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The Order Product Prime Graph for Nonabelian Metabelian Groups of Order at Most 24

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Abstract
The study of groups from a geometric perspective has lately become a focus of group theory research. Later on, as the study progressed, it led to the definition of several graphs of groups and the investigation of graphical features of finite groups. This includes the definition of the order product prime graph where the vertices of the graph are the elements of the groups, and any two vertices are adjacent if and only if the product of their orders is a prime power. A group G is metabelian if and only if there exists an abelian normal subgroup A such that the quotient group G/A is abelian. The purpose of this study is to explore the nonabelian metabelian groups of order at most 24 and to construct the order product prime graph of all nonabelian metabelian groups of order at most 24 by using the definition and related results from past research.
Keywords: Nonabelian metabelian groups; Order product prime graph; Graph of groups

1. Introduction

A metabelian group is an abelian commutator subgroup. In other words, a group is metabelian if and only if the quotient group is abelian and there is an abelian normal subgroup A. According to the concept of metabelian groups, every dihedral group with a cyclic normal subgroup of index two is metabelian. Metabelian groupings' pictures under group homomorphisms are metabelian. A metabelian group's direct product is also metabelian.

Researchers have used a variety of techniques to investigate the properties of a group as well as classify it based on its properties, which is one of the great achievements of modern mathematics. One technique that has been found to be useful is to define graphs for the groups and investigate their properties in terms of the corresponding geometric structures. The research builds a bridge between group theory and graph theory in order to obtain the properties of one in terms of the other. Devi [1] claims that the qualities of a group can be investigated by looking at the relationships between its elements or subgroups. This relationship can be thought of as the corresponding defined graph's vertex adjacency.

Cayley [2] established the study of groups in relation to graph theory in 1878, when he defined the graph that explains the abstract structure of a group generated by a set of generators. Dehn reintroduced this study in [3], calling the Cayley graph a group diagram, which led to today's geometric group theory. Beside Cayley graph, Y. Sattanathan and Kala [4] also defined another graph that is called order prime graph of groups a as a graph with the elements of groups as vertices and two vertices a, b are adjacent if and only if gcd(|a|,|b|)=1. Rajendra and Reddy expanded on this description five years later, in [5] defining general order prime graph of finite groups as a graph with the elements of groups as its vertices and two adjacent vertices a and b if and only if gcd(|a|,|b|)=1, or p where p is a prime number. Then, (|a|,|b|) is used to represent gcd(|a|,|b|).
Furthermore, Newmann defined a non-commuting graph of groups in [6], which is the complement of the commuting graph. Bertram later used the commuting graph's combinatorial features to prove three basic and nontrivial theorems on finite groups in [7]. This concept was expanded by establishing graphs

with the vertices representing subgroups of groups, according to a growing body of literature. For example, Aschbacher defined the commuting graph on subgroups of groups in [8], which led to the definition of several graphs with subgroups of groups as vertices. The permutability graph of conjugacy classes of cyclic subgroups, described by Ballester et al. in [9], is one of these graphs. Rajkumar and Devi described the permutability graph of subgroups of a given group as a graph with a vertex set consisting of all the correct subgroups of a group and two separate vertices are contiguous if the corresponding subgroups permute in [10]. The permutability graph of cyclic subgroups was presented by Rajkumar and Devi in [10], which is a graph with appropriate cyclic subgroups as vertices and two vertices are contiguous if and only if they permute. is metabelian as well.

The previous graphs of groups' vertex adjacencies are connected with a single relationship, such as the relationship between the group's generators, coprimeness, comparatively primeness, or commutativity among the group's members. There are no graphs of groups in which the vertex adjacencies are related with both primeness and commutativity within the components or subgroups of a group. As a result, a new graph of groups called the order product prime graph will be introduced in this paper. Additionally, the newly defined graphs are determined for nonabelian metabelian groups of maximum 24 orders.

2. Literature Review

2.1 Nonabelian Metabelian Group of Order at Most 24

3 **Table** Error! No text of specified style in document.**.1** All nonabelian metabelian groups of order at most 24

No.	Group	$\|G\|$	Group Presentation
1.	D_3	6	$\langle a,b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle$
2.	D_4	8	$\langle a,b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle$
3.	Q_4	8	$\langle a,b \mid a^4 = 1, a^2 = b^2 = 1, aba = b \rangle$
4.	D_5	10	$\langle a,b \mid a^5 = b^2 = 1, bab = a^{-1} \rangle$
5.	$\mathbf{Z}_3 \rtimes \mathbf{Z}_4$	12	$\langle a,b \mid a^4 = b^3 = 1, aba = a \rangle$
6.	A_4	12	$\langle a,b \mid a^2 = b^2 = c^3 = 1, ba = ab, ca = abc, cb = ac \rangle$
7.	D_6	12	$\langle a,b \mid a^6 = b^2 = 1, bab = a^{-1} \rangle$
8.	D_7	14	$\langle a,b \mid a^7 = b^2 = 1, bab = a^{-1} \rangle$
9.	D_8	16	$\langle a,b \mid a^8 = b^2 = 1, bab = a^{-1} \rangle$

10.	Quasihedral-16	16	$\langle a,b \mid a^8=b^2=1, bab=a^3\rangle$
11.	Q_8	16	$\langle a,b \mid a^8=b^2=1, aba=b\rangle$
12.	$D_4\times\mathbf{Z}_2$	16	$\langle a,b,c \mid a^4=b^2=c^2=1, ac=ca, bc=cb, bab=a^{-1}\rangle$
13.	$Q\times\mathbf{Z}_2$	16	$\langle a,b,c \mid a^4=b^4=c^2=1, b^2=a^2, ba=a^3b, ac=ca, bc=cb\rangle$
14.	Modular-16	16	$\langle a,b \mid a^8=b^2=1, ba=ba^5\rangle$
15.	B	16	$\langle a,b \mid a^4=b^4=1, ab=ba^3\rangle$
16.	K	16	$\langle a,b,c \mid a^4=b^2=c^2=1, cbc=ba^2, bab=a, ac=ca\rangle$
17.	$G_{4,4}$	16	$\langle a,b \mid a^4=b^4=1, abab=1, ba^3=ab^3\rangle$
18.	D_9	18	$\langle a,b \mid a^9=b^2=1, bab=a^{-1}\rangle$
19.	$S_3\times\mathbf{Z}_3$	18	$\langle a,b,c \mid a^3=b^2=c^3=1, bab=a^{-1}, ac=ca, bc=cb\rangle$
20.	$\left(\mathbf{Z}_3\times\mathbf{Z}_3\right)\rtimes\mathbf{Z}_2$	18	$\langle a,b,c \mid a^2=b^3=c^3=1, bc=cb, bab=a, cac=a\rangle$
21.	D_{10}	20	$\langle a,b \mid a^{10}=b^2=1, bab=a^{-1}\rangle$
22.	$Fr_{20}\cong\mathbf{Z}_5\rtimes\mathbf{Z}_4$	20	$\langle a,b \mid a^4=b^5=1, ba=ab\rangle$
23.	$\mathbf{Z}_4\rtimes\mathbf{Z}_5$	20	$\langle a,b \mid a^4=b^5=1, ba=a\rangle$
24.	$Fr_{21}\cong\mathbf{Z}_7\rtimes\mathbf{Z}_5$	21	$\langle a,b \mid a^3=b^7=1, ba=ab^2\rangle$
25.	D_{11}	22	$\langle a,b \mid a^{11}=b^2=1, bab=a^{-1}\rangle$
26.	$S_3\times\mathbf{Z}_4$	24	$\langle a,b,c \mid a^3=b^2=c^4=abab=aca^{-1}c^{-1}=bcb^{-1}c^{-1}=1\rangle$
27.	$S_3\times\mathbf{Z}_2\times\mathbf{Z}_2$	24	$\langle a,b,c \mid a^6=b^2=c^2=abab, aca^{-1}c^{-1}=bcb^{-1}c^{-1}=1\rangle$

28.	$D_4 \times \mathbf{Z}_3$	24	$\langle a,b,c \mid a^4 = b^2 = c^3 = 1, baba = 1, ac = ca, bc = cb \rangle$
29.	$Q_4 \times \mathbf{Z}_3$	24	$\langle a,b,c \mid a^4 = c^3 = 1, b^2a^2 = 1, bab = a, ca = ac, cb = bc \rangle$
30.	$A_4 \times \mathbf{Z}_2$	24	$\langle a,b,c \mid a^2 = b^3 = c^2 = 1, ab = ba, ac = ca, c = bcbc \rangle$
31.	$(\square_6 \times \square_2 \rtimes \square_2)$	24	$\langle a,b,c \mid a^3 = b^2 = c^2 = (cb)^4 = 1, ab = ba, aca = c \rangle$
32.	D_{12}	24	$\langle a,b \mid a^{11} = b^2 = 1, bab = a^{-1} \rangle$
33.	$\mathbf{Z}_2 \times (\square_3 \rtimes \square_4)$	24	$\langle a,b \mid a^4 = b^6 = 1, bab = a \rangle$
34.	$\mathbf{Z}_3 \rtimes \mathbf{Z}_8$	24	$\langle a,b \mid a^4 = b^6 = 1, bab = a \rangle$
35.	$\mathbf{Z}_3 \rtimes Q_4$	24	$\langle a,b,c \mid a^3 = b^4 = c^2 = 1, bb = cc, ab = ba, aca = c, bcb = c \rangle$

2.2. Graph of Groups

2.1.1. Fundamental Concepts and Earlier Studies on Graphs of Groups

The notion that combines group theory and graph theory allows mathematicians to analyze the properties of algebraic structures such as groups and semi-groups by building graphs for them and using graph theoretical tools to investigate their features. Many academics have explained some aspects of such algebraic structures in terms of their corresponding graphs, most notably in classifying the algebraic structures by classifying their corresponding graphs in terms of their graphical qualities. Selvakumar and Subajini [13], for example, identified all finite groups with toroidal and projective coprime graphs.

2.1.2. Prime Graph

The prime graph of a group G (Grunberg-Kegel graph) is the graph whose vertex set is the prime spectrum of G (set of all prime order elements divisors of the order of G), and two unique vertices are near if and only if their product is an order of some element of G.

Williams later reported this conclusion in [14], which included a classification of finite simple groups whose prime graph is disconnected, which are distinct from Lie-type groups with odd features. Iiyori and Yamaki [15] grouped the linked components of the prime graphs of the simple groups of Lie type over the field of even characteristic in a similar way. Maria [16] established that the diameter of the prime graph is less than or equal to five, and she classified the virtually simple groups with diameters equal to five.

Gruber et al. investigated the prime graph of solvable groups and presented the requirement for a graph to be isomorphic to the prime graph of solvable groups in [17]. Furthermore, Zhang et al. [18] used the prime graph to study almost simple groups and proved that for a prime p, the group G is isomorphic to the prime graph of solvable groups, S^p if and only if G and the prime graph of solvable groups have the same order and the prime graph of G is the same as the prime graph of solvable groups, S^p. Burness and Cavato later determined the suitable subgroups of a simple group, G and H, for which the prime graph of G is the same as the prime graph of H in [19].

2.1.2. The Order Product Prime Graph

As an extension of the prime graph, Bello [11] has defined a graph called the order product prime graph. The detail explanation on the graph is given in the following subsection. In 2021, Bello [11] has defined a new graph as an extension of the prime graph. The graph is called as the order product prime graph. Suppose G is a finite group. The order product prime graph of G is defined as a graph where vertices are elements of G and two vertices are adjacent if and only if the product of their order is a prime power.

Definition 2.12 [11] Order Product Prime Graph of a Group. Let G be a finite group, then the order product prime graph of $G, \Gamma^{opp}(G),$ is a graph whose vertices are the elements of G and two vertices x, y are adjacent if and only if $|x||y| = p^s, s \in \square$ for some prime p.

3. The Order Product Prime Graph for Nonabelian Metabelian Group of Order Less Than 24

Proof. Let D_n be the dihedral group where n are between $3 \le n \le 11.$ Recall back from Theorem 2.1. For $n = p^s$,

$$\Gamma^{opp}(G) = \begin{cases} K_{2n}, & if\ p = 2 \qquad (i) \\ K_1 + (K_{n-1} \cup K_n), & if\ p \neq 2 \qquad (ii) \end{cases}$$

If $n = 3,$ and by (ii), for $n = p^s$, where p is a prime number and $s \in \square$. In this case, $p = 3,\ s = 1,$ hence

$$\Gamma^{opp}(D_3) = K_1 + (K_2 \cup K_3)$$

Proof. Let Q_n be the quaternion group of order $2n$ which in this case are Q_4, Q_8. Recall back from Theorem 2.3. For $n = p^s$, then

$$\Gamma^{opp}(G) = \begin{cases} K_{4n}, & if\ p = 2 \qquad \text{(v)} \\ K_1 + (K_{n-1} \cup K_{1+2n}) \cup \overline{K}_{n-1}, & if\ p \neq 2 \qquad \text{(vi)} \end{cases}$$

If $n = 4$, and by (v), for $n = p^s$, where p is a prime number and $s \in \square$. In this case, $p = 2,\ s = 2$, hence

$$\Gamma^{opp}(Q_4) = K_8$$

Proposition 3.2 Let $\square_3 \times \square_4$ be a nonabelian metabelian group of order 12. Then, the order product prime graph of $\square_3 \times \square_4$ is

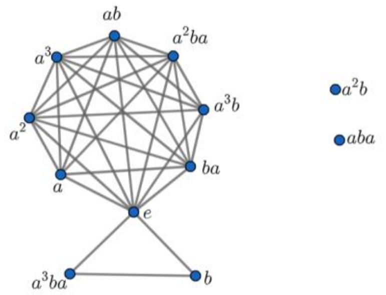

Figure Error! No text of specified style in document.**.1** Order Product Prime Graph of $\square_3 \times \square_4$

Proof. Let $\square_3 \times \square_4$ be the nonabelian metabelian group of order 12.

Given the group presentation of $\square_3 \times \square_4$ as $\langle a, b | a^4 = b^3 = 1, aba = a\rangle$ and the elements in this group are $\square_3 \times \square_4 = \{e, a^2, b, a^3ba, a, a^3, ab, a^3b, ba, a^2ba, a^2b, aba\}$ then the order of each element is $|e| = 1,\ |a^2| = 2,\ |b| = |a^3ba| = 3,\ |a| = |a^3| = |ab| = |a^3b|$

$= |ba| = |a^2ba| = 4,\ |a^2b| = |aba| = 6$.

Suppose A be the set of the elements that has order 1, 2 and 4 and B be the set of elements that has order 1 and 3 in direct product of $\square_3 \times \square_4$**.** The product of order of the elements in those set, respectively is a prime number. By Definition 2.12, there is an edge linking all the vertices of each of the sets while elements with order 6 are isolated vertices. The graph, $\Gamma^{opp}(\square_3 \times \square_4)$ is shown in Figure 3.1 above.

4. The Order Product Prime Graph for Nonabelian Metabelian Group of Order 24

Proposition 3.1. Let D_{12} be the dihedral group of order 24. Then, the order product prime graph of D_{12} are as follows:

$$\Gamma^{opp}(D_{12}) = K \ +[\cup \textstyle\bigcup_{i=2}^{2} K_2 \cup K_{15}] \cup \overline{K_6}$$

Proof. Let D_{12} be the dihedral group of order 24. Recall back from Theorem 2.2.

If $n = 12$, and by (iii), for $n = p^s$, where p is a prime number and $s \in \square$. In this case, $p_1 = 2,\ s_1 = 2$. Thus, the result holds.

Proposition 3.17. Let $S_3 \times \square_4$ be a nonabelian metabelian group of order 24. Then the order product prime graph of $S_3 \times \square_4$ is

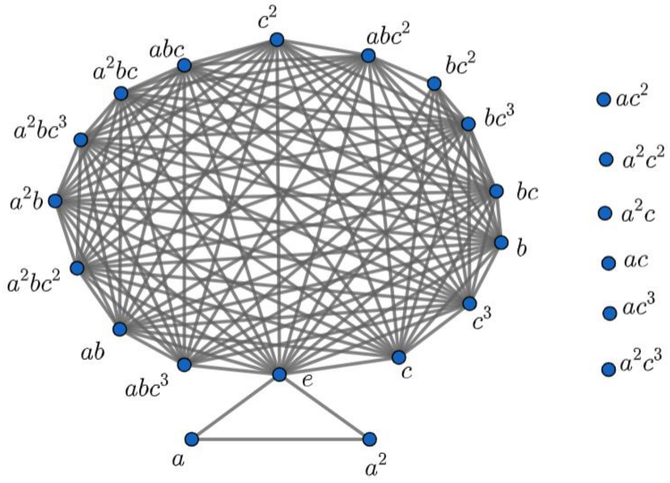

Figure Error! No text of specified style in document.**.2** Order Product Prime Graph of $S_3 \times \square_4$

Proof. Let $S_3 \times \square_4$ be the direct product group of order 24. Then, the order product prime graph of $S_3 \times \square_4$.

Given the group presentation of $S_3 \times \square_4$ as $\langle a,b,c \mid a^3 = b^2 = c^4 = abab = aca^{-1}c^{-1} = bcb^{-1}c^{-1} = 1 \rangle$ and the elements in this group are $S_3 \times \square_4$ = $\{e, a, a^2, c, c^3, bc, abc, a^2bc, bc^3, abc^3, a^2bc^3, c^2, b, ab, a^2b, bc^2, abc^2,$

$a^2bc^2, ac^2, a^2c^2, ac, a^2c, ac^3, a^2c^3\}$. Then, the order of each element is $|e| = 1, |c^2| = |b|$

$= |ab| = |a^2b| = |bc^2| = |abc^2| = |a^2bc^2| = 2, |a| = |a^2| = 3, |c| = |c^3| = |bc| = |abc| = |a^2bc| =$

$|bc^3| = |a^2bc^3| = |abc^3| = 4, |ac^2| = |a^2c^2| = 6, |ac| = |a^2c| = |ac^3| = |a^2c^3| = 12$.

Suppose A be the set for the elements that has order 1 and 2 and B be the set of elements of order 1 and 3 in $S_3 \times \square_4$. The product of order of the elements in those sets, respectively is a prime number. By Definition 2.12, there is an edge linking all the vertices of each of the sets while the elements with order 6 and 12 are isolated vertices. The graph, $\Gamma^{opp}(S_3 \times \square_4)$ is shown in Figure 3.15 above.

Proposition 3.18. Let $D_4 \times \square_3$ be a nonabelian metabelian group of order 24. Then the order product prime graph of $D_4 \times \square_3$ is

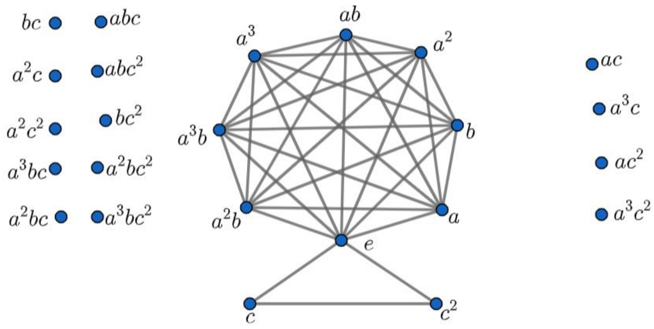

Figure Error! No text of specified style in document.**.3** Order Product Prime Graph of $D_4 \times \mathbb{Z}_3$

Proof. Let $D_4 \times \mathbb{Z}_3$ be the direct product group of order 24. Then, the order product prime graph of $D_4 \times \mathbb{Z}_3$.

Given the group presentation of $D_4 \times \mathbb{Z}_3$ as $\left\langle a,b,c \middle| a^4 = b^2 = c^3 = 1, baba = 1, ac = ca, bc = cb \right\rangle$ and the elements in this group are $D_4 \times \mathbb{Z}_3$ = $\{e, a, a^2, c, b, ab, a^2b, a^3b, c^2, a^3, a^2c, a^2c^2, bc, abc, a^2bc, a^3bc, bc^2,\ abc^2,$ $a^2bc^2, a^3bc^2, ac, a^3c, ac^2, a^3c^2\}$. Then, the order of each element is $|e| = 1,\ |a^2| = |b| =$ $|ab| = |a^2b| = |a^3b| = 2, |c| = |c^2| = 3, |a| = |a^3| = 4, |a^2c| = |a^2c^2| = |bc| = |abc| = |a^2bc| =$ $|a^3bc| = |bc^2| = |abc^2| = |a^2bc^2| = |a^3bc^2| = 6, |ac| = |a^3c| = |ac^2| = |a^3c^2| = 12$.

Suppose A be the set for the elements that has order 1, 2 and 4 and B be the set of elements of order 1 and 3 in $D_4 \times \mathbb{Z}_3$. The product of order of the elements in those sets, respectively is a prime number. By Definition 2.12, there is an edge linking all the vertices of each of the sets while the elements with order 6 and 12 are isolated vertices. The graph, $\Gamma^{opp}(D_4 \times \mathbb{Z}_3)$ is shown in Figure 3.17 above.

5. Conclusion

In this research, all the order product prime graph for nonabelian metabelian group of order at most 24 are determined and constructed. Since this research focuses on only for the order product prime graph of nonabelian metabelian group of order at most 24, it can be extended to the commuting order product prime graph for nonabelian metabelian group of order at most 24.

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