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# The Laplacian Spectrum of the Prime Order Cayley Graphs of Some Quaternion Groups

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**Abstract**
The Laplacian spectrum of a graph is the set of multiplicities of eigenvalues of the graph's Laplacian matrix. The Laplacian matrix of a graph $\Gamma$, $L(\Gamma)$, is given by $L(\Gamma) = D(\Gamma)$- $A(\Gamma)$, where $A(\Gamma)$ is the adjacency matrix and $D(\Gamma)$ is the diagonal matrix of vertex degrees of the graph. The Laplacian spectrum has been determined for various graphs such as commuting graphs of finite groups, power graph of finite groups, directed cyclic sun graph and so on. Meanwhile, a Cayley graph is a structure that is made up of vertices and edges which describes the group's presentation that is determined by a specific set of generators in which two vertices are connected by an edge in specific conditions. Extending the idea of Cayley graph, the prime order Cayley graph of a group $G$ has been introduced as a graph with the elements of $G$ as the vertices of the graph, and two distinct vertices $x$ and $y$ in $G$ are adjacent by an edge whenever $xy^{-1} \in S$, where $S$ is a subset of $G$ containing elements of prime order. This research focuses on the computation of the Laplacian spectrum of prime order Cayley graphs of the quaternion groups of order at most 32. In order to determine the Laplacian spectrum, the definition of the prime order Cayley graph is used together with the group's presentation to reconstruct the graph. Then, based on the structure of the graph, the adjacency matrix, degree matrix and Laplacian matrix are computed. Assisted by Maple software, the characteristic polynomial and the eigenvalues of the Laplacian matrix are determined. Finally, the Laplacian spectrum of the prime order Cayley graph of the group is computed based on the eigenvalues obtained. The Laplacian spectrum of the prime order Cayley graph of quaternion groups of order eight, 16 and 32 are found to be sets of eigenvalues 0 and 2 with the multiplicities to be half of the groups' order.
**Keywords** Laplacian spectrum; prime order Cayley graph; quaternion groups.

## 1. Introduction

A graph is made up of vertices that are also called as nodes or points which are connected by edges which are also called as links or lines which not necessarily straight. If the edges linked two vertices symmetrically the graph is called an undirected graph while if the edges linked two vertices asymmetrically the graph is known as a directed graph.

Let $\Gamma$ be a graph which consists a set of edges, $E(\Gamma) = \{e_1, e_2, \dots, e_n\}$ and vertices, $V(\Gamma) = \{v_1, v_2, \dots, v_n\}$ [1]. The adjacency matrix, $A(\Gamma) = [a_{ij}]$ of $\Gamma$ is an $n \times n$ matrix where $a_{ij} = 1$ if $\{v_i, v_j\} \in E(\Gamma)$ and $a_{ij} = 0$ if $\{v_i, v_j\} \notin E(\Gamma)$ [2]. The diagonal matrix of vertex degrees, $D(\Gamma)$ is the number of edges that are incident with the vertex [3]. The Laplacian matrix of $\Gamma$, $L(\Gamma)$ is defined by $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $A(\Gamma)$ is the adjacency matrix and $D(\Gamma)$ is the diagonal matrix of vertex degrees [4]**.** The roots of the characteristic polynomial of $L(\Gamma)$ are the eigenvalues. These set of eigenvalues with the multiplicities are the Laplacian spectrum, *L-Spec*($\Gamma$) [5].

In addition, a group is a set of integers under the binary operation that combines any two elements of the set to produce a third element of the set in a way such as the operation is associative, an identity element and each element has an inverse [6]. In group theory, a quaternion group of order $2^n$ can be represented by the group presentation:

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = e,\ a^{2^{n-2}} = b^2,\ ba = a^{-1}b \rangle \text{ for } n \geq 3.$$

A group is a set of objects with a rule of combination. Any two elements of the group are given so that the rule produces another group element which depends on the two elements chosen. Therefore, the information of group can be represented by a graph which is the collection of points known as vertices and lines between them known as edges. The vertices are the elements of the group and the edges are determined based on the relationships of the combination rule in the case of the graph encoding a group. This graph is called as a Cayley graph of the group [7].

A Cayley graph is a graph that encodes the abstract structure of a group. It is also known as Cayley color graph, Cayley diagram, color group or group diagram and denoted by $Cay(G, S)$ where $G$ is a finite group and $S$ is a non-empty subset of *G*. It can be concluded that Cayley graph is dependent on a specific set of generators of the group [8]. A new type of Cayley graph namely prime order Cayley graph has been introduced in [9]. Meanwhile, in [10] the prime order Cayley graphs have been constructed for quaternion groups of order $2^n$, where $n \geq 3$**.**

## 2. Literature Review

### *2.1. Basic Concepts in Group Theory*

In order to reconstruct the prime order Cayley graph of the quaternion groups, some basic concepts in group theory are presented in this section.

**Definition 2.1 [22] Group**

Let $G$ be a group together with a binary operation which is denoted as "•" that combines any two elements $a$ and $b$ to form an element denoted as $a \bullet b$. The following group axioms which known as three requirements are satisfied:

Associativity: $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for all $a, b, c$.

Identity element: There exists an element $e$ such that for every $a$, $e \bullet a = a$ and $a \bullet e = a$.

Inverse element: There exists an element $b$ for each $a$ such that $a \bullet b = e$ and $b \bullet a = e$, where $e$ is the identity element. The element $b$ is unique for each $a$ where it is called the inverse of $a$ and is commonly denoted as $a^{-1}$.

**Definition 2.2 [16] Quaternion Group**

The quaternion group of order $2^n$, $Q_{2^n}$ is

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = e,\ a^2 = b^2, ba = a^{-1}b \rangle,$$

where $n \geq 3$.

### *2.2. Preliminaries on Graph Theory*

Some preliminaries on graph theory are stated before proceeding to other points. The general definitions of a graph are given as follows,

**Definition 2.3 [20] Graph**

A graph $\Gamma$ is a pair $\Gamma = (V, E)$, where $V$ is a set of elements called vertices and $E$ is a set of paired vertices which are known as edges. A function $f$ assigns to each edge a subset $\{v_0, v_1\}$ where $v_0$ and $v_1$ are vertices, shown as follows:

$\Gamma = (V, E, f)$ or $\Gamma = (V, E)$.

**Definition 2.4 [3] Complete Graph**

A complete graph $K_n$ is a connected graph on *n* vertices where all vertices are of degree *n*-1. There is an edge between a vertex and every other vertex. A complete graph has $\frac{n(n-1)}{2}$ edges.

### *2.3. Graphs Associated to Groups*

The definition of graph of group, Cayley graph and prime order Cayley graph is given as follows.

**Definition 2.5 [24] Graph of a Group**

A graph of a group $G$, $\Gamma_G$ is an object consisting of a collection of a pair of vertices, $V$ and edges, $E$ labelled as $\Gamma_G = (V, E)$ based on geometric group theory. The elements of $G$ are the vertices of $\Gamma_G$ and the elements of $E(G)$ are the lines that join the two elements of $V(G)$.

In addition, the Cayley graph, *Cay*(*G* : *S*), where *G* is a group and *S* is a subset of *G*, is the undirected graph with vertex set *G* and set of edge. The definition of the Cayley graph is given as follows.

**Definition 2.6 [25] Cayley graph of a Group**

A Cayley graph on a group $G$ denoted as $Cay(G, S)$ is a graph with a subset $S \subseteq G \backslash \{e\}$, with $S = S^{-1} = \{s^{-1} | s \in S\}$, such that $V(Cay(G, S)) = G$, and two vertices $g$ and $h$ are adjacent if and only if $hg^{-1} \in S$. In other words, $hg^{-1} \in S$ implies that $\exists s \in S$ with $hg^{-1} = s$ or $h = sg$.

**Definition 2.7 [9] Prime Order Cayley graph**

Let $G$ be a group and $S$ be the set of prime order elements of $G$. A prime order Cayley graph, $Cay(G, S)$ is a graph where the set of vertices of the graph is the elements of $G$ and for two distinct vertices. $x$ and $y$ are connected by an edge whenever $xy^{-1} \in S$, that is $x = sy$, for some $s \in S$.

**Theorem 2.8 [10]** The prime order Cayley graph of quaternion group of order $2^n$, for $n \geq 3$, $Cay_p(Q_{2^n}, S)$ is a union of $2^{n-1}$ components of complete graph of two vertices, $K_2$.

Based on the results in [8], the prime order Cayley graphs of the quaternion groups of order at most 32 are reconstructed in order to determine the Laplacian spectrum of the graphs.

### 2.4. The Laplacian Spectrum of Graph

In the computation of Laplacian spectrum, the Laplacian matrix and adjacency matrix need to be obtained. The definition of adjacency matrix, characteristic polynomial and Laplacian spectrum are given as follows.

**Definition 2.9 [26] Adjacency Matrix**

The adjacency matrix of a graph $\Gamma$ that denoted by $A(\Gamma)$ is known as the connection matrix of a graph $\Gamma$ with $n$ vertices and no parallel edges. It is defined as in the following:

$$A(\Gamma) = \begin{cases} x_{ij=}1 \text{ if } v_i \rightarrow v_j \\ x_{ij} = 0 \text{ if } otherwise \end{cases}$$

where $v_i \rightarrow v_j$ represents the mapping from $i$th vertex to $j$th vertex.

**Definition 2.10 [3] Degree of Vertex**

The degree of a vertex, $\deg(v)$ is the number of edges that are incident with the vertex.

**Definition 2.11 [27] Diagonal Matrix of Vertex Degrees**

Let a diagonal matrix of vertex degrees, $D(\Gamma) = diag(v_1, \ldots, v_n)$ where $v_n$ is the degree of vertex.

**Definition 2.12 [19] Laplacian Matrix**

The Laplacian matrix of a graph $\Gamma$ is given by $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $A(\Gamma)$ is adjacency matrix and $D(\Gamma)$ is diagonal matrix of vertex degrees of a $\Gamma$.

**Definition 2.13 [28] Characteristic Polynomial**

The characteristic polynomial of a graph $\Gamma$ is $\det(\lambda I - A)$ where *A* is any $n \times n$ matrix and denoted by $f(\lambda)$.

**Definition 2.14 [19] Laplacian Spectrum**
The Laplacian spectrum of a graph $\Gamma$ denoted by *L-Spec*$(\Gamma)$is the set $\{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_n^{k_n}\}$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the Laplacian matrix of $\Gamma$ with multiplicities $k_1, k_2, \dots, k_n$ respectively.

## 3. The Laplacian Spectrum of the Prime Order Cayley Graph of Quaternion Group of Order Eight

### *3.1 Reconstruction of the Prime Order Cayley Graph of the Quaternion Group of Order Eight*

The group presentation of $Q_8$ as follows:

$$Q_8 = \langle a, b \,|\, a^4 = e, a^2 = b^2, ba = a^{-1}b \rangle.$$

Hence, the elements of $Q_8$ are:

$$Q_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$

The order of each element of $Q_8$ is calculated to determine which element has prime order. The order of the eight elements in $Q_8$ is shown as follows,

| $g_1 \in Q_8$ | $e$ | $a$ | $a^2$ | $a^3$ | $b$ | $ab$ | $a^2b$ | $a^3b$ |
|---|---|---|---|---|---|---|---|---|
| $\lvert g_1 \rvert$ | 1 | 4 | 2 | 4 | 4 | 4 | 4 | 4 |

By the definition of prime order Cayley graph given in Definition 2.7, the set of vertices of $Cay_p(Q_8, S_1)$ is:
$V(Cay_p(Q_8, S_1)) = Q_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. The subset of $Q_8$ with the element of prime order is $S_1 = \{a^2\}$. Let $x, y \in Q_8$. Vertex $x$ is connected to vertex $y$, denoted as $x{\sim}y$ if $xy^{-1} \in S$ which implies that exists $s \in S$ such that $xy^{-1} = S$ or $x = sy$. Therefore, the set of edges of $Cay_p(Q_8, S_1)$ is given as follows:

$$E(Cay_p(Q_8, S_1)) = \{\{e, a^2\}, \{a, a^3\}, \{b, a^2b\}, \{ab, a^3b\}\}.$$

### *3.2 The Laplacian Matrix of Prime Order Cayley Graph of the Quaternion Group of Order Eight*

In Definition 2.12, the Laplacian matrix of $\Gamma$, $L\left(Cay_p(Q_8, S_1)\right)$ is given by

$$L\left(Cay_p(Q_8, S_1)\right) = D\left(Cay_p(Q_8, S_1)\right) - A\left(Cay_p(Q_8, S_1)\right)$$

Thus, firstly, the adjacency matrix of prime order Cayley graph of quaternion group of order eight, $Q_8$ is determined based on the set of edges of $Cay_p(Q_8, S_1)$ and the structure of the graph. Based on the Definition 2.10, the entry for adjacency matrix, $a_{ij} = 1$ if the pair of elements are connected by an edge and $a_{ij} = 0$ if the elements are not connected by an edge where *i* denoted as row and *j* denoted as column. Therefore, the adjacency matrix of $Cay_p(Q_8, S_1)$ is

$$A\left(Cay_p(Q_8, S_1)\right) = \begin{array}{c} \begin{matrix} e & a & a^2 & a^3 & b & ab & a^2b & a^3b \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \begin{matrix} e \\ a \\ a^2 \\ a^3 \\ b \\ ab \\ a^2b \\ a^3b \end{matrix}$$

In addition, the diagonal matrix of vertex degrees of $Cay_p(Q_8, S_1)$ is determined as follows:

$$D(\Gamma) = \begin{cases} d(i), & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

where $d(i)$ is the degree of the vertex $i$. Based on the prime order Cayley graph of $Q_8$, $Cay_p(Q_8, S_1)$, the degree of all vertices is 1 since only one edge is connected to each vertex.

Therefore, the diagonal matrix of vertex degrees of $Cay_p(Q_8, S_1)$ is

$$D\left(Cay_p(Q_8,S_1)\right)=\begin{array}{c}\begin{matrix} e & a & a^2 & a^3 & b & ab & a^2b & a^3b \end{matrix}\\ \begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\end{bmatrix}\end{array}\begin{matrix} e\\ a\\ a^2\\ a^3\\ b\\ ab\\ a^2b\\ a^3b\end{matrix}$$

Based on the Definition 2.12, the Laplacian matrix of prime order Cayley graph of $Q_8$ is

$$L\left(Cay_p(Q_8,S_1)\right)=\begin{bmatrix}1&0&-1&0&0&0&0&0\\0&1&0&-1&0&0&0&0\\-1&0&1&0&0&0&0&0\\0&-1&0&1&0&0&0&0\\0&0&0&0&1&0&-1&0\\0&0&0&0&0&1&0&-1\\0&0&0&0&-1&0&1&0\\0&0&0&0&0&-1&0&1\end{bmatrix}$$

### *3.3 The Laplacian Spectrum of $Cay_p(Q_8,S_1)$*

In order to compute the Laplacian spectrum of $Cay_p(Q_8,S_1)$, the eigenvalue of Laplacian matrix is determined that assists by Maple software. Hence, the characteristic polynomial of $L\left(Cay_p(Q_8,S_1)\right)$ is

$$f(\lambda)=\lambda^8-8\lambda^7+24\lambda^6-32\lambda^5+16\lambda^4$$

Thus, the eigenvalues are $\lambda=0$ with multiplicity 4 and $\lambda=2$ with multiplicity 4 as well. Based on Definition 2.14, $L-Spec(\Gamma)=\{\lambda_1^{k_1},\lambda_2^{k_2},\dots,\lambda_n^{k_n}\}$, the Laplacian spectrum of prime order Cayley graph of $Q_8$, $L\text{-}Spec\left(Cay_p(Q_8,S_1)\right)$ is $\{0,0,0,0,2,2,2,2\}=\{0^4,2^4\}$.

Therefore, the Laplacian spectrum of prime order Cayley graph of quaternion group of order eight is determined to be the set of eigenvalues of 0 and 2 with the multiplicities four.

## 4. The Laplacian Spectrum of the Prime Order Cayley Graph of Quaternion Group of Order 16

### *4.1 Reconstruction of the Prime Order Cayley Graph of the Quaternion Group of Order 16*

The group presentation of $Q_{16}$ is

$$Q_{16}=\langle a,b\,|\,a^8=e,\ a^4=b^2, ba=a^{-1}b\rangle.$$

Hence, the elements of $Q_{16}$ are:

$$Q_{16}=\{e,a,a^2,a^3,a^4,a^5,a^6,a^7,b,ab,a^2b,a^3b,a^4b,a^5b,a^6b,a^7b\}.$$

The order of each element of $Q_{16}$ is calculated to determine which element has prime order. The order of the sixteen elements in $Q_{16}$ is shown as follows,

| $g_2\in Q_{16}$ | $e$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ | $a^7$ | $b$ | $ab$ | $a^2b$ | $a^3b$ | $a^4b$ | $a^5b$ | $a^6b$ | $a^7b$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert g_2\rvert$ | 1 | 8 | 4 | 8 | 2 | 8 | 4 | 8 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |

By the definition of prime order Cayley graph given in Definition 2.7, the set of vertices of $Cay_p(Q_{16},S_2)$ is:

$V(Cay_p(Q_{16},S_2))=Q_{16}=\{e,a,a^2,a^3,a^4,a^5,a^6,a^7,b,ab,a^2b,a^3b,a^4b,a^5b,a^6b,a^7b\}$. The subset of $Q_{16}$ with the element of prime order is $S_2=\{a^4\}$. Let $x,y\in Q_{16}$. Vertex $x$ is connected to vertex $y$, denoted as $x{\sim}y$ if $xy^{-1}\in S$ which implies that exists $s\in S$ such that $xy^{-1}=S$ or $x=sy$. Therefore, the set of edges of $Cay_p(Q_{16},S_2)$ is given as follows:

$$E(Cay_p(Q_{16},S_2)=\{\{e,a^4\},\{a,a^5\},\{a^2,a^6\},\{a^3,a^7\},\{a^4b,b\},\{a^5b,ab\},\{a^2b,a^6b\},\{a^3b,a^7b\}\}.$$

### *4.2 The Laplacian Matrix of Prime Order Cayley Graph of the Quaternion Group of Order 16*

In Definition 2.12, the Laplacian matrix of $\Gamma$, $L(Cay_p(Q_{16},S_2)\,)$ is given by

$$L(Cay_p(\mathrm{Q}_{16}, S_2)) = D(Cay_p(\mathrm{Q}_{16}, S_2)) - A(Cay_p(\mathrm{Q}_{16}, S_2))$$

Thus, the adjacency matrix of prime order Cayley graph of quaternion group of order 16, $Q_{16}$ is determined by using the group presentation of $Q_{16}$. Based on the Definition 2.10, the entry for adjacency matrix, $a_{ij} = 1$ if the pair of elements are connected by an edge and $a_{ij} = 0$ if the elements are not connected by an edge where *i* denoted as row and *j* denoted as column. Therefore, the adjacency matrix of $Cay_p(\mathrm{Q}_{16}, S_2)$ is

$A(Cay_p(Q_{16}, S_2)) =$

| $e$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ | $a^7$ | $b$ | $ab$ | $a^2b$ | $a^3b$ | $a^4b$ | $a^5b$ | $a^6b$ | $a^7b$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $e$ |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^2$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^3$ |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^4$ |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^5$ |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^6$ |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^7$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | $b$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | $ab$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | $a^2b$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | $a^3b$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^4b$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | $a^5b$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | $a^6b$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | $a^7b$ |

In addition, the diagonal matrix of vertex degrees of $Cay_p(\mathrm{Q}_{16}, S_2)$ is determined as follows:

$$D(\Gamma) = \begin{cases} d(i), & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

where $d(i)$ is the degree of the vertex $i$. Based on the prime order Cayley graph of $Q_{16}$, $Cay_p(\mathrm{Q}_{16}, S_2)$ , the degree of all vertices is 1 since only one edge is connected to each vertex. Therefore, the diagonal matrix of vertex degrees of $Cay_p(\mathrm{Q}_{16}, S_2)$ is

$D(Cay_p(Q_{16}, S_2)) =$

| $e$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ | $a^7$ | $b$ | $ab$ | $a^2b$ | $a^3b$ | $a^4b$ | $a^5b$ | $a^6b$ | $a^7b$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $e$ |
| 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a$ |
| 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^2$ |
| 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^3$ |
| 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^4$ |
| 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^5$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^6$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $a^7$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $b$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | $ab$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | $a^2b$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | $a^3b$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | $a^4b$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | $a^5b$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | $a^6b$ |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | $a^7b$ |

Based on the Definition 2.12, the Laplacian matrix of prime order Cayley graph of $Q_{16}$ is

$$L(Cay_p(Q_{16}, S_2)) = \begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1
\end{bmatrix}$$

### *4.3 The Laplacian Spectrum of* $Cay_p(Q_{16}, S_2)$

In order to compute the Laplacian spectrum of $Cay_p(Q_{16}, S_2)$, the eigenvalue of Laplacian matrix is determined with the help of Maple software. Hence, the characteristic polynomial of $L(Cay_p(Q_{16}, S_2))$ is calculated as follows:

$$f(\lambda) = (\lambda^2 - 2\lambda)^8$$

Thus, the eigenvalues are $\lambda = 0$ with multiplicity 8 and $\lambda = 2$ with multiplicity 8 as well. Based on Definition 2.14, $L - Spec(\Gamma) = \{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_n^{k_n}\}$, the Laplacian spectrum of prime order Cayley graph of $Q_{16}$, $L\text{-}Spec(Cay_p(Q_{16}, S_2))$ is $\{0,0,0,0,0,0,0,0,2,2,2,2,2,2,2,2\} = \{0^8, 2^8\}$.
Therefore, the Laplacian spectrum of prime order Cayley graph of quaternion group of order 16 is determined to be the set of eigenvalues of 0 and 2 with the multiplicities eight.

## 5. The Laplacian Spectrum of the Prime Order Cayley Graph of Quaternion Group of Order 32

### *5.1 The Prime Order Cayley Graph of the Quaternion Group of Order 32*

The group presentation of $Q_{32}$ as follows:

$$Q_{32} = \langle a, b \mid a^{16} = e, a^8 = b^2, ba = a^{-1}b\rangle.$$

Hence, the elements of $Q_{32}$ are:

$$Q_{32} = \begin{Bmatrix} e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}, \\ b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b, a^8b, a^9b, a^{10}b, a^{11}b, a^{12}b, a^{13}b, a^{14}b, a^{15}b \end{Bmatrix}.$$

The order of each element of $Q_{32}$ is calculated to determine which element has prime order. The order of the thirty-two elements in $Q_{32}$ is shown as follows,

| $g_3 \in Q_{32}$ | $e$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ | $a^7$ | $a^8$ | $a^9$ | $a^{10}$ | $a^{11}$ | $a^{12}$ | $a^{13}$ | $a^{14}$ | $a^{15}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert g_3 \rvert$ | 1 | 16 | 8 | 16 | 4 | 16 | 8 | 16 | 2 | 16 | 8 | 16 | 4 | 16 | 8 | 16 |

| $g_3 \in Q_{32}$ | $b$ | $ab$ | $a^2b$ | $a^3b$ | $a^4b$ | $a^5b$ | $a^6b$ | $a^7b$ | $a^8b$ | $a^9b$ | $a^{10}b$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\lvert g_3 \rvert$ | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |

| $g_3 \in Q_{32}$ | $a^{11}b$ | $a^{12}b$ | $a^{13}b$ | $a^{14}b$ | $a^{15}b$ |
|---|---|---|---|---|---|
| $\lvert g_3 \rvert$ | 4 | 4 | 4 | 4 | 4 |

By the definition of prime order Cayley graph given in Definition 2.7, the set of vertices of $Cay_p(Q_{32}, S_3)$ is: $V(Cay_p(Q_{32}, S_3)) = Q_{32} = \begin{Bmatrix} e, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9, a^{10}, a^{11}, a^{12}, a^{13}, a^{14}, a^{15}, \\ b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b, a^8b, a^9b, a^{10}b, a^{11}b, \\ a^{12}b, a^{13}b, a^{14}b, a^{15}b \end{Bmatrix}$.

The subset of $Q_{32}$ with the element of prime order is $S_3 = \{a^8\}$. Let $x, y \in Q_{32}$. Vertex $x$ is connected to vertex $y$, denoted as $x \sim y$ if $xy^{-1} \in S$ which implies that exists $s \in S$ such that $xy^{-1} = S$ or $x = sy$. Therefore, the set of edges of $Cay_p(Q_{32}, S_3)$ is given as follows:

$$E(Cay_p(Q_{32}, S_3)) =$$
$$\begin{Bmatrix} \{e, a^8\}, \{a, a^9\}, \{a^2, a^{10}\}, \{a^3, a^{11}\}, \{a^4, a^{12}\}, \{a^5, a^{13}\}, \{a^6, a^{14}\}, \{a^7, a^{15}\}, \\ \{b, a^8b\}, \{ab, a^9b\}, \{a^2b, a^{10}b\}, \{a^3b, a^{11}b\}, \{a^4b, a^{12}b\}, \{a^5b, a^{13}b\}, \{a^6b, a^{14}b\}, \{a^7b, a^{15}b\} \end{Bmatrix}..$$

### *5.2 The Laplacian Matrix of Prime Order Cayley Graph of the Quaternion Group of Order 32*

In Definition 2.12, the Laplacian matrix of $\Gamma$, $L(Cay_p(Q_{32}, S_3))$ is given by

$$L(Cay_p(Q_{32}, S_3)) = D(Cay_p(Q_{32}, S_3)) - A(Cay_p(Q_{32}, S_3))$$

Thus, the adjacency matrix of prime order Cayley graph of quaternion group of order 32, $Q_{32}$ is determined by using the group presentation of $Q_{32}$. Based on the Definition 2.10, the entry for adjacency matrix, $a_{ij} = 1$ if the pair of elements are connected by an edge and $a_{ij} = 0$ if the elements are not connected by an edge where *i* denoted as row and *j* denoted as column.

Therefore, the adjacency matrix of $Cay_p(Q_{32}, S_3)$ is

$A\left(Cay_p(Q_{32}, S_3)\right) =$

| | $e$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ | $a^7$ | $a^8$ | $a^9$ | $a^{10}$ | $a^{11}$ | $a^{12}$ | $a^{13}$ | $a^{14}$ | $a^{15}$ | $b$ | $ab$ | $a^2b$ | $a^3b$ | $a^4b$ | $a^5b$ | $a^6b$ | $a^7b$ | $a^8b$ | $a^9b$ | $a^{10}b$ | $a^{11}b$ | $a^{12}b$ | $a^{13}b$ | $a^{14}b$ | $a^{15}b$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $e$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^5$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^6$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^7$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^8$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^9$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{10}$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{11}$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{12}$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{13}$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{14}$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{15}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $ab$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^2b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $a^3b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $a^4b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $a^5b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $a^6b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $a^7b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| $a^8b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^9b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{10}b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{11}b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{12}b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{13}b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{14}b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{15}b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

In addition, the diagonal matrix of vertex degrees of $Cay_p(\mathrm{Q}_{32}, S_3)$ is determined as follows:

$$D(\Gamma) = \begin{cases} d(i), & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

where $d(i)$ is the degree of the vertex $i$. Based on the prime order Cayley graph of $Q_{32}$, $Cay_p(\mathrm{Q}_{32}, S_3)$, the degree of all vertices is 1 since only one edge is connected to each vertex. Therefore, the diagonal matrix of vertex degrees of $Cay_p(\mathrm{Q}_{32}, S_3)$ is

$D\left(Cay_p(Q_{32}, S_3)\right) =$

| | $e$ | $a$ | $a^2$ | $a^3$ | $a^4$ | $a^5$ | $a^6$ | $a^7$ | $a^8$ | $a^9$ | $a^{10}$ | $a^{11}$ | $a^{12}$ | $a^{13}$ | $a^{14}$ | $a^{15}$ | $b$ | $ab$ | $a^2b$ | $a^3b$ | $a^4b$ | $a^5b$ | $a^6b$ | $a^7b$ | $a^8b$ | $a^9b$ | $a^{10}b$ | $a^{11}b$ | $a^{12}b$ | $a^{13}b$ | $a^{14}b$ | $a^{15}b$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $e$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a$ | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^2$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^3$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^4$ | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^5$ | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^6$ | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^7$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^8$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^9$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{10}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{11}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{13}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{14}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{15}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $ab$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^2b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^3b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^4b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^5b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^6b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^7b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^8b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^9b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $a^{10}b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| $a^{11}b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| $a^{12}b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| $a^{13}b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $a^{14}b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| $a^{15}b$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

Based on the Definition 2.12, the Laplacian matrix of prime order Cayley graph of $Q_{32}$ is

$$L\left(Cay_p(Q_{32},S_3)\right) =$$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

### *5.3* The Laplacian Spectrum of $Cay_p(\mathbf{Q}_{32}, S_3)$

In order to compute the Laplacian spectrum of $Cay_p(\mathrm{Q}_{32}, S_3)$, the eigenvalue of Laplacian matrix is determined with the help of Maple software. Hence, the characteristic polynomial of $L\left(Cay_p(\mathrm{Q}_{32}, S_3)\right)$ is calculated as follows:

$$f(\lambda) = (\lambda^2 - 2\lambda)^{16}$$

Thus, the eigenvalues are $\lambda = 0$ with multiplicity 16 and $\lambda = 2$ with multiplicity 16 as well. Based on Definition 2.14, $L - Spec(\Gamma)$ = $\{\lambda_1^{k_1}, \lambda_2^{k_2}, \dots, \lambda_n^{k_n}\}$, the Laplacian spectrum of prime order Cayley graph of $Q_{32}$, $L\text{-}Spec\left(Cay_p(\mathrm{Q}_{32}, S_3)\right)$ is $\{0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\} = \{0^{16}, 2^{16}\}$.

Therefore, the Laplacian spectrum of prime order Cayley graph of quaternion group of order 32 is determined to be the set of eigenvalues of 0 and 2 with the 16.

**Conclusion**

In this research, the prime order Cayley graphs of the quaternion groups of order eight, 16 and 32 are reconstructed based on the groups' presentation and the definition of the prime order Cayley graph. From the graphs, the adjacency matrix, degree matrix and Laplacian matrix are determined. Then, the Laplacian spectrum of the prime order of the quaternion groups of order eight, 16 and 32 are computed. Based on Theorem 2.8, the prime order Cayley graph of the quaternion group of order $2^n$ is the union of $2^{n-1}$ components of $K_2$. Thus, it can be observed that every vertex in the graph has degree one, which implies that the diagonal matrix of vertex degrees for the graph is equal to identity matrix. Therefore, it can be concluded that

$$L\left(Cay_p(\mathrm{Q}_{2^n}, S_i)\right) = I_{2^n} - A\left(Cay_p(\mathrm{Q}_{2^n}, S_i)\right),$$

Furthermore, the Laplacian spectrum of the prime order of the quaternion groups depend on the characteristic polynomial of Laplacian matrix of the graph. It concluded that the Laplacian spectrum of these graphs are the sets of the eigenvalues of 0 and 2 with multiplicities to be half of the groups' order.

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