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Mathematical Modelling of an Aquaponics System in the Presence of Nitrite

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Abstract
The purposes of this research are to analyze and formulate a mathematical model for aquaponics system in the presence of nitrite. The systems can be solved by using the system of ordinary differential equation (ODE) based on the assumptions that related to the system. In this case of study, the eigenvalue method used in order to solve the aquaponics problems. In addition, there are three cases obtained which are (i) only plant survive, (ii) only nitrate survives and (iii) coexistences where all are survive. The eigenvalues, eigenvector and general solution for each equilibrium point can be obtained by using the eigenvalue method. Hence, mathematical model for aquaponics system in the presence of nitrite are solved by using eigenvalue method. The relation between population of fish, concentration of ammonia, concentration of nitrite, concentration of nitrate, and population of plant can be analysed and identified based on the model assumption for system of ordinary differential equation and eigenvalues number. The general solution for each cases that involving the relation between population of fish, concentration of ammonia, concentration of nitrite, concentration of nitrate, and population of plant are determined. Lastly, the stability of the equilibrium point for each relation can be determined and checked based on the general solution and phase portrait of the system.
Keywords: aquaponics; system ordinary differential equation; eigenvalues; eigenvector; phase portrait; stability; general solution.

1. Introduction

Problems frequently arise when it comes to aquaponics system that known as an integration between a growing fish and plant in the same recirculating aquaculture system. The aquaponics system is developed to help and solve problems that related to aquaculture and hydroponics because this system will contribute to their own knowledge and benefits.

As we known, aquaponics system does involve with a nitrogen cycle process. The nitrogen cycle well-explain the relationship between the population of fish and the population of plants. Fish survive with the help of plants and plants also survive with the help of fish. They fill their needs among each other in order to grow in most suitable condition and environment. Therefore, mathematical approach like system of ordinary differential equation has been used in order to analyze and calculate the variables involved in aquaponics system.

2. Literature Review

2.1 Aquaponics System

Aquaponics is a food production system that uses nutrient-rich water to fertilize the plants. Aquaponics system is a combination of two system which are aquaculture and hydroponic system that grow fish and plant together in one integrated system [1,2,3,4,5,6,7,8]. Both are about a growing plant with or without soil and a growing fish in recirculating aquaculture system.

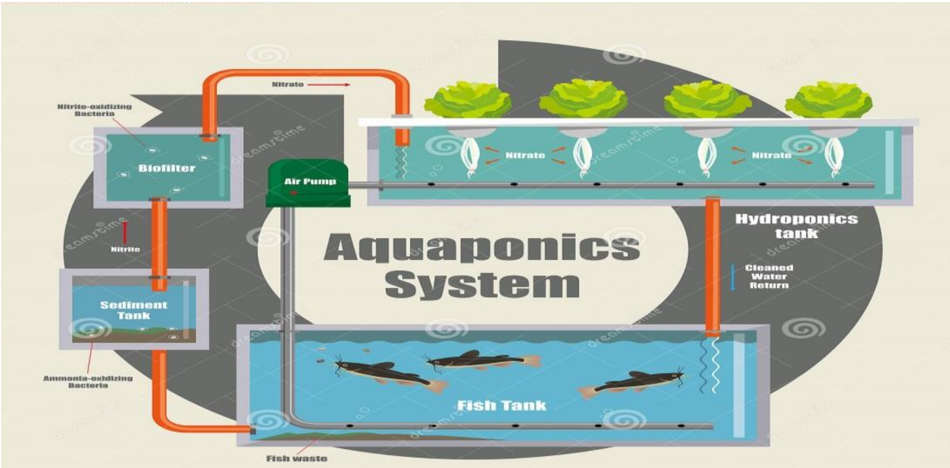

Figure 1 A simplified aquaponics system illustration [9].

2.2 Nitrogen Cycle

The most important part in aquaponics system is the living organisms such as the presence of nitrifying bacteria. Nutrifying bacteria work together with nitrogen cycle to turn fish waste into plant food [10]. The nitrogen cycle started when fish produce waste in the form of ammonia. Aquaponics system depends on the bacteria called Nitromonas bacteria that convert ammonia into nitrite [10]. This step is needed in order to keep the fish healthy by removing the excess ammonia from the water. The big amount of nitrite can interrupt the oxygen uptake by the fish which can slower the fish growth [10]. Next step in the cycle involve bacteria called Nitrospora [10]. These bacteria will take the nitrite and turn it into nitrate [10]. Then the fish waste turns into a highly nutrious food for the hydroponics plants. The plants absorb the nitrate and grow healthy and leave the water clean for the growth of fish [10]. We can say that without bacteria, both fish and plant will suffer to growth healthy.

3. Methodology

3.1. Linearizing Non-Linear System of Ordinary Differential Equation

In order to solve nonlinear function, we need to linearize the function by using the Jacobian matrix.

The non-linear system of two ordinary differential equations system is

$$\frac{dx}{dt} = f_1(x,y) = a_{11}g_1(x,y) + a_{12}g_2(x,y) \qquad (1)$$

$$\frac{dy}{dt} = f_2(x,y) = a_{21}g_3(x,y) + a_{22}g_4(x,y) \qquad (2)$$

where a_{ij} is related to partial derivatives of f_1 and f_2 , make up the coefficient of the 2×2 Jacobian matrix, $\mathrm{J}(\bar{x}_{0,}\bar{y}_0)$. Since equation (1) and (2) are nonlinear equation, we can solve them by using Jacobian matrix as below

$$\mathrm{J}(\bar{x}_{0,}\bar{y}_0) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \frac{df_1}{dx} & \frac{df_1}{dy} \\ \frac{df_2}{dx} & \frac{df_2}{dy} \end{pmatrix}.$$

3.2 Eigenvalue Method

Eigenvalue method is used to define the stability of the system through the value of eigenvalues.Consider a homogeneous linear equation as

$$x' = Ax,$$

where A is a 2×2 matrix.

Eigenvalue method is described as

$$Av = \lambda v$$

where v is an eigenvector and λ is eigenvalue. We found that the eigenvalue of the Jacobian matrix, J in two dimensions is as follow

$$(A - \lambda I)v = 0\ .$$

Let take matrix A as

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Then, the determinant matrix becomes

$$\det \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0\ .$$

The characteristic equation gives

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0$$
$$\lambda^2 - (a_{11} + a_{12})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$$
$$\lambda^2 - \text{trace}(A)\lambda + \text{determinant}(A) = 0$$

Therefore, the eigenvalue can be obtained by

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\delta}}{2}$$

where

$\tau = \text{trace } A = a_{11} + a_{12}$

$\partial = \text{determinant } A = a_{11}a_{22} - a_{12}a_{21}$

$\delta = \text{discriminant } A = \tau^2 - 4\partial$

The general solution is given by

$$x(t) = c_1\overrightarrow{v_1}e^{\lambda_1 t} + c_2\overrightarrow{v_2}e^{\lambda_2 t}$$

where $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ are linearly independent eigenvectors.

3.3 Stability of Equilibrium Point

Stability can be classified into two which are real and complex eigenvalues. Real eigenvalues can be categorized into group namely positive, negative and opposite sign. Meanwhile, complex eigenvalues are positive and negative real part with pure imaginary.

The three properties as follows

$$\tau = \text{trace } A = a_{11} + a_{12}$$
$$\partial = \text{determinant } A = a_{11}a_{22} - a_{12}a_{21}$$
$$\delta = \text{discriminant } A = \tau^2 - 4\partial$$

can be used to determine the eigenvalues by (3.2).

Therefore, the equilibrium can be classified into six cases which are:

1. Unstable node for $\tau > 0$ and $\partial > 0$
2. Saddle node for $\tau < 0$ and $\partial > 0$
3. Saddle point for $\partial < 0$
4. Neutral center for $\tau^2 < 4\partial$ and $\tau = 0$

5. Unstable spiral $\tau^2 < 4\partial$ and $\tau > 0$
6. Stable spiral $\tau^2 < 4\partial$ and $\tau < 0$

3.4 Real and Distinct Eigenvalue

The general solution of the system is

$$\vec{X}(t) = c_1\vec{v}_1 e^{\lambda_1 t} + c_2\vec{v}_2 e^{\lambda_2 t}$$

where

- v_1 and v_2 are linearly independent eigenvectors.
- $\vec{X}_1 = \vec{v}_1 e^{\lambda_1 t}$ and $\vec{X}_2 = \vec{v}_2 e^{\lambda_2 t}$

The eigenvalues can be classify into three categories which are

1. If both eigenvalues are negative $(\lambda_1 < 0, \lambda_2 < 0)$, then it is asymptotically stable.
2. If both eigenvalues are positive $(\lambda_1 > 0, \lambda_2 > 0)$, then it is unstable.
3. If eigenvalues are opposite sign $((\lambda_1 < 0, \lambda_2 > 0)$ or $(\lambda_1 > 0, \lambda_2 < 0)$, then it is unstable.

3.5 Complex Eigenvalue

For $\lambda_1, \lambda_2 = a \pm bi$, we differentiate into the following cases:

1. If $a > 0$, then it is spiral point and unstable.
2. If $b > 0$, then it is spiral point and asymptotically stable.
3. If $\lambda_1 = bi, \lambda_2 = -bi$, then it is center and stable.

3.6 Aquaponics Model Assumptions

The assumption obtained from [11]. This research will be considered nitrite as important concentration between ammonia and nitrate. The assumptions are made as follows:

i. The aquaponics ecosystem is a closed environment.
ii. The fish population increases at some natural survival rate $\left(\frac{\textbf{Births}}{\textbf{Deaths}}\right)$, hindered by a carrying capacity due to the limited tank space.
iii. There is additional fish decay due to increased ammonia presence in the water until it reaches a critical ammonia level where no fish will survive. This can be reasonably modeled by using a ratio $\left(\frac{\textbf{Ammonia Present}}{\textbf{Toxic Ammonia Level}}\right)$.
iv. Ammonia is present in the system exclusively due to fish waste and hence grows at a rate proportional to the population of fish. It decays due to its conversion to nitrite at a rate proportional to its concentration.
v. Nitrite is present in the system as it arises through the nitrogen cycle. It grows at a rate proportional to the ammonia in the system, and decays at a rate proportional to its own concentration.
vi. Nitrate grows at a rate proportional to the level of nitrite, and decays due to plant uptake at a rate which interacts with the total nitrate and the total plant population.
vii. Plants grow at a constant rate hindered by a carrying capacity indicative of the limited surface area of the system.
viii. Modelling the concentrations of ammonia, nitrite, and nitrate in the system will capture any other relationships due to the bacteria present in the nitrogen cycle.
ix. The system is well mixed so the nitrogen cycle occurs naturally and plants have even access to nitrate.

x. The systems are established in a way that water pH and temperature are within a healthy range for the fish.

3.7 Derivation of Aquaponics System

From the assumption, we can generate model for aquaponics system. There are several variables will represent the term in this system which are:

- $F(t)$ is representing the population of fish.
- $A(t)$ is representing the concentration of ammonia (in mg).
- $Z(t)$ is representing the concentration of nitrite (in mg).
- $N(t)$ is representing the concentration of nitrate (in mg).
- $P(t)$ is representing the population of the plant.

3.7.1 Model for Population of Fish

Regarding the assumptions (ii) and (iii), population of fish will affected due the limitation of the surface of tank. We can say that, the population of fish will decrease as the concentration of ammonia increase. This is because the fish population cannot survive since the water became toxic as the ammonia increase. Thus, the model can be formulated as

$$\frac{dF}{dt} = a_1\left(1 - \frac{F}{K_F}\right) - \frac{A}{K_A}F \ .$$

3.7.2 Model for Concentration of Ammonia

The presence of ammonia is come from fish waste and food that are not eaten by fish. So, the concentration of ammonia is proportional to the fish populations as population of fish increases then concentration of ammonia will also increases. But to be noted that, the concentration of ammonia influenced by the nitrification process that converts ammonia into nitrite. This process will decrease the concentration of ammonia in the water. Hence, the model can be represented as

$$\frac{dA}{dt} = a_2F - a_3A \ .$$

3.7.3 Model for Concentration of Nitrite

From assumption (v), nitrite is present in the system through nitrification process in nitrogen cycle. Its grow at a rate of proportional to the ammonia in the system. Nitrite decays at a rate proportional to its own concentration. Then, the model can be written as

$$\frac{dZ}{dt} = a_4A - a_5Z \ .$$

3.7.4 Model for Concentration of Nitrate

The concentration of nitrate grows at a rate of proportional to the concentration of nitrate. However, the concentration of nitrate also grows at a rate of proportional to the concentration of ammonia. The change of concentration of nitrate is directly proportional to the concentration of ammonia. Since the nitrate act as food that helping the plant to grow, the concentration of nitrate decay due to the plant uptake. Thus, the model can be represented as

$$\frac{dN}{dt} = a_6Z - a_7NP \qquad .$$

3.7.5 Model for Population of Plant

From assumption, the plant grows at a constant rate. The plant grows increase as the nitrate is increase. The nitrate act as a food, so if the nitrate increase, the plant will get enough nutrient to grow up and the plant population will increase as well. The rate of plant grows are prevented from the capacity indicative of the limited space of the system. By considering assumption (vi), we will obtain the model for the growth rate of the plant where can be written as

$$\frac{dP}{dt} = a_8\left(1 - \frac{P}{K_P}\right)NP \ .$$

4. Result and Analysis

Result and analysis are done by using the parameter estimation as below [11]

Table 1 Data of parameter estimation

PARAMETERS	VALUE
a_1	0.0124
a_2	0.1
a_3	0.94
a_4	2.7
a_5	5.0
a_6	3.384
a_7	0.92
a_8	0.056
K_A	12500
K_F	250
K_P	300

4.1 Equilibrium Points

The equilibrium points are obtained by letting the equation equal to zero.

4.1.1 The Equilibrium Point for Population of Fish, F and Concentration of Ammonia, A

Case I: $F = 0$, $A = 0$ and
Case II: $F = 213.3856$, $A = 22.7006$.

4.1.2 The Equilibrium Point for Concentration of Ammonia, A and Concentration of Nitrite, Z

Case I: $A = 0$, $Z = 0$.
Case II: $A = 22.7006$, $Z = 11.523$.

4.1.3 The Equilibrium Point for Concentration of Nitrite, Z and Concentration of Nitrate, N

Case I: $A = 0$, $Z = 0$ then $N = 0$.
Case II: $A = 22.7006$, $Z = 11.523$, then $N = 0.296$.

4.1.4 The Equilibrium Point for Concentration of Nitrate, N and Population of Plant, P

Case I: $Z = 0$, $N = 0$, $P = P$.
Case II: $Z = 0$, $N = N$, $= 0$.
Case III: $Z = 11.523$, $N = 0.296$, $P = 300$.

4.2 Solution for Population of Fish, F and Concentration of Ammonia, A

The general Jacobian matrix *A* and *Z* is

$$J(A,Z) = \begin{pmatrix} -0.94 & 0 \\ 2.7 & -5 \end{pmatrix}$$

4.2.1 Solution for Case I

For Case I, equilibrium point is at (0,0), hence the Jacobian matrix gives

$$J(0,0) = \begin{pmatrix} -0.94 & 0 \\ 2.7 & -5 \end{pmatrix} = C_2.$$

To obtain the eigenvalues, determinant matrix is determined as

$$\begin{vmatrix} -0.94 - \lambda & 0 \\ 2.7 & -5 - \lambda \end{vmatrix} = 0$$

$(-0.94 - \lambda)(-5 - \lambda) = 0.$
Therefore, the eigenvalues are

$\lambda_5 = -0.94$ and $\lambda_6 = -5.$

Based on the stability properties, the system is asymptotically stable since $\lambda_6 < \lambda_5 < 0$.

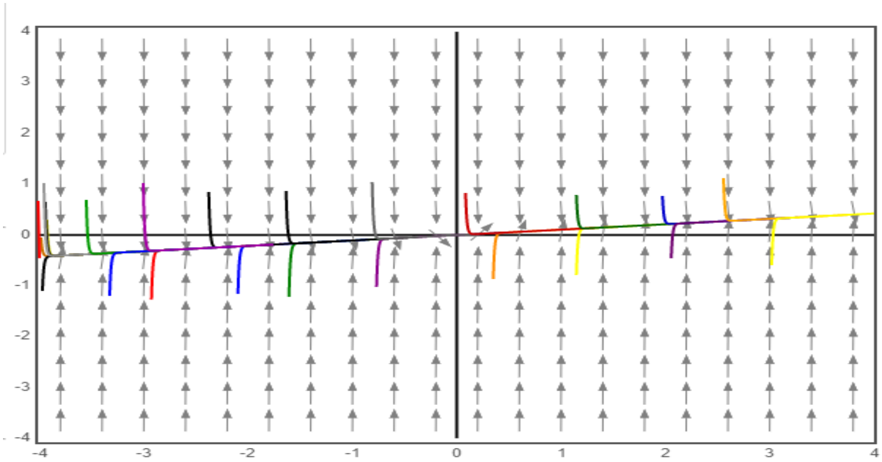

Figure 2 The phase portrait for concentration of ammonia versus population of fish (Case I).

The Figure 2 shows that the system is unstable since the trajectory is keeping away and away from the equilibrium point(0,0). We can say that the system is not complete and unstable because of the nonexistence population of fish and concentration of ammonia.

For $\lambda_1 = 0.0124,$

$$\overrightarrow{X_1} = \begin{pmatrix} 1 \\ \frac{0.1}{0.9524} \end{pmatrix} e^{0.0124t}.$$

For $\lambda_2 = -0.94$,

$\overrightarrow{X_2} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} e^{-0.94t}$.

Hence, the general solution for Case I is

$$\begin{pmatrix} F_1 \\ A_1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ \frac{0.1}{0.9524} \end{pmatrix} e^{0.0124t} + c_2 \begin{pmatrix} 0 \\ 5 \end{pmatrix} e^{-0.94t}.$$

$F_1 = c_1 e^{0.0124t}$,

$A_1 = c_1(0.1050)e^{0.0124t} + c_2(5)e^{-0.94t}$.

4.2.2 Solution for Case II

The equilibrium point is at$(213.3856\ ,\ 22.7006)$. Then, the Jacobian matrix becomes

$$J(213.3856\ ,\ 22.7006) = \begin{pmatrix} -0.0106 & -0.0171 \\ 0.1 & -0.94 \end{pmatrix} = D_1.$$

To get eigenvalues, determinant matrix is determined as

$$\begin{vmatrix} -0.0106 - \lambda & -0.0171 \\ 0.1 & -0.94 - \lambda \end{vmatrix} = 0.$$

$(-0.0106 - \lambda)(-0.94 - \lambda) + 0.0017 = 0$,

$\lambda_3 = -0.0124$ and $\lambda_4 = -0.9440$.

Based on stability properties, the system is asymptotically stable since $\lambda_3 < \lambda_4 < 0$.

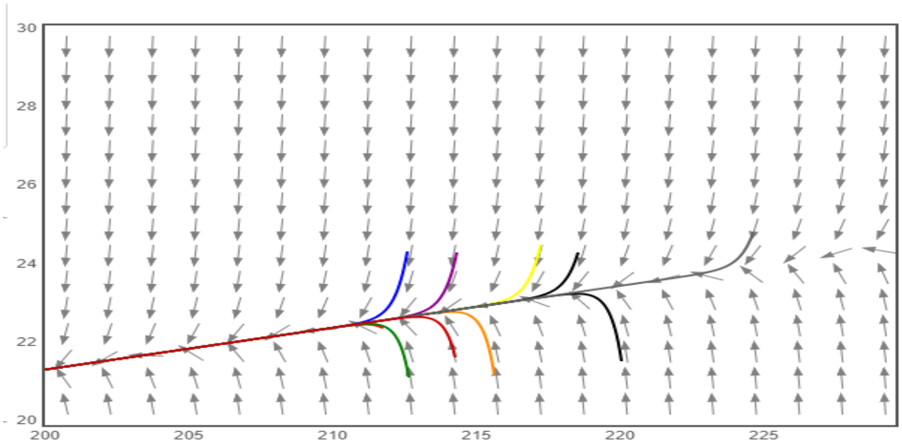

Figure 3 The phase portrait for concentration of ammonia versus population of fish (Case II)

Based on the Figure 3, the system is asymptotically stable since the direction of trajectory is approaching to equilibrium point$(213.3856{,}22.7006)$. We can say that the system is asymptotically stable since there is exist the population of fish and concentration of ammonia.

For $\lambda_3 = -0.0124$,
$\overrightarrow{X_3} = \begin{pmatrix} 1 \\ 0.0901 \end{pmatrix} e^{-0.0124t}$.
For $\lambda_4 = -0.9440$,

$\overrightarrow{X_4} = \begin{pmatrix} 0.0253 \\ 1 \end{pmatrix} e^{-0.9440t}$.

Hence, the general solution for Case II is
$\begin{pmatrix} F_2 \\ A_2 \end{pmatrix} = c_3 \begin{pmatrix} 1 \\ 0.0901 \end{pmatrix} e^{-0.0124t} + c_4 \begin{pmatrix} 0.0253 \\ 1 \end{pmatrix} e^{-0.9440t}$.

$F_2 = c_3 e^{-0.0124t} + c_4(0.0253)e^{-0.9440t}$,
$A_2 = c_3(0.0901)e^{-0.0124t} + c_4 e^{-0.9440t}$.

4.3 Solution for Concentration of Ammonia, *A* and Concentration of Nitrite, *Z*

4.3.1 Solution for Case I

For Case I, equilibrium point is at (0,0), hence the Jacobian matrix gives

$J(0,0) = \begin{pmatrix} -0.94 & 0 \\ 2.7 & -5 \end{pmatrix} = C_2$.

To obtain the eigenvalues, determinant matrix is determined as

$$\begin{vmatrix} -0.94 - \lambda & 0 \\ 2.7 & -5 - \lambda \end{vmatrix} = 0$$

$(-0.94 - \lambda)(-5 - \lambda) = 0$.
Therefore, the eigenvalues are

$\lambda_5 = -0.94$ and $\lambda_6 = -5$.

Based on the stability properties, the system is asymptotically stable since
$\lambda_6 < \lambda_5 < 0$.

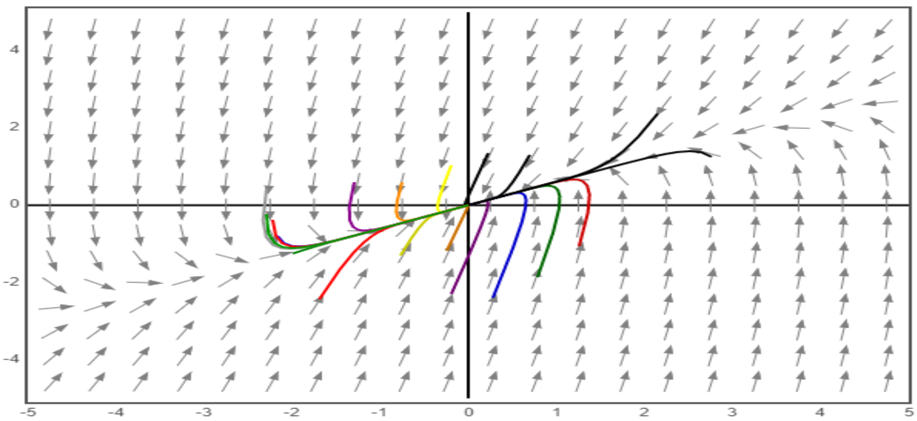

Figure 4 The phase portrait for concentration of nitrite versus concentration of ammonia (Case I)

The Figure 4 shows that the system is asymptotically stable since the direction of trajectory are approaching the equilibrium point$(0,0)$.

For $\lambda_5 = -0.94$,
$\overrightarrow{X_5} = \begin{pmatrix} 1.5307 \\ 1 \end{pmatrix} e^{-0.94t}$.
For $\lambda_6 = -5$,
$\overrightarrow{X_6} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-5t}$.

Therefore, the general solution for Case I is

$$\begin{pmatrix} A_1 \\ Z_1 \end{pmatrix} = c_5 \begin{pmatrix} 1.5307 \\ 1 \end{pmatrix} e^{-0.94t} + c_6 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-5t}.$$

$$A_1 = c_5(1.5307)e^{-0.94t},$$

$$Z_1 = c_5 e^{-0.94t} + c_6 e^{-5t}.$$

4.3.2 Solution for Case II

The equilibrium point for Case II is at $(26.4523, 14.2696)$. Hence, the Jacobian matrix becomes

$$J(26.4523,14.2696) = \begin{pmatrix} -0.94 & 0 \\ 2.7 & -5 \end{pmatrix} = C_2.$$

.

To obtain the eigenvalues, determinant matrix is determined as

$$\begin{vmatrix} -0.94 - \lambda & 0 \\ 2.7 & -5 - \lambda \end{vmatrix} = 0$$

$$(-0.94 - \lambda)(-5 - \lambda) = 0.$$

$\lambda_7 = -0.94$ and $\lambda_8 = -5$.

Based on the stability properties, the system is asymptotically stable since
$\lambda_6 < \lambda_5 < 0$.

For eigenvectors, the eigenvectors for Case II are likewise with eigenvectors for Case I. Thus, the general solution for Case I is

$$\begin{pmatrix} A_2 \\ Z_2 \end{pmatrix} = c_7 \begin{pmatrix} 1.5307 \\ 1 \end{pmatrix} e^{-0.94t} + c_8 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-5t}.$$

$$A_2 = c_7(1.5307)e^{-0.94t},$$

$$Z_2 = c_7 e^{-0.94t} + c_8 e^{-5t}.$$

4.4 Solution for Concentration of Nitrite, *Z* and Concentration of Nitrate, *N*

The general Jacobian matrix for *Z* and *N* is

$$J(Z,N) = \begin{pmatrix} -5 & 0 \\ 0 & -0.92P \end{pmatrix}.$$

4.4.1 Solution for Case I

For Case I, equilibrium point is at $(0,0)$. Therefore, the Jacobian matrix becomes

$$J(0,0) = \begin{pmatrix} -5 & 0 \\ 0 & -0.92P \end{pmatrix} = C_3.$$

In order to get the eigenvalues, determinant matrix is determined as

$$\begin{vmatrix} -5-\lambda & 0 \\ 0 & -0.92P-\lambda \end{vmatrix} = 0.$$

$$(-5-\lambda)(-0.92P-\lambda) = 0.$$

$\lambda_9 = -5$ and $\lambda_{10} = -0.92P$.

Based on the stability properties, the system is asymptotically stable since both eigenvalues less than zero, $\lambda_9 < \lambda_{10} < 0$.

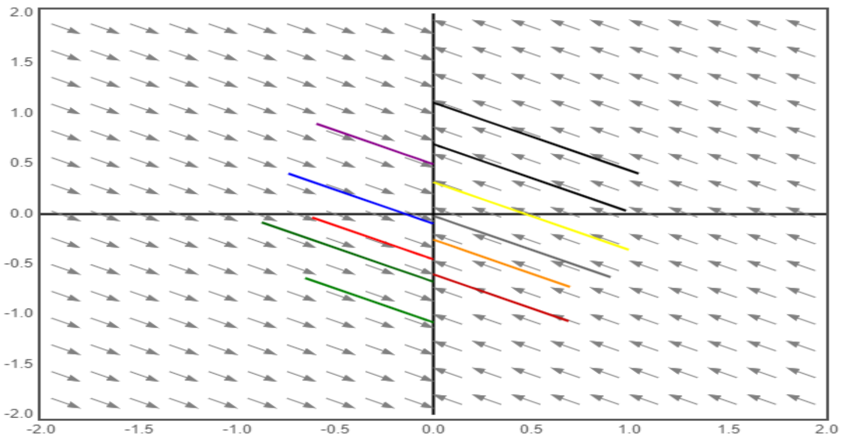

Figure 5 The phase portrait for concentration of nitrate versus concentration of nitrite (Case I).

Based on Figure 5, we can see that the direction of arrows seems moving toward the equilibrium point $(0,0)$.Therefore, the system is asymptotically stable.

For $\lambda_9 = -5$,

$$\overrightarrow{X_9} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-5t}.$$

For $\lambda_{10} = -0.92P$,

$$\overrightarrow{X_{10}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-0.92Pt}.$$

Therefore, the general solution for Case I is

$$\begin{pmatrix} Z_1 \\ N_1 \end{pmatrix} = c_9 \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-5t} + c_{10} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-0.92Pt}.$$

$$Z_1 = 2c_9 e^{-5t}$$

$$N_1 = c_{10} e^{-0.92Pt}$$

4.4.2 Solution for Case II

The equilibrium point is at $\left(14.2696, \frac{84.0376}{P}\right)$.

$J\left(14.2696, \frac{84.0376}{P}\right) = \begin{pmatrix} -5 & 0 \\ 0 & -0.92P \end{pmatrix} = D_3.$

The eigenvalues and eigenvectors for Case II are likewise with Case I since variable *Z* and *N* in Jacobian matrix does not change.

$\lambda_9 = -5$ and $\lambda_{10} = -0.92P$.

Based on the stability properties, the system is asymptotically stable since both eigenvalues less than zero, $\lambda_9 < \lambda_{10} < 0$. The phase portrait is likewise with phase portrait for Case I.

Hence, the general solution for Case II is

$$\begin{pmatrix} Z_2 \\ N_2 \end{pmatrix} = c_{11} \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-5t} + c_{12} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-0.92Pt}$$

$$Z_2 = 2c_{11}e^{-5t}$$

$N_2 = c_{12}e^{-0.92Pt}$.

4.5 Solution for Concentration of Nitrate, *N* and Population of Plant, *P*

The general Jacobian matrix for *N* and *P* is

$$J(N,P) = \begin{pmatrix} -0.92P & -0.92N \\ 0.056P\left(1 - \frac{P}{300}\right) & 0.056N\left(1 - \frac{P}{300}\right) - a_8\frac{NP}{300} \end{pmatrix}.$$

4.5.1 Solution for Case I

The equilibrium point for Case I is at $(0, P)$. Hence, Jacobian matrix becomes

$$J(0,P) = \begin{pmatrix} -0.92P & 0 \\ 0.056P\left(1 - \frac{P}{300}\right) & 0 \end{pmatrix}.$$

In order to get the eigenvalues, determinant matrix is determined as

$$\begin{vmatrix} -0.92P - \lambda & 0 \\ 0.056P\left(1 - \frac{P}{300}\right) & -\lambda \end{vmatrix} = 0$$

$$(-\lambda)(-0.92P - \lambda) = 0$$

$\lambda_{13} = 0$ and $\lambda_{14} = -0.92P$.

The existence population of plant,*P* will be considered whether it is decline or incline in population. $\lambda_{14} = -0.92P$ can be either negative or positive eigenvalue since we do not know exact value of *P.*

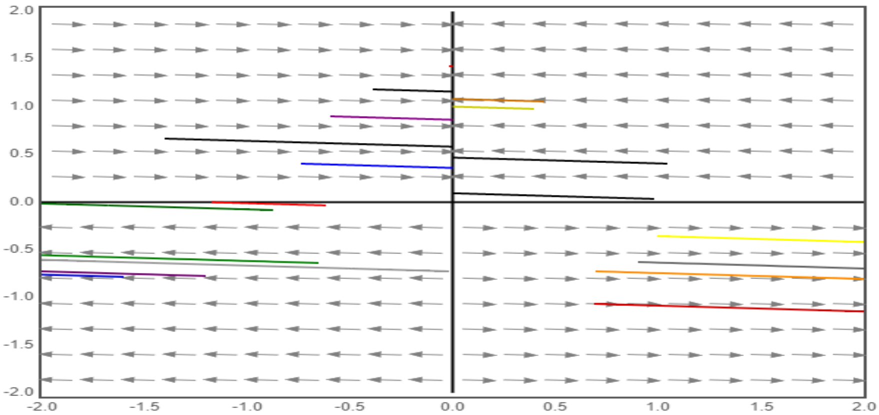

Figure 6 The phase portrait for population of plant versus concentration of nitrate (Case I).

From Figure 6, we can see that when *P* is negative, *N* are keep away from zero. This is because, when the plant population decline, there always exist the concentration of the nitrate. But, when *P* is positive, *N* are closed to zero and there exist the population of plant and concentration of nitrate will be affected the system.

For $\lambda_{13} = 0$,
$\overrightarrow{X_{13}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
For $\lambda_{14} = -0.92P$,
$\overrightarrow{X_{14}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-0.92Pt}$.

Therefore, the general solution for Case I is

$$\begin{pmatrix} N_1 \\ P_1 \end{pmatrix} = c_{13} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_{14} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-0.92Pt},$$

$N_1 = c_{14} e^{-0.92Pt}$,
$P_1 = c_{13}$.

4.5.2 Solution for Case II

The equilibrium point for Case II is at$(N, 0)$. Thus, the Jacobian matrix for *N* and *P* becomes

$$J(N, 0) = \begin{pmatrix} 0 & -0.92N \\ 0 & 0.056N \end{pmatrix}.$$

To find eigenvalues, determinant matrix is determined as

$$\begin{vmatrix} -\lambda & -0.92N \\ 0 & 0.056N - \lambda \end{vmatrix} = 0,$$

$$(-\lambda)(0.056N - \lambda) = 0,$$

$\lambda = 0$ and $\lambda = 0.056N$.

Same as before, variable *N* can be either negative or positive.

For $= 0$,
$\overrightarrow{X_{15}} = \binom{1}{0}$.

For $\lambda = 0.056N$,
$\overrightarrow{X_{16}} = \binom{16.4286}{1} e^{0.056N\,t}$.

Therefore, the general solution for Case II is

$$\binom{N_2}{P_2} = c_{15}\binom{1}{0} + c_{16}\binom{16.4286}{1} e^{0.056N\,t}.$$

$$N_2 = c_{15} + 16.4286c_{16}e^{0.056N\,t}$$

$P_2 = c_{16}e^{0.056N\,t}$.

4.5.3 Solution for Case III

The equilibrium point is $(0.2801,300)$. The Jacobian matrix for *N* and *P* will becomes
$J(0.2801,300) = \begin{pmatrix} -276 & -0.2561 \\ 0 & -0.0156 \end{pmatrix}$.

To find eigenvalues,

$$\begin{vmatrix} -276-\lambda & -0.2561 \\ 0 & -0.0156-\lambda \end{vmatrix} = 0,$$

$$(-276-\lambda)(-0.0156-\lambda) = 0,$$

$\lambda_{17} = -276$ and $\lambda_{18} = -0.0156$.

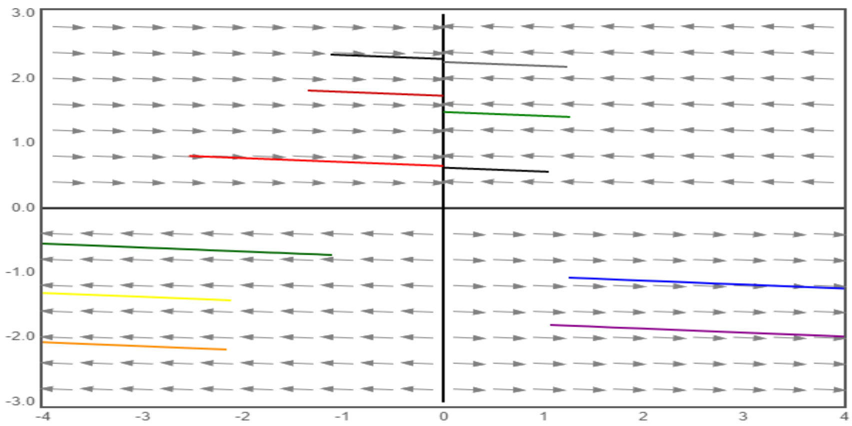

Figure 7 The phase portrait for population of plant versus concentration of nitrate (Case III).

Based on phase portrait in Figure 7, we see that when *N* is negative and *P* is positive, the arrow will moving toward the line $= 0$, when both *N* and *P* are negative, the arrow will go away from line $N = 0$. We can conclude that, when the concentration of nitrate decrease, the plant population can be either decrease or increase. But, when *N* is positive and *P* is positive, the arrow are approach line $= 0$. Here, we can say that, the plant population is exist but the concentration of nitrate will affected the system.

For $\lambda_{17} = -276$,

$\overrightarrow{X_{17}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-276t}$.

For $\lambda_{18} = -0.0156$,
$\overrightarrow{X_{18}} = \begin{pmatrix} -0.0009 \\ 1 \end{pmatrix} e^{-0.0156t}$.
Therefore, the general solution for Case III is

$\begin{pmatrix} N_3 \\ P_3 \end{pmatrix} = c_{17} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-276t} + c_{18} \begin{pmatrix} -0.0009 \\ 1 \end{pmatrix} e^{-0.0156t}$.

$$N_3 = c_{17}e^{-276t} - 0.0009c_{18}e^{-0.0156t}$$

$$P_3 = c_{18}e^{-0.0156t}$$

5. Conclusion

Mathematical model for aquaponics system is formulated by using the assumption from [19]. The combination of assumption helps to form the system of ordinary differential equation (ODE). Since the system of ordinary differential equation is nonlinear, the Jacobian matrix has been used in order to linearize the nonlinear system of ODE. Eigenvalue method can be used when the system of ODE is linear system. Hence, the eigenvalue and eigenvector can be obtained. The eigenvalue help us to determine the stability of equilibrium point meanwhile the eigenvector used to determine the general solution of the system. The phase portrait also plotted in order to visualize the shape and behaviour of the trajectory, so the stability can be checked. After analyzing the data, we found that the fish population, concentration of ammonia, concentration of nitrite, concentration of nitrate and plant population are independently coexistence in the aquaponics system.

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