Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia
Vol. 9, 2022, page 278-285

Probabilities and Graph of Prime Ideal Based Product of Some Commutative Rings

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Abstract
In mathematics, there are areas known as algebra which consists of a discipline called ring theory. Ring theory has been applied in various fields such as computer science and cryptography by previous researchers. In this study, probability theory and graph theory are applied to ring theory. A new probability, namely the prime ideal based product probability, is introduced and applied on a commutative ring which is the ring of diagonal matrices over integers. The prime ideal based probability is defined as the probability that the product of two elements in a ring is an ideal. In this context, an ideal of a ring is a special subset of its elements. In this research, the ring of 2×2 diagonal matrices over integers modulo three and integers modulo four are considered. The prime ideal based product probability for these two types of rings is calculated using its definition. Furthermore, a new graph, called the prime ideal based product graph is also introduced, where the vertices of this graph are the elements of the ring, and two vertices are connected if their product is in the ideal. The prime ideal based product graph is then constructed for the two rings in the scope of this research. Finally, two graph properties, namely the clique number and the chromatic number are also found in this study. The results show that the clique number and the chromatic number for the prime ideal based product graph for the 2×2 diagonal matrices over integers modulo three, $M_2(\mathbb{Z}_3)$, are both equal to six. Meanwhile, the clique number and the chromatic number for the prime ideal based product graph for the 2×2 diagonal matrices over integers modulo four, $M_2(\mathbb{Z}_4)$, are both equal to eight.

Keywords: Ring; Ideal; Diagonal Matrix; Probability Theory; Clique Number; Chromatic Number

1. Introduction

A ring R is a set with two binary operations such that R is an abelian group under addition that satisfies associative law under multiplication and distributive law over addition [1]. On the other hand, in a commutative ring R, for every $a, b \in R$ an ideal I is a prime ideal of R if and only if $ab \in I$ implies $a \in I$ or $b \in I$ [2]. In this study, probability theory and graph theory are applied to ring theory, specifically to the ring of diagonal matrices over integers of certain modulo.

Besides, the concept of a zero divisor graph had been introduced in which the zero divisor graph of R, denoted as $\Gamma(R)$, is an undirected graph whose vertices are the nonzero zero-divisors of R with two distinct vertices x and y joined by an edge if and only if $xy = 0$ [3]. Meanwhile, the definition of an ideal based zero divisor graph is an undirected graph $\Gamma(R)$ with vertices $\{x \in I \mid xy \in R - I$ for some $y \in R - I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$ [4].

Meanwhile, the use of prime ideal on a specific topic related to prime ideal based product probability and prime ideal based product graph of commutative rings is not discovered yet. Therefore, to fulfill this study gap, the objective of this article are to study and compute the probability by using the definition of prime ideal based probability which is the probability that the product of two elements in a commutative ring which is a diagonal matrix is in the prime ideal.

In this article, the study focuses on rings and probability with the introduction of a new type of probability formula. The study starts by defining a new definition which is the prime ideal based probability.

2. Preliminaries

In this section, some basic definitions of ring theory, probability theory and graph theory are provided.

Ring theory is the study of rings in abstract algebra. A ring is a fundamental algebraic structure that is made up of a set and two binary operations called addition and multiplication. Ring theory investigates the structure of rings, their properties, particular classes of rings and their applications.

Definition 2.1: [5] Ring
A structure $(R,+,\cdot)$ is a ring if R is a non-empty set with two binary operations, addition and multiplication and the following axioms are satisfied.

i. $(R,+,\cdot)$ is an abelian group.
ii. Associativity of multiplication.
$$(a\cdot b)\cdot c = a\cdot(b\cdot c)$$
iii. For all $a,b,c \in R$, satisfy left and right distributive law,
$$a\cdot(b+c) = a\cdot b + a\cdot c$$
$$(a+b)\cdot c = a\cdot c + b\cdot c$$

The set R is a commutative group under addition. This sort of group is often referred to as abelian (rather than commutative) and is named after the famous Norwegian mathematician N. H. Abel (1890-1829), who studied a class of algebraic equations related to commutative groups [1]. Meanwhile (vi) and (vii) introduced a semigroup which is the set S with a binary operation of multiplication satisfying the associative law [1]. As in (viii), a ring is a set R with a binary operation that is connected by distributive law. In conclusion, a ring R is an abelian group under addition and satisfies associative law under multiplication and distributive laws.

Definition 2.2: [6] Commutative Ring
The ring $(R,+,\cdot)$ is said to be commutative if it satisfies commutativity for multiplication that is, a ring in which the equation $a\cdot b = b\cdot a$ holds for all the elements of a,b of the ring.

Next, some definition on equivalent concepts for ideals related to finite ring is stated as follows.

Definition 2.3: [2] Ideal
A subring I of a ring R is called a (two-sided) ideal of R if for every $r \in R$ and every $a \in I$ both ra and ar are in I.

Theorem 2.1 [7] Ideal test
A non-empty subset I of a ring R is called an ideal if

1. $\forall a,b \in I$ then $a-b \in I$.
2. $\forall a \in I, r \in R$ then $ra \in I$ and $ar \in I$.

Definition 2.4: [2] Prime Ideal
In a commutative ring R, an ideal U is a prime ideal if and only if $a,b \in R$ and $ab \in U$ implies $a \in U$ or $b \in U$.

Definition 2.5: [7] Maximal Ideal
A maximal ideal of a commutative ring R is a proper ideal of R such that, whenever B is an ideal of R and $A \subseteq B \subseteq R$, then $B = A$ or $B = R$.

Proposition 2.1 [7] The ideals in $\mathbb{Z}_n$ are precisely the sets of form $< a >$ where a divides n.

Following that, several concepts of probability theory in groups and rings are introduced. In group theory, probability theory is commonly studied. Gustafson [8] presented the commutativity degree of a group in 1973 as one of the instances. The definition is as follows:

Definition 2.6: [8] Commutativity Degree
The probability that a random element in a group commute with each other in the same group is denoted as:

$$P(R) = \frac{|\{(x, y) \in R \times R | xy = yx\}|}{|R|^2}.$$

This concept has been determined by many researchers such as MacHale [9] for finite rings. In this article, a new probability is defined namely the prime ideal based product probability and the results are stated in section four.

In addition, new concept of graph associated to a ring is introduced namely prime ideal based product graph and the details is in section four as well. Besides, two graph properties namely the clique number and chromatic number is defined as the following.

Definition 2.7: [10] Clique Number
A complete Γ_{sub} is called a clique. The greatest size of a clique in an undirected graph Γ is called clique number of Γ and is denoted by $\omega(\Gamma)$.

Definition 2.8: [10] Chromatic number
The smallest number of colors needed to color the vertices of Γ so that no two adjacent vertices share the same color is the chromatic number and is usually denoted by $\chi(\Gamma)$.

3. Methodology

This research begins with the study of several ideas which is the prime ideal, probabilities of ideal and the graph related to probability related to prime ideal. Simultaneously, the probability of two elements in some commutative rings have product in the prime ideal is computed using its definition.

After that, several ideas in graph theory are investigated. The concepts in graph theory is examined using the computation from the results of probability. Besides, the determination of graph properties which are the clique number and chromatic number are done with the assistance of Maple software.

4. Results and discussions

This section consists of two parts, the first part focuses on a new probability namely the prime ideal based product probability of rings of diagonal matrix $M_2(Z_3)$ and $M_2(Z_4)$, have the product in its prime ideal. In the second part, a notion of graph associated to rings is introduced, namely the prime ideal based product graph. From the graph obtained, two graph properties are determined which are the clique number and chromatic number.

4.1 The Probability that Two Elements in Some Commutative Rings Have Product in Prime Ideal

In this section, the probability that a random pair in some commutative rings that have the product in the prime ideal, which is subsequently called prime ideal based product probability is introduced as in the following definition.

Definition 4.1 Prime Ideal Based Product Probability.

Let *R* be a ring with prime ideal. The prime ideal based product probability that two elements are chosen at random from a ring *R* that have the product in *I* is:

$$\tilde{P}(R) = \frac{\{(x, y) \in R \times R | x \cdot y \in I\}}{|R \times R|}$$

It can also be written in the following general formula:

$$\tilde{P}(R) = \frac{|R \times R| - |\omega \times \omega|}{|R|^2}$$

where $\omega = \{a | a \in R$ and $a \notin I\}$.

4.1.1 Prime Ideal Based Product Probability of ring of 2×2 *Diagonal Matrices Over Integer Modulo n,* $M_2(Z_n)$

In this subsection, the ring of 2×2 diagonal matrices over integer modulo n, $M_2(\mathbb{Z}_3)$ and $M_2(\mathbb{Z}_4)$ are studied. The results obtained are stated as follows.

Proposition 4.1 Let $M_2(\mathbb{Z}_3)$ be a 2×2 diagonal matrix over integer modulo three and its ideal $I = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a = 0 \text{ } or \text{ } b = 0, a, b \in \mathbb{Z}_3\right\}$. Then, the ring $M_2(\mathbb{Z}_3)$ has 65 pairs of elements where the multiplication of these elements is in prime ideal and the prime ideal based product probability is $\tilde{P}(R) = \frac{65}{81}$.

Proof By using Definition 4.1, the prime ideal based product probability is calculated. Since $0 \cdot A = A \cdot 0 = 0$ for all $A \in M_2(Z_3)$, there are 17 pairs for the cases of $0 \cdot A$. In addition, another 48 pairs of prime ideal based product probability are determined by considering the cases when $xy \in I$ imply either $x \in I$ or $y \in I$ from Definition 2.4. Hence, by using definition 4.1, the prime ideal based product probability is,

$$\tilde{P}(M_2(\mathbb{Z}_3)) = \frac{|9 \times 9| - |4 \times 4|}{|9|^2} = \frac{65}{81}.$$

Proposition 4.2 Let $M_2(Z_4)$ be a 2×2 diagonal matrix over integer modulo four and its ideal $I = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a = 0 \text{ } or \text{ } b = 0, a, b \in \mathbb{Z}_4\right\}$. Then, the ring $M_2(\mathbb{Z}_4)$ has 175 pairs of elements where the multiplication of these elements is in prime ideal and the prime ideal based product probability is $\tilde{P}(R) = \frac{175}{256}$.

Proof By using Definition 4.1, the prime ideal based product probability is calculated. Since $0 \cdot A = A \cdot 0 = 0$ for all $A \in M_2(Z_4)$, there are 31 pairs for the cases of $0 \cdot A$. In addition, another 144 pairs of prime ideal based product probability are determined by considering the cases when $xy \in I$ imply either $x \in I$ or $y \in I$ from Definition 2.4. Hence, by using definition 4.1, the prime ideal based product probability is,

$$\tilde{P}(M_2(\mathbb{Z}_4)) = \frac{|16 \times 16| - |9 \times 9|}{|16|^2} = \frac{175}{256}.$$

4.2. Prime Ideal Based Product Graph of Some Commutative Rings

In this section, a new notion of a graph associated to a ring is introduced namely prime ideal based product graph. This graph is constructed for some commutative rings namely $M_2(\mathbb{Z}_3)$ and $M_2(\mathbb{Z}_4)$. From the graph obtained, two graph properties are determined which are clique number and chromatic number. The definition of the prime ideal based product graph is given as in the following.

Definition 4.2 Prime Ideal Based Product Graph

Let R be a commutative ring and I be a prime ideal of R. The undirected graph $\Gamma_p(R)$ with vertices $\{x \in R | xy \in I \text{ for } y \in R\}$ where distinct vertices x and y are adjacent if and only if $xy \in I$ is called the prime ideal based product graph.

4.2.1. Prime Ideal Based Product Graph 2×2 *Diagonal Matrices,* $M_2(\mathbb{Z}_3)$ and $M_2(\mathbb{Z}_4)$

In this subsection, the results on the prime ideal based product graph of $M_2(\mathbb{Z}_3)$ and $M_2(\mathbb{Z}_4)$ are provided in the form of propositions.

Proposition 4.3 Let $M_2(\mathbb{Z}_3)$ be a 2×2 diagonal matrix over integer modulo three. Then the prime ideal based product graph of $M_2(\mathbb{Z}_3)$, $\Gamma_p(M_2(\mathbb{Z}_3))$ is an undirected graph with nine vertices and 30 edges.

Proof Let $M_2(\mathbb{Z}_3) = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \; a, b \in \mathbb{Z}_3\right\}$. The graph is constructed based on the prime ideal based product probability found in Proposition 4.1. Recall from Definition 4.2, prime ideal based product graph is an undirected graph where its vertices are the elements in R and two vertices are adjacent if and only if the product of the vertices is in prime ideal.

Since $M_2(\mathbb{Z}_3)$ has nine elements, then the prime ideal based product graph of $M_2(\mathbb{Z}_3)$, $\Gamma_p(M_2(\mathbb{Z}_3))$ has nine vertices. Each vertex of $M_2(\mathbb{Z}_3)$ is given as follows where the elements are denoted as 1 to 9, respectively.

$$\begin{aligned} M_2(Z_3) &= \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right\} \\ &= \{1,2,3,4,5,6,7,8,9\} \end{aligned}$$

Then, the edges of the graph $\Gamma_p(M_2(\mathbb{Z}_3))$ are constructed based on the vertices of $M_2(\mathbb{Z}_3)$. From Proposition 4.1, it has been found that $M_2(\mathbb{Z}_3)$ has 65 pairs of elements where the product is in the prime ideal. Since the prime ideal based product graph is a simple graph, all $x, y \in R$, in which $x = y$ are excluded to avoid creating a self-loop on all vertices. Hence it is found that $\Gamma_p(M_2(\mathbb{Z}_3))$ has 30 edges. Therefore, $\Gamma_p(M_2(\mathbb{Z}_3))$ is an undirected graph with nine vertices and 30 edges, as shown in Figure 1.

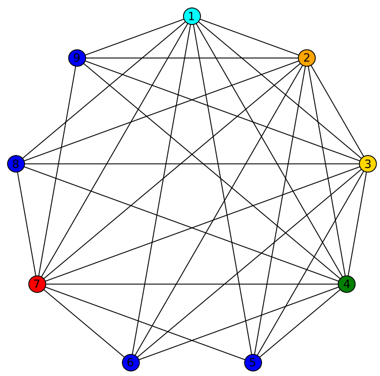

Figure 1 The prime ideal based product graph of $\Gamma_p(M_2(\mathbb{Z}_3))$

Recall from Definition 2.7 where a complete Γ_{sub} is called a clique. The, greatest size of a clique in an undirected graph Γ is called clique number of Γ and is denoted by $\omega(\Gamma)$. Since the prime ideal based product graph of $\Gamma_p(M_2(\mathbb{Z}_3))$ is an undirected graph where the maximum clique number is $\{1,2,3,4,5,7\}$, therefore, the clique number is six. Besides that, by Definition 2.8, six colours can be applied to the vertices $\Gamma_p(M_2(\mathbb{Z}_3))$ such that no two adjacent vertices share the same colour. Therefore, the chromatic number, $\chi\left(\Gamma_p(M_2(\mathbb{Z}_3))\right)$ is six.

Proposition 4.4 Let $M_2(\mathbb{Z}_4)$ be a 2×2 diagonal matrix over integer modulo three. Then the prime ideal based product graph of $M_2(\mathbb{Z}_4)$, $\Gamma_p(M_2(\mathbb{Z}_4))$ is an undirected graph with 16 vertices and 84 edges.

Proof Let $M_2(\mathbb{Z}_4) = \left\{\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \ a, b \in \mathbb{Z}_4\right\}$. The graph is constructed based on the prime ideal based product probability found in Proposition 4.2. Recall from Definition 4.2, prime ideal based product graph is an undirected graph where its vertices are the elements in R and two vertices are adjacent if and only if the product of the vertices is in prime ideal.

Since $M_2(\mathbb{Z}_4)$ has sixteen elements, then the prime ideal based product graph of $M_2(\mathbb{Z}_4)$, $\Gamma_p(M_2(\mathbb{Z}_4))$ has sixteen vertices. Each vertex of $M_2(\mathbb{Z}_4)$ are as follows where the elements is denoted as 1 to 16, respectively.

$$\begin{aligned} M_2(Z_4) &= \left\{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \right. \\ &\left. \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right\} \\ &= \{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16\}. \end{aligned}$$

Then, the edges of the graph $\Gamma_p(M_2(\mathbb{Z}_4))$ are constructed based on the vertices of $M_2(\mathbb{Z}_4)$. From Proposition 4.2, it has been found that $M_2(\mathbb{Z}_4)$ has 175 pairs of elements where the product is in the prime ideal. Since the prime ideal based product graph is a simple graph, all $x, y \in R$, in which $x = y$ are excluded to avoid creating a self-loop on all vertices. Hence it is found that $\Gamma_p(M_2(\mathbb{Z}_4))$ has 84 edges. Therefore, $\Gamma_p(M_2(\mathbb{Z}_4))$ is an undirected graph with 16 vertices and 84 edges, as shown in Figure 2.

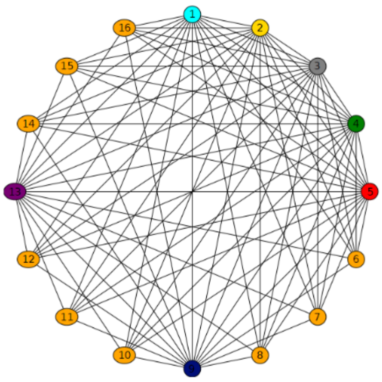

Figure 2 The prime ideal based product graph of $\Gamma_p(M_2(\mathbb{Z}_4))$

By Definition 2.7, the prime ideal based product graph of $\Gamma_p(M_2(\mathbb{Z}_4))$ is an undirected graph where the maximum clique number is $\{1,2,3,4,5,6,9,13\}$. Therefore, the clique number is eight. Besides that, by Definition 2.8, eight colours can be applied to the vertices $\Gamma_p(M_2(\mathbb{Z}_4))$ such that no two adjacent vertices share the same colour. Therefore, the chromatic number, $\chi\left(\Gamma_p(M_2(\mathbb{Z}_4))\right)$ is eight.

Conclusion

This article consists of two parts, the first part focuses on a new probability namely the prime ideal based product probability of rings of diagonal matrix $M_2(Z_3)$ and $M_2(Z_4)$, and the second part focusses on graph associated to ring is introduced namely prime ideal based product graph. Finally, two properties of the graph are obtained which are clique number and chromatic number. The results are summarized in Table 1.

Ring	**Prime ideal based product probability**	**Prime ideal based product graph**		**Graph properties**	
		Number of vertices	**Number of edges**	**Clique number**	**Chromatic number**
$M_2(\mathbb{Z}_3)$	$\frac{65}{81}$	9	30	6	6
$M_2(\mathbb{Z}_4)$	$\frac{175}{256}$	16	84	8	8

Table 1 The summary of the results

Acknowledgement

The authors would like to acknowledge Universiti Teknologi Malaysia (UTM) and Ministry of Higher Education Malaysia (MoHE) for the financial support through UTM Fundamental Research Grant (UTMFR) Vote Number 20H70 and Fundamental Research Grant Scheme (FRGS/1/2020/STG06/UTM/01/2). Special thanks to Nurhidayah Zaid and Nur Athirah Farhana Omar Zai for their assistance.

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