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Adaptive Runge Kutta of Fehlberg Method on Displacement of Spring due to Opposite Force

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Abstract
A spring mass damping is generally consisting of springs, dashpots and masses. The displacement of the mass spring damping system due to an opposite force is the main concern for this study. These systems often can be found in chemical processes, control theory, electromagnetism, thermodynamics, and many other problems of physics, engineering, and the various works cited therein. The efficiency of the approximation of the second order ordinary differential equations by Runge Kutta Order 4 method and Fehlberg method is analyzed in this study. Adaptive algorithm for the solution of ordinary differential equation systems as Fehlberg method by using Matlab aid to reduce computational time. Matlab built in function ode45 is also being interpreted in terms of accuracy and effectiveness. The results obtained clarified that the smaller the step size for Runge Kutta Order 4, the higher the accuracy of the approximation method. However, smaller step size may require longer computational time and Fehlberg method may become the key of success as it modified and changed the step size according to the tolerance level that had been set up. The Fehlberg method have the greatest accuracy compared to Runge Kutta Order 4 method. Matlab built in function of ode45 may become alternative way to find the approximation in easier way compared to Fehlberg method as it can improves the accuracy by lowering the tolerance level for the error terms. Furthermore, Runge KuttaOrder 4 can be used for the numerical solutions when the expectation for the accuracy level is low. In conclusion, Fehlberg method can lower the computational time by altering step size may greatly improve the efficiency of finding the solution for mass spring damping system.

Keywords: Runge Kutta Order 4; Fehlberg method; ode45; Mass-spring damping system; error performance

1. Introduction

Ordinary differential equations are extensively employed in many fields such as mechanics, astronomy, physics and in the sector of chemistry and biology. By solving the ordinary differential equation using the approximation of RK4, the ODEs can be solved numerically. An adaptive Runge Kutta, especially Fehlberg method is well known and widely implemented for solving the ODEs. Solving a spring mass damping system using approximation by Runge Kutta Fehlberg method is the concern of this study.

In the cases of having a normal Runge Kutta method may be hard and requires more computation time when solving a hard problem. Adaptive Runge Kutta methods plays important role when solving this kind of initial value problems which it implements some kind of variable step size and have some automatic error control to minimize the work while obtaining a user-defined accuracy. This will probably increase the efficiency of getting the solution numerically. By solving the differential equations using analytic solution may sometimes lead to a complicated solution.

Besides, the computational results on solving the vibrating of mass spring due to opposition force is important as it used in many fields. The application of software Matlab should be considered as it will be more convenient.

The modelling of spring mass damping is generally consisting of springs, dashpots and masses [1]. The spring is having a spring constant to describe on the stiffness of the spring. The dashpot which is also called damper which resisting the motion via viscous friction. The motion spring is closely related to the Newton's second law of motion which states that the bigger the mass, the greater the force required to make an acceleration [4]. Besides, the vibration of the spring damping system also one of the studies of this paper. The vibration of the system is a large concern for engineers during operations [6].

This research aims to (1) obtain numerical approximations of Runge Kutta Order 4 and Fehlberg Runge Kutta method and ode45 for solving the vibrating of mass spring due to opposition force. and (2) obtain exact solutions of vibrating of mass spring due to opposition force equation using the method of undetermined coefficients. (3) evaluate the performance and stability of ode45, Runge Kutta Order 4 and Fehlberg Runge Kutta in solving non-homogenous second order ordinary differential equation.

2. Literature Review

2.1. Vibration of mass spring damper system

Vibration can be defined as "the cyclical change in the position of an object as it moves alternately to one side and the other of some reference or datum position" [2]. Mass spring damping can be denoted as scalar second order differential equation of the form

$mx'' + ax + kx = F(x)$

where:

a = damping coefficient.

m = mass attached to the lower end of the spring

k = spring constant.

$F(x)$= externally impressed force.

2.2 Solving Ordinary Differential Equations

Sometimes, general methods of integrating ordinary differential equations are not sufficient. Integrating the ODEs will sometimes lead to an analytical solution but some functions are difficult in distinguishing the solution. Some solutions with special properties and criteria need to be considered. Hence, boundary value problems which are closely related to ordinary differential equations are being studied. These studies of the functions later led to modern numerical methods.

2.2.1 Analytic Solution

By the reference of Lagrange on the problem of determining an integrating factor for the general linear equation, mathematicians found that some conditions under which the order of a linear differential equation could be lowered. Reduction of order is a method in which applicable for finding the general solutions of linear differential equations.

Variation of parameter method can be viewed as a brilliant improvement of the reduction of order method for solving nonhomogeneous linear ODE. The method of undetermined coefficients also been widely used to find the particular function and complementary function.

2.2.2 Numerical Solution

Several works in numerical solutions of initial value problems using Runge Kutta method have been carried out. Many authors have attempted to solve initial value problems (IVP) to obtain high accuracy with speed by using numerous methods.

2.2.2.1. Runge Kutta Order 4 Method

Runge Kutta are a family of implicit and explicit iterative methods. The accurate solutions of initial value problems for ordinary differential equations are studied by many researchers to determine the error between the numerical method on solving the ODEs [5]. This Runge Kutta method extended the approximation method of Euler to be more capable of greater accuracy. The idea of Runge Kutta was an approximate solution with improved formulas as the midpoint and trapezoidal rules. The Runge Kutta method was derived from Taylor series where the Taylor series can be denoted as below:

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2}y_n'' + \frac{h^3}{6}y_n''' + \cdots$$

If applying it at Runge Kutta method, the equation will become

$$y_{n+1} = y_n + \sum_{i=1}^{q} \omega_i k_i \text{ with } k_i = (x)_n + ha_i,\ y_n + h\sum_{j=1}^{q} \beta_{ij} k_i$$

2.2.2.2 Runge Kutta Fehlberg Method

Runge Kutta Fehlberg Method is an algorithm in numerical methods for solving ordinary differential equations. It was introduced by Erwin Fehlberg and it is the improved formula based on the Runge Kutta methods. By performing one extra calculation, the error in the solution can be estimated and controlled by using the higher-order embedded method that allows for an adaptive step size to be determined automatically.

The Runge Kutta Fehlberg Method also been modified to solve second order ordinary differential equations. This method is more direct and more efficient than the traditionally method of reducing the problems into system of first order ODEs. It requires no preliminary calculation required before obtaining the next iterations [3].

3. Analytical and Numerical Solution of Mass Damping Spring System

3.1 Formulation of model of the mass damping spring system

The methodology of solving the spring mass damping system requires to follow the Hooke's Law and Newton's Second Law of motion. Hooke's Law states that the restoring force F exerted by a spring when it is stretched or compressed is proportional to the distance that it is stretched or compressed. The gravitational forces also been taken into consideration. When summing all the forces acted on the spring, the equation are as follows:

$$mx''(t) + x'(t) + kx = F(t) = F_0 \cos \omega t.$$

$$F = F_1 + F_2 + F_3 + F_4$$

$$m\frac{d^2x}{dt^2} = mg - kx - mg - a\frac{dx}{dt} = F(t)$$

$$m\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = F(t)$$

3.2 Analytical Method of Spring Mass Damping System due to Opposition Force

There will be two cases for the solution

CASE 1

If there are no external forces exerted within the system, Let $x(t) = e^{rt}$ be a solution, the equation is become

$$r_{1,} r_2 = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

and the solution is

$$e^{rt}\left(mr^2 + \gamma r + k\right) = 0$$

There will be three solutions since there are three types of roots needed to be considered for this solution. However, the solutions tend to zero as time approaches to infinity. These analytic solutions are transient solution.

$x_p(t) = a\cos\omega t + b\sin\omega t$

CASE 2
If there is external force exerted within the system, then the steady state solution is

The method of undetermined coefficients was used to determine the derivative of the solution

$x_p'(t) = -a\omega \sin\omega t + b\omega \cos\omega t$
$x_p''(t) = -a\omega^2 \cos\omega t - b\omega^2 \sin\omega t$
Substitute it into $x_p(t) = a\cos\omega t + b\sin\omega t$

$$\left(\left(k - m\omega^2\right)a + b\gamma\omega\right)\cos\omega t + \left(-\gamma\omega a + \left(k - m\omega^2\right)b\right)\sin\omega t = F_0\cos\omega t$$

Setting $\omega_0 = \sqrt{\frac{k}{m}}$ and the denominator of *a* and *b* one not equal to zero

$$a = \frac{F_0 m\left(\omega_0 - \omega^2\right)}{m^2\left(\omega_0^2 - \omega^2\right) + \omega^2\gamma^2}$$

$$b = \frac{F_0 m\gamma\omega\left(\omega_0 - \omega^2\right)}{m^2\left(\omega_0^2 - \omega^2\right) + \omega^2\gamma^2}$$

the particular solution is

$$x_p(t) = \frac{F_0 m\left(\omega_0 - \omega^2\right)\cos\omega t + F_0\gamma\omega\left(\omega_0 - \omega^2\right)\sin\omega t}{m^2\left(\omega_0^2 - \omega^2\right) + \omega^2\gamma^2}$$

3.3 Runge Kutta Order 4 Method

The development of the Runge Kutta Order 4 (RK4) method use four approximations to the slope. The Runge Kutta method finds approximate value of $y(t)$ for a given t.

$$k_1 = hf\left(y(t_0), t_0\right)$$

$$k_2 = hf\left(y(t_0) + \frac{h}{2}, t_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(y(t_0) + \frac{h}{2}, t_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(y(t_0) + h, t_0 + k_3\right)$$

$$y\left(t_0 + h\right) = y(t_0) + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

3.4 Fehlberg Method as Adaptive Runge Kutta method

To improve the accuracy in the solution of an ODEs is solving the problem twice and compare answers at the mesh points corresponding to the larger step size. But this requires a significant amount of computation for the smaller step size and must be repeated if it is determined that the agreement is not good enough. The Runge-Kutta-Fehlberg method also denoted RKF45 is one of the types of adaptive Runge Kutta. The six values are needed to be compute for every iteration.

$$k_1 = hf(t_k, y_k)$$

$$k_2 = hf(t_k + \frac{1}{4}h, y_k + \frac{1}{4}k_1)$$

$$k_3 = hf(t_k + \frac{3}{8}h, y_k + \frac{3}{32}k_1 + \frac{9}{32}k_2)$$

$$k_4 = hf(t_k + \frac{12}{12}h, y_k + \frac{1392}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3)$$

$$k_5 = hf(t_k + h, y_k + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4)$$

$$k_6 = hf(t_k + \frac{1}{2}h, y_k - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5)$$

$$y_{k+1} = y_k + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4101}k_4 - \frac{1}{5}k_5$$

$$z_{k+1} = y_k + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6$$

If the two approximations, z_{k+1} *and* y_{k+1} are in close agreement, the approximation is accepted and the step size is not changed. If the two solutions do not agree to a specified accuracy, the step size is altered by formula below:

$$h_{n+1} = h_n s = h_n \left(\frac{|h_{tol}|}{2|z_{k+1} - y_{k+1}|} \right)^{\frac{1}{4}}$$

where the h_{n+1} is the new h, h_n is the current h, and h_{tol} is the tolerance level of h. The step size will not changing only when the specified error control tolerance is met.

3.5 Matlab Programming

MATLAB is a computer program that issue the user with a simple and convenience platform for performing many types of calculations. Apart from that, it is usually used to solve differential equations and it is a practical way and can be considered as quick and easy.

This numerical problem shows that dealing with ordinary differential equations of the mathematical models is much more efficient and accurate in term of the MATLAB programming. The numerical problem can be solved by key in the formula of the Fourth order of Runge-Kutta Method. The coding for Fehlberg can be easily found on Matlab library on web. The initial condition of the problem and the solution will be calculated accurately by MATLAB.

A build in source code of ode45 was also been used to tabulate the error from the analytical solutions. Hence, the Matlab programme had improve in reducing the time spending on calculating the solution and reduce the chance of making careless mistake or round-off error.

4. Results and discussion

4.1. Experimental Setting

There are some constants used to approximate the solution
mass, m = 1kg, stiffness, k = 9, omega, ω = 1ms^{-1} and force, F = 2Nm^{-2}

Initial Condition
Displacement of the spring, x(0) = 0.5m Velocity of the spring, x'(0) = 0
Ending time, t_max = 10 Initial time, t_0 = 0

Some software also been used to aid with this study. Matlab programming help to tabulate and find the approximation of the solution. Microsoft Excel also been implemented to plot the error graphs between solutions.

4.2. Analytical solution for mass spring damping system

Given $mx'' + \gamma x' + kx = F_0\cos(\omega t)$

$\omega_0 = \sqrt{\frac{k}{m}}$,

$Let\ 2\beta\omega_0 = \frac{\gamma}{m}$

$Then,\ \delta = \frac{\gamma}{2m}$

$x'' + 2\beta\omega_0 x' + \omega_0^2 x = F_0\cos(\omega t)$

$x'' + 2\delta x' + \omega_0^2 x = F_0\cos(\omega t)$

When $\omega_0 > \delta$,

$$x(t) = e^{-\delta t}\left(A\cos\omega_D t + B\sin\omega_D t\right) + R\cos(\omega t - \theta)$$
$$x'(t) = e^{-\delta t}\left((\alpha)\cos\omega_D t - (\beta)\sin\omega_D t\right) - R\omega\sin(\omega t - \theta)$$

where

$$\omega_D = \sqrt{\omega_0^2 - \delta^2}$$

$$R = \frac{\frac{F_0}{m}}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + (2\delta\omega)^2}}$$

$$\tan\theta = \frac{2\delta\omega}{\omega_0^2 - \omega^2}$$

$$\alpha = \omega_D B - \delta A$$

$$\beta = \omega_D A + \delta B$$

The analytic solution for the mass damper spring system can be obtained by

$$x = e^{-\delta t}\left(A\cos\left(\sqrt{\omega_0^2 - \delta^2}\,t\right) + B\sin\left(\sqrt{\omega_0^2 - \delta^2}\,t\right) + \frac{F}{m\left(\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + (2\delta\omega)^2}\right)}\cos\left(\omega t - \tan^{-1}\left(\frac{2\delta\omega}{\omega_0^2 - \omega^2}\right)\right)\right)$$

given that $\delta = 0.5\left(\frac{k}{m}\right)$

4.3 *Comparison of error for Different Step Size of Runge Kutta Order 4 Method and ode45 with Analytical Solution*

A graph was plotted to ilustrate the error between the approximation solution using ode45 and different step size of RK4 with exact solution.

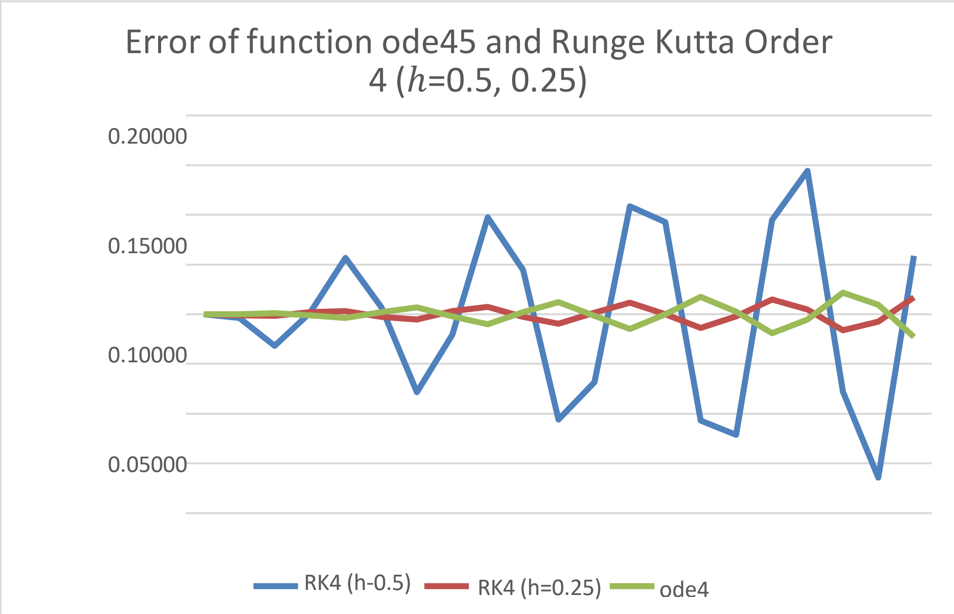

Figure 1: Error of function ode45 and Runge Kutta Order 4 (h=0.5, 0.25)

From the graph, it is clearly illustrated that the Runge Kutta Order 4 Method approximation error with h = 0.25 is smaller than h = 0.5. From the plot, deduction can be made where smaller step size will carry a closer approximation to exact solutions.

4.4 Solving of Mass Spring Damper Ordinary Differential Equation Using ode45 and Fehlberg Method

A graph of the approximate solution for mass spring damper system using Fehlberg method was plotted.

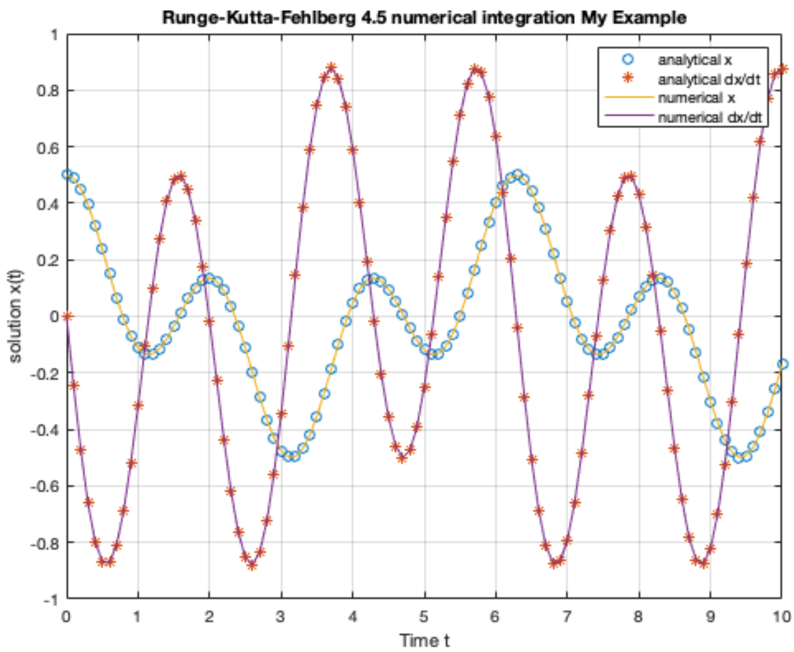

Figure 2: Approximation of Runge Kutta Fehlberg Method

The graph is then validated and verified by using Matlab built-in ode45 function.

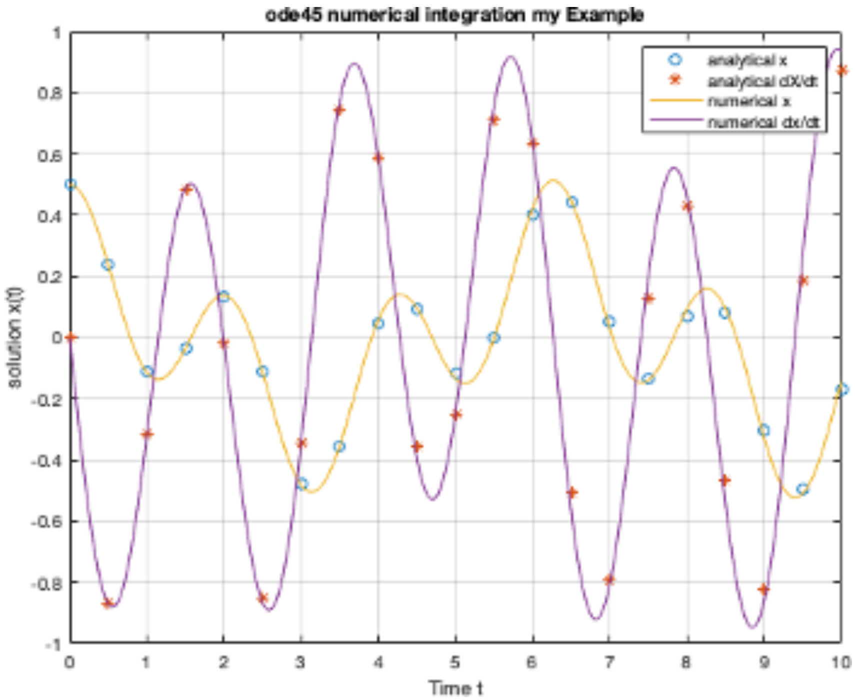

Figure 3: Approximation of solution using ode45

From the graph plotted using Matlab, some deduction can be made for solving this mass spring damper ordinary differential equation. Fehlberg method might not be the best method in solving mass spring problem. This is because Fehlberg method will need much of iterations when encountered to this mass damping spring problem. However, a Matlab built in function of ode45 can further aids with this kind of approximation.

From the output of Matlab graph, the approximation solutions can only be obtained and acceptable only when the step size is lower. The results of plotting for the approximation of RK4 not very efficient in illustrating the solution of this problem.

4.5. Comparison of Fehlberg Method and Runge Kutta Order 4 solution

The results of estimated parameters accuracy using the MCSE and standard deviation values is provided in Table 1.

Based on Table 1, it shows that the MCSE value is < 5% × standard deviation. Therefore, it can be concluded that the estimated parameters of PHMM three-state using Bayesian approach are accurate.

Table 1: Approximation of Runge Kutta Order 4 Method and Fehlberg Method

Iteration,	Time	RK4 Method (h=0.5)	Time	Fehlberg Method
1	0	0.5000	0.5000	0.7506
2	0.5	1.0532	1.1488	1.0053
3	1	1.1971	1.7980	0.5840
4	1.5	0.9558	2.4808	-0.2588
5	2	0.4248	3.2697	-1.1573
6	2.5	-0.2441	3.9857	-1.4306
7	3	-0.8738	4.7461	-0.9752
8	3.5	-1.3020	5.4572	-0.0623

9	4	-1.4190	6.2421	0.9558
10	4.5	-1.1932	6.9624	1.4054
11	5	-0.6780	7.7440	1.1041
12	5.5	0.0015	8.4516	0.2645
13	6	0.6796	9.2086	-0.7619
14	6.5	1.1907	10.0000	-1.3833
15	7	1.4099		
16	7.5	1.2836		
17	8	0.8430		
18	8.5	0.1959		
19	9	-0.4993		
20	9.5	-1.0722		
21	10	-1.3826		

From the result tabulated above, the approximation of Fehlberg method is definitely easier and faster to get the results for time 10. This showed that the efficiency of Fehlberg method is greater than RK4 method. This is because the time will be varying when the step size keeps varying. The step size is getting larger as the tolerance level is still meet. From the table, number of iterations needed to obtain the approximation of time 10 by using Runge Kutta Order 4 method is 20 while only 14 iterations needed for Fehlberg method.

Conclusion

Numerical methods are influential in solving for initial value problems of nonlinear ordinary differential equations. Solving for initial value problems of linear ordinary differential equations might be difficult sometimes. Thus, it would be hard and difficult to solve initial value problems of nonlinear ordinary differential equations without implementing numerical methods.

The main results of this study can be highlighted into two different areas which are Runge Kutta Order 4 Method and Fehlberg method. A numerical method which has smaller step size will have the higheraccuracy level for approximating the mass spring damping system. Fehlberg method sometimes work more efficient if the significance of accuracy level is high for non-stiff problem.

In general, Fehlberg method is one of the most effective methods to achieve a higher accuracy level of solution with less computational time. It can be widely implemented in industries or organisations in finding all the modelling problems. Matlab programming can aid on the computational time of finding approximation of ordinary differential equation especially the built-in function of ode45.

The error analysis of this study is to determine the effectiveness and performance of the Fehlberg method as an alternative for Runge Kutta Order 4 Method. Fehlberg method seemed to be the alternative way when the expectation accuracy level is high. Moreover, Runge Kutta Order 4 Method can be used with smaller step size to approximate the solution if the significance of accuracy is not high

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