Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia
Vol. 9, 2022, page 73-81

Numerical Solutions of Diabetes Mellitus Type 1 Model Using Runge-Kutta-Fehlberg Method

Farrah Qhairunniza Md Affandi*, Shazirawati Mohd Puzi
Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia
*Corresponding author: farrah.qhairunniza@graduate.utm.my

Abstract
The purpose of this study is to solve and compare Diabetes Mellitus Model Type 1 by using Runge-Kutta method order of 4 (RK4) and The Runge-Kutta-Fehlberg method (RK45) which an adaptive method for RK4 and develop a programming using MATLAB programming software. Type 1 diabetes, also known as insulin-dependent diabetes or juvenile diabetes, is a chronic illness in which the pancreas produces little or no insulin. A model structure selection based on differential equations obtain from the Diabetes Mellitus Model Type 1 has been proposed in this study to reduce the case population of Diabetes and to compare the trends and result between solving using RK4 and RK45. The Runge-Kutta method is a popular and successful approach for solving differential equations' initial-value issues. The key benefits of Runge-Kutta methods are that they are simple to build, relatively stable, and self-starting. Adaptive approaches are intended to provide an estimate of a single Runge–Kutta step's local truncation error. The Runge–Kutta–Fehlberg method is a numerical analysis methodology for solving ordinary differential equations. Results were compared between RK4 and RK45 method. It was discovered that with similar number of parameters, in most cases, RK45 has less iteration of h than RK4 method since RK45 is to attain a preset level of precision in the solution with the least amount of computing work. The result obtains from the MATLAB programming and manual calculation show that the iteration of h in RK45 is lesser than RK4. As conclusion, it shows that RK45 is more effective in Diabetes Mellitus Model Type 1 than RK4. For future research can use RK45 to solve any Ordinary Differential Equations (ODE) that have many step size and want to reduce the time to solve the problems.

Keywords: Runge-Kutta method order of 4 (RK4); The Runge-Kutta-Fehlberg method (RK45); Diabetes Mellitus Model Type 1; MATLAB programming; Ordinary Differential Equations (ODE).

1. Introduction

Diabetes mellitus is a group of diseases that impact body's ability to utilise blood sugar (glucose). Because glucose is a significant source of energy for the cells that make up muscles and tissues, it is essential to health. It's also the primary source of energy for brain. Diabetes, regardless of a kind that, can result in an excess of sugar in the blood. Type 1 diabetes and type 2 diabetes are both chronic diabetic diseases.

Type 1 diabetes, also known as insulin-dependent diabetes or juvenile diabetes, is a chronic illness in which the pancreas produces little or no insulin. Insulin is a hormone that allows sugar (glucose) into cells for energy production. Despite continued research, there is no treatment for type 1 diabetes. To avoid problems, treatment emphasizes on controlling blood sugar levels with insulin, food, and lifestyle changes.

To solve the Diabetes Mellitus Model Type 1 in this research will use mathematical model using The Runge-Kutta Method. Mathematical models are representations of mathematics that arise through mathematical modelling. Differential equations systems may be used to solve a variety of issues in the field of mathematics. One of them is diabetes mellitus, a condition that is affecting an increasing number of people throughout the world each year [1].This study is for solving Diabetes Mellitus Model Type 1 using Runge-Kutta method order of 4 (RK4) and The Runge-Kutta-Fehlberg method (RK45) which is an adaptive method for RK4. RK4 is the most often used The Runge-Kutta method. The results from both RK4 and RK45 for Diabetes Mellitus Model Type 1 will be compared.

2. Literature Review

2.1. Runge-Kutta method order of 4 (RK4)

The Runge-Kutta method is a popular and successful approach for solving differential equations' initial-value issues. The Runge-Kutta method may be used to build high-order accurate numerical methods from functions without the requirement for high-order derivatives. Runge-Kutta's applications were limited to issues that could be solved by hand. However, with the introduction of modern computers, Runge-Kutta procedures gained fresh prominence. The fundamental aim, the manufacture of efficient and reliable differential equation software, became established as the driving force for research efforts on Runge-Kutta methods, and progress was divided across theory and practice.

Many generalisations of Runge-Kutta methods have been investigated over the last 30 years. These include Byrne and Lambert's pseudo-Runge-Kutta methods, as well as a variety of multistep methods that contain off-step points in the same manner as Runge-Kutta methods do. Jackiewicz and Tracogna's two-step Runge-Kutta methods have recently been presented. These are all instances of generic linear techniques, and within this huge family of methods, many more extensions of Runge-Kutta methods are available [2].

2.2. Adaptive Runge-Kutta method

Adaptive approaches are intended to provide an estimate of a single Runge–Kutta step's local truncation error. This is accomplished by the use of two ways, one with order p and the other with order $p-1$. They have a lot in common in terms of intermediary phases. The step size is adjusted during integration to keep the predicted error below a user-defined threshold: A step is repeated with a smaller step size if the mistake is too great; if the error is significantly smaller, the step size is raised to save time. This yields a (almost) optimum step size, reducing computing time. Furthermore, the user does not have to waste time determining the proper step size.There are a few types of method which are The Runge–Kutta–Fehlberg method, The Bogacki–Shampine method, The Cash–Karp method and The Dormand–Prince method. The Runge–Kutta–Fehlberg method's benefit is based on error management and time step adjustment to keep the error below a particular threshold. By comparing the current technique to various previous methods in a numerical experiment, it is demonstrated that the current method outperforms them in terms of computing speed and accuracy [3].

2.2.1. The Runge-Kutta-Fehlberg method (RK45)

In this research, The Runge-Kutta-Fehlberg method will be use. One technique to ensure correctness in an I.V.P. solution is to solve it twice with step sizes h and $\frac{h}{2}$ and compare results at the mesh points corresponding to the bigger step size. However, with the lower step size, this necessitates a large amount of calculation, which must be repeated if the agreement is found to be inadequate. It has a method for determining whether the correct step size h is being utilized. Two distinct approximations for the answer are created and compared at each stage. The approximation is approved if the two responses are almost identical. The step size is adjusted if the two responses do not agree to a defined accuracy. The step size is raised if the answers agree to more significant digits than necessary.

The Runge–Kutta–Fehlberg method is a numerical analysis methodology for solving ordinary differential equations. It was created by Erwin Fehlberg, a German mathematician, and is based on the Runge–Kutta method family. The double step approach needs a smaller step-by-step evaluation than the one-step method, and it is referred to as the prediction corrector method by practically everyone. One of the double-step approaches for achieving more stability.

2.3. Diabetes Mellitus Model Type 1

Type 1 diabetes is a disorder in which your immune system attacks the cells in your pancreas that produce insulin. These are referred to as β -cells. Because the disorder is most commonly diagnosed in adolescents and teenagers, it was previously known as juvenile diabetes. Secondary diabetes is like type 1, but your β -cells are destroyed by something other than your immune system, such as a disease or an injury to your pancreas.

To enhance the quality of life and prognosis of those affected, major research efforts are needed to obtain early diagnosis, reduce cell loss, and create better treatment choices. Both genetic and environmental variables are linked to the development of Diabetes Mellitus Type 1. Children with a history of several infections had a lower probability of developing cell-targeted autoimmunity in Russia than children with the same genetic risk but fewer illnesses. Approximately 0.3–0.5 percent of children in the general population are likely to have two or more islet-targeting autoantibodies during their childhood [4].

The model of Diabetes Mellitus Type 1,

$$\frac{dx_1}{dt} = pA - \frac{\beta_1 x_1 x_3}{x_1 + x_2 + x_3} - \mu x_1 \tag{1}$$

$$\frac{dx_2}{dt} = (1-p)A - \frac{\beta_2 x_2 x_3}{x_1 + x_2 + x_3} + \gamma x_3 - \mu x_3 \tag{2}$$

$$\frac{dx_3}{dt} = \frac{\beta_1 x_1 x_3}{x_1 + x_2 + x_3} + \frac{\beta_2 x_2 x_3}{x_1 + x_2 + x_3} - (\mu + \gamma + \delta) x_3 \tag{3}$$

where A is constant,

$x_1 =$ healthy human compartments with no diabetes gene in their blood,

$x_2 =$ healthy human compartments with disease-carrying genes diabetes in their blood,

$x_3 =$ diabetic infected human compartments with diabetes in their blood,

$\beta =$ rate of proportion infected,

$\mu =$ natural rate death,

$\delta =$ death rate due to diabetes,

$\gamma =$ infected human population recovers,

$p =$ the proportion of humans born healthy

3. Methodology

3.1. Proposed method

The-Runge-Kutta-Fehlberg method is the most suitable to solve the Diabetes Mellitus Model Type 1. It's a numerical analysis algorithm for solving ordinary differential equations numerically. It was created by Erwin Fehlberg, a German mathematician, and is based on the Runge–Kutta technique family.

3.2. Runge-Kutta method order of 4 (RK4)

Runge-Kutta method of Order 4 (RK4) is the most common method of Runge-Kutta. Derivation of Runge-Kutta method of Order 2 (RK2) is the same with derivation of Runge-Kutta method of Order 4

(RK4). Firstly, Taylor expansion need to be construct from the ordinary differential equation and the initial condition which is in (4) and (5),

$$\frac{dy}{dx} = f(x, y) \tag{4}$$

$$y(0) = y_0 \tag{5}$$

where the derivation for the ordinary differential equation is as (6), (7) and (8),

$$\frac{d^2y}{dx^2} = f^{'}(x, y) \tag{6}$$

$$\frac{d^3y}{dx^3} = f^{''}(x, y) \tag{7}$$

$$\frac{d^4y}{dx^4} = f^{'''}(x, y) \tag{8}$$

Next, Taylor expansion can be construct from the equation (4), (5), (6), (7) and (8)

$$y_{i+1} = y_i + \frac{dy}{dx}|_{x_i, y_i} (x_{i+1}, x_i) + \frac{1}{2!}\frac{d^2y}{dx^2}|_{x_i, y_i} (x_{i+1} - x_i)^2 + \frac{1}{3!}\frac{d^3y}{dx^3}|_{x_i, y_i} (x_{i+1} - x_i)^3 +$$

$$\frac{1}{4!}\frac{d^4y}{dx^4}|_{x_i, y_i} (x_{i+1} - x_i)^4 + O(x_{i+1} - x_i)^5 \tag{9}$$

By substituting (4), (5), (6), (7) and (8) with $x_{i+1} - x_i = h$ into (9)

$$y_{i+1} = y_i + f(x_i, y_i)h + \frac{1}{2!} f^{'}(x_i, y_i)h^2 + \frac{1}{3!} f^{''}(x_i, y_i)h^3 + \frac{1}{4!} f^{'''}(x_i, y_i)h^4 \tag{10}$$

After expanding equation (10), the result obtain for general equation are

$$y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = hf(x_i, y_i)$$
$$k_2 = hf(x_i + \frac{h}{2}, y_i + \frac{k_1}{2})$$
$$k_3 = hf(x_i + \frac{h}{2}, y_i + \frac{k_2}{2})$$
$$k_4 = hf(x_i + h, y_i + k_3)$$

3.3. The Runge-Kutta-Fehlberg method (RK45)

For the numerical solution of initial-value problems with oscillating solutions, a Runge-Kutta approach is proposed. In order to produce reliable approximations along the intervals employed, traditional Runge-Kutta techniques use tiny step sizes for the integration of equations describing free oscillations. This RK45 method is more likely as Runge-Kutta method of order 5 or higher order. By using the same ordinary differential equation from (3.2), we will have the adaptive of RK4 such as

$$y_{i+1} = y_i + \frac{25}{216}k_1 + \frac{1408}{2565}k_2 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5 \tag{11}$$

where

$$k_1 = hf(x_i, y_i)$$
$$k_2 = hf(x_i + \frac{h}{4}, y_i + \frac{k_1}{4})$$
$$k_3 = hf(x_i + \frac{3h}{8}, y_i + \frac{3k_1}{32} + \frac{9k_2}{32})$$
$$k_4 = hf(x_i + \frac{12h}{13}, y_i + \frac{1932k_1}{2197} - \frac{7200k_2}{2197} + \frac{7296k_3}{2197})$$
$$k_5 = hf(x_i + h, y + \frac{439k_1}{216} - 8k_2 + \frac{3680k_3}{513} - \frac{845k_4}{4104})$$

With adding an additional term, the equation would be like

$$k_6 = hf(x + \frac{h}{2}, y - \frac{8k_1}{27} + 2k_2 - \frac{3544k_3}{2565} + \frac{1859k_4}{4104} - \frac{11k_5}{40})$$
$$z_{i+1} = z_i + \frac{16k_1}{135} + \frac{6656k_3}{12825} + \frac{28561k_4}{56430} - \frac{9k_5}{50} + \frac{2k_6}{55} \tag{12}$$

Next, subtracting (12) and (11) to measure an error,

$$error = | z_{i+1} - y_{i+1} | \tag{13}$$

The optimal step size h_n can be determined by multiplying the scalar times the current step size h. The scalar h_n is

$$h_n = (\frac{tol(h)}{2(error)})^{\frac{1}{4}} \approx 0.84(\frac{tol(h)}{(error)})^{\frac{1}{4}} \tag{14}$$

by substituting (13) into (14),

$$h_n = (\frac{tol(h)}{2| z_{i+1} - y_{i+1} |})^{\frac{1}{4}} \approx 0.84(\frac{tol(h)}{| z_{i+1} - y_{i+1} |})^{\frac{1}{4}} \tag{15}$$

It has a method for determining whether the correct step size h is being utilised. Two distinct approximations for the answer are created and compared at each stage. The approximation is approved if the two answers are almost identical. The step size is adjusted if the two answers do not agree to a defined accuracy. The step size is raised if the answers correspond to more significant digits than necessary.

4. Results and discussion

4.1 Comparison between trends RK4 and RK45

Computationally calculation from MATLAB for RK4 show that number of healthy human infected with Diabetes Mellitus Type 1 is slightly decreasing since healthy human have a better lifestyle by controlling what they eat and regularly exercise. Healthy lifestyle shows that from 100 person that be counted just decreasing to 99.9729 person and we know that 2.79% decreasing rate of person remain healthy and the balance infected with Diabetes Mellitus Type 1. Figure 1 below shows that the trend of healthy human that infected with Diabetes Mellitus Type 1.

Computationally calculation from MATLAB for RK45 show that the number of healthy humans after 5 years is decreasing since healthy human tends to have healthy lifestyle and can reduce the risk of diabetes. Figure 2 below show that the number of healthy humans that is not suffer from diabetes drop with rate of 25.1%.

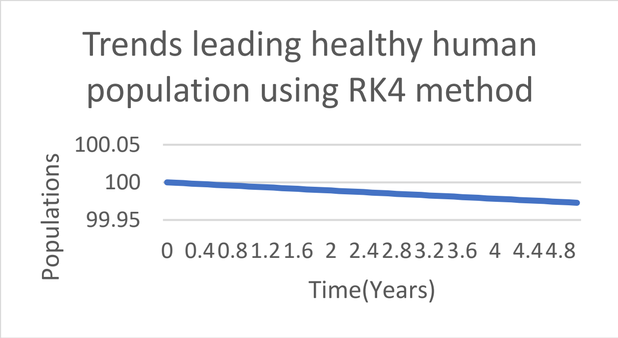

Figure 1 Trends leading healthy human population using RK4 method

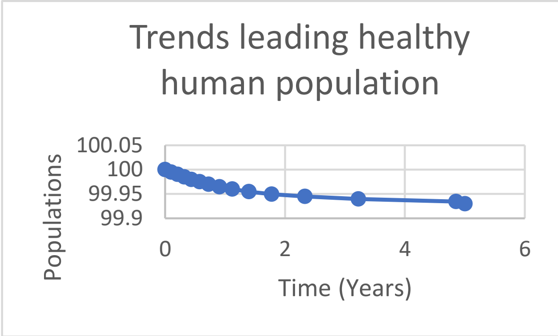

Figure 2 Trends leading healthy human population using RK45 method method

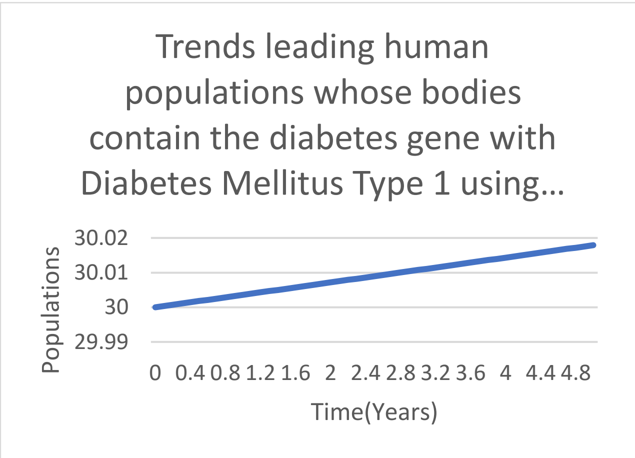

Figure 3 Trends leading human populations whose bodies contain the diabetes gene with Diabetes Mellitus Type 1 using RK4 method

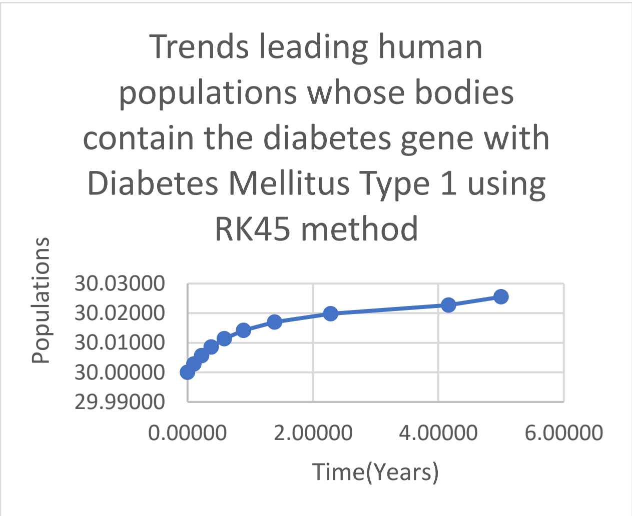

Figure 4 Trends leading human populations whose bodies contain the diabetes gene with Diabetes Mellitus Type 1 using RK45 method

Based on Figure 3 above trends show that there were an increasing number in population for x_2. Initial number of human tests was 30 person and after 5 years it shows that the number of human increases to 30.0179 person. It shows that the increasing with rate 1.79% of human whose bodies contain the diabetes gene with Diabetes Mellitus Type 1. From Figure 4, the graph obtain show that the number of human populations whose bodies contain the diabetes gene with Diabetes Mellitus Type 1 has increase to 2.549% which is from initials with 30 person and after 5 years the number changed to 30.02549. Humans that already have the genes in their chromosome is highly tends to get the disease than human that is healthy.

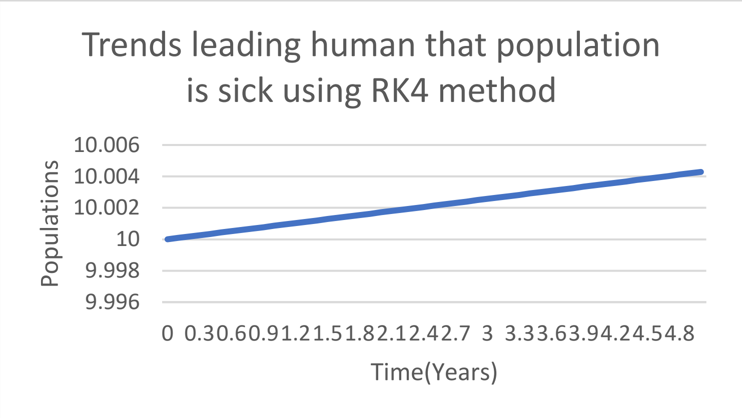

Figure 5 Trends leading human that population is sick using RK4 method

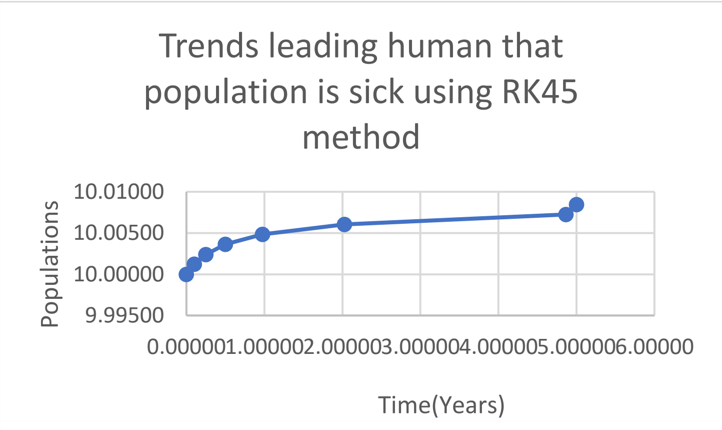

Figure 6 Trends leading human that population is sick using RK45 method

The trends in Figure 5, it is show that the number of populations of human that is sick is increasing. Initial number of humans that is sick is 10 but after 5 years the number has increase with rate 0.43%. Humans that have genes of diabetes in their bodies tends to have the disease more than humans who have another disease that is initially sick. Initials number of humans in Figure 6 is 10 and increase by rate of 0.845% by 5 years. The number is increase slightly because human that is sick, they may not have any diabetes gene in their body and their antibody can fight with diabetes but not other disease.

4.2. Comparison between RK4 and RK45

Table 1: Iteration of x_1 by RK4

i	t	x_1
0	0	100
1	0.1	99.99946
2	0.2	99.99891
3	0.3	99.99837
4	0.4	99.99783
5	0.5	99.99729
6	0.6	99.99674
.	.	.
.	.	.
.	.	.
.	.	.
47	4.7	99.9745
48	4.8	99.97395
49	4.9	99.97341
50	5	99.97287

Table 2: Iteration of x_1 RK45

i	t	x_1
0	0	100
1	0.10000	99.99498
2	0.20528	99.98996
3	0.31754	99.98494
4	0.43921	99.97992
5	0.57373	99.97490
6	0.72626	99.96988
7	0.90471	99.96486
8	1.12187	99.95984
9	1.39942	99.95482
10	1.77658	99.9498
11	2.32998	99.94478
12	3.22362	99.93976
13	4.85039	99.93474
14	5	99.92972

Both of the Table 1and Table 2 shows the iteration of the step size. In Table 1, the iteration of x_1 reach until iteration number 50 while in Table 2, the iteration was reduced to 14 iterations. The duration still 5 years but the iteration can be reduced by using RK45. RK45 is a method that can reduce the time step to make the calculation shorter than RK4.

Conclusion

As a conclusion, RK45 shows that it can reduced the step size into the shorter way than RK4. RK45 is suitable to solve the problem that have many data so that it can also reduce the time taken to solve the problem. RK4 is suitable to use if the data from the problem is less. So, it can solve it without making any error. RK45 can reduced its step size by using the optimal step size formula that require the users to calculate the error and tolerance before change the step size.

Acknowledgement

The researcher would like to thank all people who have supported the research and give support to write the paper.

References

[1] Side, S., Astari, G. P., Pratama, M. I., Irwan, & Sanusi, W. (2019). Numerical Solution of Diabetes Mellitus Model without Genetic Factors with Treatment using Runge Kutta Method. *Journal of Physics: Conference Series*, *1244*(1), 012021.

[2] Butcher, J. (1997). An introduction to "Almost Runge-Kutta" methods. *Applied Numerical Mathematics*, *24*(2–3), 331–342.

[3] Nwankwo, C., & Dai, W. (2021). An adaptive and explicit fourth order Runge–Kutta–Fehlberg method coupled with compact finite differencing for pricing American put options. *Japan Journal of Industrial and Applied Mathematics*, *38*(3), 921–946.

[4] Katsarou, A., Gudbjörnsdottir, S., Rawshani, A., Dabelea, D., Bonifacio, E., Anderson, B. J., Jacobsen, L. M., Schatz, D. A., & Lernmark, K. (2017). Type 1 diabetes mellitus. *Nature Reviews Disease Primers*, *3*(1).