Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia
Vol. 8, 2022, page 19-25

The Order Prime Permutability Graph of Cyclic Subgroups of Nonabelian Metabelian Group of Order Less Than 24

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Abstract
A graph of groups is an object in geometric group theory that consists of a collection of groups indexed by the vertices and edges of a graph. The vertices of the graphs are represented by the elements or subgroups in the presentation, and the edges are defined by a specific rule. The scope of this study focuses on the cyclic subgroups of all nonabelian metabelian groups of order 24 where a metabelian group is a group whose commutator subgroup is abelian. Previous studies have completed the list of nonabelian metabelian groups of order at most 24 and the order prime permutability graph of cyclic subgroups of finite group has been defined. Therefore, in this research, the cyclic subgroups of nonabelian metabelian groups of order less than 24 are determined. Futhermore, the order prime permutability graph of cyclic subgroups of all nonabelian metabelian groups of order less than 24 are constructed by using the definition and previous related results from past research. The results obtained show that the graphs of the groups are either connected, complete or regular.
Keywords: Metabelian Group; Commutator Group; Cyclic Subgroup; Graph of Group; Order Prime Permutability Graph

1. Introduction

A group *G* is metabelian if and only if there exists an abelian normal subgroup $H \triangleleft G$ such that G/H is abelian.

The scope of this research is on a metabelian groups. In 2010, Abdul Rahman [1] has listed 74 groups of order at most 24 with their group presentation and there are 59 groups of order less than 24 and 15 groups of order 24. All groups of order less than 24 are metabelian and there exist two groups for groups of order less than 24 are not metabelian. But, only 35 groups are nonabelian. From all these groups, only 25 groups have order less than 24 that will be discussed in this research.

In 1878, Cayley [2] invented a method of groups in relation to graph theory. He defined the graph that explains the abstract structure of a group generated by a set of generators. Beside the graph that established by Cayley [2], many graphs of groups come into existence. There is another graph that defined by Sattanathan and Kala [3] is called order prime graph of groups as a graph with the elements of groups as vertices and two vertices a, b are adjacent if and only if $\gcd(|a|,|b|) = 1$.

In 2021, Bello [4] has introduced some new graphs of finite groups which is order prime permutability graph of cyclic subgroups where the graph is defined by using definition and theorems that will be discussed in the next section. As a continuation, this research will determine the order prime permutability graph of cyclic subgroups of nonabelian metabelian groups of order less than 24.

This research aims to (1) study the nonabelian metabelian groups of order less than 24, (2) identify all cyclic subgroups of nonabelian metabelian groups of order less than 24 and (3) construct the order prime permutability graph of cyclic subgroups of all nonabelian metabelian groups of order less than 24.

2. Literature Review

2.1. Nonabelian Metabelian Group of Order Less Than 24

The list of all nonabelian metabelian groups of order less than 24 [2] are listed below in Table 1.

Table 1: List of all nonabelian metabelian groups of order less than 24

No.	Group	$\lvert G \rvert$	Group Presentation
1.	D_3	6	$\langle a,b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle$
2.	D_4	8	$\langle a,b \mid a^4 = b^2 = 1, bab = a^{-1} \rangle$
3.	Q_4	8	$\langle a,b \mid a^4 = 1, a^2 = b^2 = 1, aba = b \rangle$
4.	D_5	10	$\langle a,b \mid a^5 = b^2 = 1, bab = a^{-1} \rangle$
5.	$\mathbf{Z}_3 \times \mathbf{Z}_4$	12	$\langle a,b \mid a^4 = b^3 = 1, aba = a \rangle$
6.	A_4	12	$\langle a,b \mid a^2 = b^2 = c^3 = 1, ba = ab, ca = abc, cb = ac \rangle$
7.	D_6	12	$\langle a,b \mid a^6 = b^2 = 1, bab = a^{-1} \rangle$
8.	D_7	14	$\langle a,b \mid a^7 = b^2 = 1, bab = a^{-1} \rangle$
9.	D_8	16	$\langle a,b \mid a^8 = b^2 = 1, bab = a^{-1} \rangle$
10.	Quasihedral-16	16	$\langle a,b \mid a^8 = b^2 = 1, bab = a^3 \rangle$
11.	Q_8	16	$\langle a,b \mid a^8 = b^2 = 1, aba = b \rangle$
12.	$D_4 \times \mathbf{Z}_2$	16	$\langle a,b,c \mid a^4 = b^2 = c^2 = 1, ac = ca, bc = cb, bab = a^{-1} \rangle$
13.	$Q \times \mathbf{Z}_2$	16	$\langle a,b,c \mid a^4 = b^4 = c^2 = 1, b^2 = a^2, ba = a^3b, ac = ca, bc = cb \rangle$
14.	Modular-16	16	$\langle a,b \mid a^8 = b^2 = 1, ba = ba^5 \rangle$
15.	B	16	$\langle a,b \mid a^4 = b^4 = 1, ab = ba^3 \rangle$
16.	K	16	$\langle a,b,c \mid a^4 = b^2 = c^2 = 1, cbc = ba^2, bab = a, ac = ca \rangle$
17.	$G_{4,4}$	16	$\langle a,b \mid a^4 = b^4 = 1, abab = 1, ba^3 = ab^3 \rangle$
18.	D_9	18	$\langle a,b \mid a^9 = b^2 = 1, bab = a^{-1} \rangle$
19.	$S_3 \times \mathbf{Z}_3$	18	$\langle a,b,c \mid a^3 = b^2 = c^3 = 1, bab = a^{-1}, ac = ca, bc = cb \rangle$
20.	$(\mathbf{Z}_3 \times \mathbf{Z}_3)\mathbf{Z}_2$	18	$\langle a,b,c \mid a^2 = b^3 = c^3 = 1, bc = cb, bab = a, cac = a \rangle$
21.	D_{10}	20	$\langle a,b \mid a^{10} = b^2 = 1, bab = a^{-1} \rangle$

22.	$Fr_{20} \cong \mathbf{Z}_5 \times \mathbf{Z}_4$	20	$\langle a,b \mid a^4 = b^5 = 1, ba = ab \rangle$
23.	$\mathbf{Z}_5 \times \mathbf{Z}_5$	20	$\langle a,b \mid a^4 = b^5 = 1, ba = a \rangle$
24.	$Fr_{21} \cong \mathbf{Z}_7 \times \mathbf{Z}_5$	21	$\langle a,b \mid a^3 = b^7 = 1, ba = ab^2 \rangle$
25.	D_{11}	22	$\langle a,b \mid a^{11} = b^2 = 1, bab = a^{-1} \rangle$

2.2. Cyclic Subgroups

Definition 1 [2] Cyclic Subgroup. Suppose that *G* is a finite group and *H* be a subgroup of *G. H* is called cyclic subgroup if there is an element *a* in *G* such that $H = \{a^n \mid n \in Z\}$. Such an element *a* is called a generator of *H* and it is denoted as $H = \langle a \rangle$

All cyclic subgroups of nonabelian metabelian group of order less than 24 will be shown in Table 2 in the next chapter.

2.3. Order Prime Permutability Graph of Cyclic Subgroups

The concept for graphs of groups is further studied by Bello [5] on introducing another graph of cyclic subgroups of finite groups. Moreover, the general presentations of the graph of cyclic groups, dihedral groups and generalized quaternion groups.

The definition of the order prime permutability graph of cyclic subgroups of finite groups has defined by Bello [4] in 2021 in Definition 2.

Definition 2 [4] The Order Prime Permutability Graph of Cyclic Subgroups of a Finite Group. Let G be a finite group, then the order prime permutability graph of cyclic subgroups of a group G, $\Gamma^{opc}(G)$, is a graph having the proper subgroups of G as its vertices and two vertices *H* and *K* are adjacent if and only if $|H||K| = p^s$, $s \in \mathbb{N}$ where H and K are permuting cyclic subgroups of G.

Remark on Definition 2. As can be seen, the definition that has been defined by Bello [4] has consider the permuting subgroup that come from the proper subgroups which are both cyclic and non-cyclic of the groups as the vertices of the graph. In this research, only cyclic subgroups are considered as the vertices of the graph.

The order prime permutability graph of cyclic subgroups of dihedral groups is given by Bello [5] in 2021 in following Theorem 1.

Theorem 1. [4] Let G be the dihedral group of degree n, for $n = p^s$, where $s \in \mathbb{N}$, where p is a prime number. Then the order prime permutability graph of cyclic subgroups,

$$\Gamma^{opc}(G) = \begin{cases} K_2 + (K_{s-1} \cup \frac{n}{2} K_2) \cup \overline{K}_{\Sigma \frac{n}{\Upsilon}}, & if \quad p = 2, \\ K_1 + (K_s \cup \overline{K}_n) \cup \overline{K}_{\Sigma \frac{n}{\Upsilon}}, & otherwise, \end{cases} \qquad (1)$$

where γ is the non-trivial proper divisors of n and $\overline{K_0}$ is considered as empty graph without vertex.

The order prime permutability graph of cyclic subgroups of generalized quartenion groups is given by Bello [4] in 2021 in following Theorem 2.

Theorem 2.2. [4] Let G be the generalized quartenion group of order $4n$, for $n = p^s$, where $s \in \mathbb{N}$, where p is a prime number. Then the order prime permutability graph of cyclic subgroups,

$$\Gamma^{opc}(G) = \begin{cases} K_2 + \left[K_s \cup \frac{n}{2} K_2 \right] \cup \overline{K}_{\Sigma\phi(n)}, & if \quad p = 2, \\ K_1 + \left[(K_1 + \overline{K}_n) \cup K_s \right] \cup \overline{K}_{s+\Sigma\phi(n)}, & otherwise, \end{cases} \quad (2)$$

where p is a prime number and $\Phi(n) = \{x \in \square : x \equiv 0 \bmod n, \ for \ 1 < x < n\}$.

Remark on Theorem 2.1 and Theorem 2.2. As mention before, in this research, only cyclic subgroups are considered to be the vertices of the graph. Hence, there will be no isolated vertices for the graph of D_n and Q_{4n} presented in this research. Hence, there will be no isolated vertices in the graph presented and $\overline{K_n}$ at the of each of general presentation in the theorems above.

3. The Cyclic Subgroups of All Nonabelian Metabelian Groups of Order Less Than 24

3.1. Introduction

The cyclic subgroups of nonabelian metabelian groups of order less than 24 are determined manually by using Cayley table and Definition 1.

3.2. Cyclic Subgroups of Nonabelian Metabelian Group of Order Less Than 24

The list of all cyclic subgroups of all nonabelian metabelian groups of order less than 24 are listed in Table 2 below.

Table 2: List of all cyclic subgroups of nonabelian metabelian groups of order less than 24

No.	Group	Cyclic Subgroups
1.	D_3	$\langle 1\rangle, \langle a\rangle, \langle b\rangle, \langle ab\rangle, \langle a^2b\rangle$
2.	D_4	$\langle 1\rangle, \langle a\rangle, \langle a^2\rangle, \langle b\rangle, \langle ab\rangle, \langle a^2b\rangle, \langle a^3b\rangle$
3.	Q_4	$\langle 1\rangle, \langle a\rangle, \langle a^2\rangle, \langle b\rangle, \langle ab\rangle$
4.	D_5	$\langle 1\rangle, \langle a\rangle, \langle b\rangle, \langle ab\rangle, \langle a^2b\rangle, \langle a^3b\rangle, \langle a^4b\rangle$
5.	$\mathbf{Z}_3 \times \mathbf{Z}_4$	$\langle 1\rangle, \langle a\rangle, \langle a^2\rangle, \langle b\rangle, \langle ab\rangle, \langle ba\rangle, \langle a^2b\rangle$
6.	A_4	$\langle 1\rangle, \langle a\rangle, \langle b\rangle, \langle c\rangle, \langle ab\rangle, \langle ac\rangle, \langle bc\rangle, \langle ac^2\rangle$
7.	D_6	$\langle 1\rangle, \langle a\rangle, \langle b\rangle, \langle ab\rangle, \langle a^2b\rangle, \langle a^3b\rangle, \langle a^4b\rangle, \langle a^5b\rangle$
8.	D_7	$\langle 1\rangle, \langle a\rangle, \langle b\rangle, \langle ab\rangle, \langle a^2b\rangle, \langle a^3b\rangle, \langle a^4b\rangle, \langle a^5b\rangle, \langle a^6b\rangle$
9.	D_8	$\langle 1\rangle, \langle a\rangle, \langle a^2\rangle, \langle a^4\rangle, \langle b\rangle, \langle ab\rangle, \langle a^2b\rangle, \langle a^3b\rangle, \langle a^4b\rangle, \langle a^5b\rangle, \langle a^6b\rangle, \langle a^7b\rangle$
10.	Quasihedral-16	$\langle 1\rangle, \langle a\rangle, \langle a^2\rangle, \langle a^4\rangle, \langle b\rangle, \langle ab\rangle, \langle a^2b\rangle, \langle a^3b\rangle, \langle a^4b\rangle, \langle a^6b\rangle$
11.	Q_8	$\langle 1\rangle, \langle a\rangle, \langle a^2\rangle, \langle a^4\rangle, \langle b\rangle, \langle ab\rangle, \langle a^2b\rangle, \langle a^3b\rangle, \langle a^4b\rangle, \langle a^6b\rangle$

12.	$D_4 \times \mathbf{Z}_2$	$\langle 1\rangle,\langle a\rangle,\langle a^2\rangle,\langle b\rangle,\langle c\rangle,\langle ab\rangle,\langle a^2b\rangle,\langle a^3b\rangle,\langle ac\rangle,\langle a^2c\rangle,$ $\langle bc\rangle,\langle abc\rangle,\langle a^2bc\rangle,\langle a^3bc\rangle$
13.	$Q \times \mathbf{Z}_2$	$\langle 1\rangle,\langle a\rangle,\langle a^2\rangle,\langle b\rangle,\langle c\rangle,\langle ab\rangle, \langle ac\rangle,\langle a^2c\rangle, \langle bc\rangle,\langle abc\rangle$
14.	Modular-16	$\langle 1\rangle,\langle a\rangle,\langle a^2\rangle,\langle a^4\rangle,\langle b\rangle, \langle ab\rangle,\langle a^2b\rangle,\langle a^3b\rangle,\langle a^4b\rangle,\langle a^6b\rangle$
15.	*B*	$\langle 1\rangle,\langle a\rangle,\langle a^2\rangle,\langle b\rangle,\langle b^2\rangle,\langle ab\rangle,\langle ab^2\rangle,\langle a^2b\rangle,\langle a^2b^2\rangle,\langle a^3b\rangle$
16.	*K*	$\langle 1\rangle,\langle a\rangle,\langle a^2\rangle,\langle b\rangle,\langle ab\rangle,\langle a^2b\rangle,\langle c\rangle, \langle ac\rangle,\langle a^2c\rangle,\langle bc\rangle,\langle abc\rangle,\langle a^3bc\rangle$
17.	$G_{4,4}$	$\langle 1\rangle,\langle a\rangle,\langle a^2\rangle,\langle b\rangle,\langle b^2\rangle,\langle ab\rangle, \langle ab^2\rangle,\langle a^2b\rangle,\langle a^2b^2\rangle,\langle ba\rangle,\langle a^2ba\rangle,\langle a^3b\rangle$
18.	D_9	$\langle 1\rangle,\langle a\rangle,\langle a^3\rangle,\langle b\rangle,\langle ab\rangle,\langle a^2b\rangle, \langle a^3b\rangle,\langle a^4b\rangle,\langle a^5b\rangle,\langle a^6b\rangle,\langle a^7b\rangle,\langle a^8b\rangle$
19.	$S_3 \times \mathbf{Z}_3$	$\langle 1\rangle,\langle a\rangle,\langle a^2\rangle,\langle b\rangle,\langle c\rangle,\langle ab\rangle, \langle a^2b\rangle,\langle ac\rangle,\langle a^2c\rangle,\langle bc\rangle,\langle abc\rangle,\langle a^2bc\rangle$
20.	$(\mathbf{Z}_3 \times \mathbf{Z}_3)\mathbf{Z}_2$	$\langle 1\rangle,\langle a\rangle,\langle b\rangle,\langle ab\rangle,\langle ba\rangle,\langle c\rangle,\langle ac\rangle,$ $\langle bc\rangle,\langle abc\rangle,\langle bac\rangle,\langle abac\rangle,\langle ca\rangle,\langle bca\rangle,\langle abaca\rangle$
21.	D_{10}	$\langle 1\rangle,\langle a\rangle,\langle a^2\rangle,\langle a^5\rangle,\langle b\rangle,\langle ab\rangle,\langle a^2b\rangle,$ $\langle a^3b\rangle,\langle a^4b\rangle,\langle a^5b\rangle,\langle a^6b\rangle,\langle a^7b\rangle,\langle a^8b\rangle,\langle a^9b\rangle.$
22.	$Fr_{20} \cong \mathbf{Z}_5 \times \mathbf{Z}_4$	$\langle 1\rangle,\langle a\rangle,\langle b\rangle,\langle b^2\rangle,\langle ab\rangle,\langle a^2b\rangle, \langle a^3b\rangle,\langle a^4b\rangle,\langle ab^2\rangle,\langle a^3b^2\rangle$
23.	$\mathbf{Z}_5 \times \mathbf{Z}_5$	$\langle 1\rangle,\langle a\rangle,\langle a^2\rangle,\langle b\rangle,\langle ab\rangle, \langle a^2b\rangle,\langle ba\rangle,\langle ab^2\rangle,\langle b^2a\rangle$
24.	$Fr_{21} \cong \mathbf{Z}_7 \times \mathbf{Z}_5$	$\langle 1\rangle,\langle a\rangle,\langle b\rangle,\langle ab\rangle, \langle bab\rangle,\langle a^2b\rangle,\langle ba\rangle,\langle a^2ba^2\rangle$
25.	D_{11}	$\langle 1\rangle,\langle a\rangle,\langle b\rangle,\langle ab\rangle,\langle a^2b\rangle,\langle a^3b\rangle,$ $\langle a^4b\rangle,\langle a^5b\rangle,\langle a^6b\rangle,\langle a^7b\rangle,\langle a^8b\rangle,\langle a^9b\rangle,\langle a^{10}b\rangle$

4. The Order Prime Permutability Graph of Cyclic Subgroups of All Nonabelian Metabelian Group of Order Less Than 24

4.1. Introduction

The determination of the order prime permutability graph of cyclic subgroups of all nonabelian metabelian groups of order less than 24 are determined from Theorem 1, Theorem 2 and Definition 2 will be discussed next.

4.2. The Order Prime Permutability Graph of Cyclic Subgroups of Special Groups of Nonabelian Metabelian Groups of Order Less Than 24

The order prime permutability graph of cyclic subgroups of special groups of nonabelian metabelian group of order less than 24 are given. This special group consist of dihedral groups and general quaternion groups that fulfilled the theorems. The result is proven by using Theorem 1 and Theorem 2.

4.2.1. The Order Prime Permutability Graph of Cyclic Subgroups of Dihedral Groups

The dihedral group, *D*$_n$ that will be obtained in this subsection are the dihedral group that satisfies the Theorem 1, where the dihedral group of degree n for $n = p^s$, where $s \in \mathbb{N}$, where p is a prime number.

Let D_n be the dihedral groups of degree n where $n = 4$. Then the order prime permutability graph of cyclic subgroups of D_4 are given as follows:

$$\Gamma^{opc}(D_4) = K_2 + (K_1 \cup 2K_2) \cup \overline{K}_2$$

If $n = 4,\ \ p = 2, s = 2$. Since $p = 2$, by using (i) and $n = 4$. Thus, the results hold.

4.2.2. The Order Prime Permutability Graph of Cyclic Subgroups of Quartenion Groups

The generalized quartenion group, Q_n that will be obtained in this subsection are the quartenion group that satisfies the Theorem 2 where the quartenion group of degree $4n$ where $n = p^s$ and $s \in \mathbb{N}$, where p is a prime number.

Let Q_n be the dihedral groups of degree n where $n = 4$. Then the order prime permutability graph of cyclic subgroups of Q_8 are given as follows:

$$\Gamma^{opc}(Q_8) = K_2 + (K_2 \cup 2K_2) \cup \overline{K}_5$$

If $n = 4,\ p = 2, s = 2$ by using (i) since $p = 2$. Thus the result following holds.

4.3. The Order Prime Permutability Graph of Cyclic Subgroups of Other Groups of Nonabelian Metabelian Groups of Order Less Than 24

4.3.1. The Order Prime Permutability Graph of Cyclic Subgroups of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_2$

Consider $G = (\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_2$ of order 18, then $G = \{1, a, b, ab, ba, aba, c, ac, bc, abc, bac, abac, ca, aca, bca, abca, baca, abaca\}$. The cyclic subgroups of G are; $\langle 1\rangle, \langle a\rangle, \langle b\rangle, \langle ab\rangle, \langle ba\rangle, \langle c\rangle, \langle ac\rangle, \langle bc\rangle, \langle abc\rangle, \langle ca\rangle, \langle bca\rangle, \langle baca\rangle, \langle abaca\rangle$. The order prime permutability graph of cyclic subgroups of G given in Figure 1.

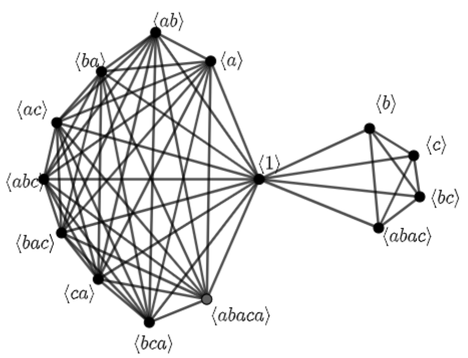

Figure 1: Order Prime Permutability Graoh of Cyclic Subgroups of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_2$

Proof. All the cyclic subgroups of $(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_2$ are the vertices of the graphs. The order of cyclic subgroups for $(\mathbb{Z}_3 \times \mathbb{Z}_3) \times \mathbb{Z}_2$ are; $|\langle 1 \rangle| = 1$, $|\langle a \rangle| = |\langle ab \rangle| = |\langle ba \rangle| = |\langle ac \rangle| = |\langle abc \rangle| = |\langle bac \rangle| = |\langle ca \rangle| = |\langle bca \rangle| = |\langle abaca \rangle| = 2$ and $|\langle b \rangle| = |\langle c \rangle| = |\langle bc \rangle| = |\langle abac \rangle| = 3.$ From the definition, where the two vertices that are adjacent if $|H||K| = p^s, s \in \mathbb{Z}$ and p is the prime number. The two vertices are adjacent with each other where the order of cyclic subgroups are 1 and 2 and 1 and 3 the product of order for each cyclic subgroups produce the $p^s, s \in \mathbb{Z}$ where p is the prime number; $|1||2| = 2^1$ and $|1||3| = 3^1$. Thus, the order of cyclic subgroups of order 3 are only adjacent with the order of cyclic subgroups of 1. The product of order cyclic subgroups of 2 and 3 are $|2||3| = 6^1$, where 6 are not prime numbers. Hence, there are no isolated vertices. Thus, the results hold.

5. Conclusion

All cyclic subgroups and order prime permutability graph of cyclic subgroups of nonabelian metabelian groups of order less than 24 have been determined in this full research. The cyclic subgroups of all nonabelian metabelian group have been identify by using Definition 1. The order prime permutability graph of cyclic subgroups have been determined by using between Theorem 1, Theorem 2 and Definition 2 by Bello [4] in 2021.

Acknowledgement

The researcher would like to thank all people who have supported the research, and would like to express gratitude to the supervisor, family, friends, siblings and all colleagues for never ending support in this beautiful journey on completing this research.

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