and (25)).

$$ad_{11} - 2bd_{11} = 0,$$
$$(2-2a)d_{11} - (2-2b-a)d_{11} = 0.$$

Hence, from the above system, it can be concluded that $d_{11} = 0.$ Therefore, $d_{12} = 0.$ In a nutshell, the value of derivative that is obtained is $d_{11} = d_{12} = d_{21} = d_{22} = 0$ which implies that the derivation is trivial.

iii) Now consider $c = 0$ and $b = 1,$ Equation (7) to Equation (12) will reduce into the following system:

$$ad_{11} + 2d_{12} - (1-a)d_{21} = 0, \tag{28}$$
$$2(1-a)d_{11} - ad_{12} - (1-a)d_{22} = 0, \tag{29}$$
$$ad_{21} + d_{22} = 0, \tag{30}$$
$$(1-a)d_{21} = 0, \tag{31}$$
$$-d_{21} = 0, \tag{32}$$
$$d_{22} = 0. \tag{33}$$

From Equation (32) and Equation (33), it is clearly show that $d_{12} = d_{22} = 0$. Then, substitute the value of d_{21} and d_{22} into the Equation (28) and (29) becomes

$$ad_{11} + 2d_{12} = 0, \tag{34}$$
$$(2-2a)d_{11} - ad_{12} = 0. \tag{35}$$

By solving Equation (34), the value of d_{11} will obtained which is $d_{11} = -\frac{2d_{12}}{a}$. So by substitute the value of d_{11} into Equation (35) and gained

$$(2-2a)(-\frac{2}{a}d_{12}) - ad_{12} = 0$$
$$-\frac{d_{12}(4-4a+a^2)}{a} = 0$$

Now let consider that

$$a^2 - 4a + 4 = 0$$
$$(a-2)^2 = 0$$
$$a = 2$$

Thus, the value obtained is $a = 2$ which contradicts with the condition in Equation (3). Hence, it is concluded that $d_{12} = 0.$ Therefore, $d_{11} = d_{12} = d_{21} = d_{22} = 0$ and implies that the derivation is trivial. From all these cases, it is shown that the theorem is proved. However, there exists a positive result on non-trivial derivation as the following example.

Example 5.1 Let *D* be a derivative for genetic algebra *A*. By assuming the coefficients of heredity, $a = b = c = 0,$ there exist non-trivial derivation.

Proof Let *D* be a derivative for genetic algebra *A*. By using the same system of equation (Equation (7) to Equation (12)) and assume that $a = b = c = 0.$ Then, the system will reduced and become:

$$-d_{21} = 0, \tag{36}$$
$$2d_{11} + 2d_{12} - d_{22} = 0, \tag{37}$$
$$-d_{21} = 0, \tag{38}$$
$$d_{11} + d_{12} + d_{21} = 0, \tag{39}$$
$$d_{21} = 0, \tag{40}$$
$$-2d_{21} + d_{22} = 0. \tag{41}$$

From the system of equations, it is clearly show that $d_{21}=0$ and substitute the value of d_{21} into Equation (41). It concludes that $d_{22}=0$. Hence, it implies that $d_{11}=-d_{12}$. Therefore, it is show that there exists non-trivial derivation.

Remark 4: If $c=0,$ the system of equations (Equation (7) to Equation (12)) reduce to Equation (19) to Equation (24). This system has been considered by Afini [4] in which the non-trivial derivation is exists. Therefore, Theorem 2 is generalizing and the result obtained.

Conclusion

This paper studies the associativity and derivation of two-dimensional genetic algebra. The conditions for two-dimensional genetic algebra to be associative are determined. Furthermore, the condition of trivial derivations of two-dimensional genetic algebra are also described.

References

[1] Abdulghafor R, Almotairi S, Almohamedh H, Almutairi B, Bajahzar A, Almutairi SS. A Nonlinear Convergence Consensus: Extreme Doubly Stochastic Quadratic Operators for Multi-Agent Systems. Symmetry. 2020 Apr 3;12(4):540.

[2] Abdulghafor R, Almotairi S, Almohamedh H, Turaev S, Almutairi B. Nonlinear Consensus Protocol Modified from Doubly Stochastic Quadratic Operators in Networks of Dynamic Agents. Symmetry. 2019 Dec 15;11(12):1519.

[3] Abdulghafor R, Turaev S, Abubakar A, Zeki A. The Extreme Doubly Stochastic Quadratic Operators on Two Dimensional Simplex. 2015 4th International Conference on Advanced Computer Science Applications and Technologies (ACSAT). 2015 Dec;

[4] Afini N, Rosli M, Fadillah A. On Algebraic Properties of Two-Dimensional b-Bistochastic Genetic Algebra. 2021;4:72–80.

[5] Badocha M, Bartoszek W. Quadratic stochastic operators on Banach lattices. Positivity. 2017 Sep 18;22(2):477–92.

[6] Bernstein S. Solution of a mathematical problem connected with the theory of heredity. The Annals of Mathematical Statistics. 1942 Mar 1;13(1):53-61.

[7] El-Qader HA, Ghani ATA, Qaralleh I. On Genetic Algebra and its Transformation to some Evolution Algebras in Dimension Four. Journal of Physics: Conference Series. 2020 Apr;1529:042093.

[8] Ganikhodjaev N, Hisamuddin HH. Associativity in inheritance or are there associative populations. Malaysian Journal of Science. 2008;27(2):131-6.

[9] Ganikhodzhaev R, Mukhamedov F, Pirnapasov A, Qaralleh I. Genetic Volterra algebras and their derivations. Communications in Algebra. 2017 Aug 30;46(3):1353–66.

[10] Ganikhodzhaev R, Mukhamedov F, Rozikov U. Quadratic Stochastic Operators And Processes: Results And Open Problems. Infinite Dimensional Analysis, Quantum Probability And Related Topics. 2011 Jun;14(02):279–335.

[11] Holgate P. Jordan algebras arising in population genetics. Proceedings of the Edinburgh Mathematical Society. 1967 Dec;15(4):291–4.

[12] Holmes S, Hofbauer J, Sigmund K. Evolutionary Games and Population Dynamics. Journal of the American Statistical Association. 2000 Jun;95(450):688.

[13] Ladra M, Rozikov UA. Evolution algebra of a bisexual population. Journal of Algebra. 2013 Mar;378:153–72.

[14] Lotka AJ. UNDAMPED OSCILLATIONS DERIVED FROM THE LAW OF MASS ACTION. Journal of the American Chemical Society. 1920 Aug;42(8):1595–9.

[15] Lyubich YI, Akin E, Karpov AE, Vulis D. Mathematical structures in population genetics. Berlin: Springer-Verlag; 1992 Jan.

[16] Mukhamedov F, Saburov M, Qaralleh I. Classification of $\xi^{(s)}$-Quadratic Stochastic Operators on 2D simplex. Journal of Physics: Conference Series. 2013 Apr 26;435:012003.

[17] Mukhamedov F, Saburov M, Qaralleh I. On ξ(s)-Quadratic Stochastic Operators on Two-Dimensional Simplex and Their Behavior. Abstract and Applied Analysis. 2013;2013:1–12.

[18] Plank M. Hamiltonian structures for the *n*-dimensional Lotka–Voltera equations. Journal of Mathematical Physics. 1995 Jul;36(7):3520–34.

[19] Qaralleh I, Ahmad MZ, Alsarayreh A. Associative and derivation of genetic algebra generated from ξ(s)-QSO. AIP Conference Proceedings. 2016;
[20] Reed ML. Algebraic structure of genetic inheritance. Bulletin of the American Mathematical Society [Internet]. 1997 Apr 1;34(02):107–31. Available from: https://www.ams.org/journals/bull/1997-34-02/S0273-0979-97-00712-X/S0273-0979-97-00712-X.pdf
[21] Rozikov UA, Shoyimardonov SK. Ocean ecosystem discrete time dynamics generated by ℓ-Volterra operators. International Journal of Biomathematics. 2019 Feb;12(02):1950015.
[22] Rozikov UA, Velasco MV. A discrete-time dynamical system and an evolution algebra of mosquito population. Journal of Mathematical Biology. 2018 Nov 13;78(4):1225–44.
[23] Rozikov UA, Zhamilov UU. F-quadratic stochastic operators. Mathematical Notes. 2008 Apr;83(3-4):554–9.
[24] Saburov M. Nonergodic Quadratic Stochastic Operators. Mathematical Notes. 2019 Jul;106(1-2):142–5.
[25] Takeuchi Y. Global dynamical properties of Lotka-Volterra systems. World Scientific; 1996 Apr 13.
[26] Usmonov JB, Kodirova MA. A quadratic stochastic operator with variable coefficients. Matematika Instituti Byulleteni Bulletin of the Institute of Mathematics Бюллетень Института. 2020(3):98-107.