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Analytical Solutions of Conformable Advection-Dispersion Transport in Porous Media

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Abstract

The conformable derivative is used to propose the advection-dispersion equation (ADE) for an accurate description of fluid flow and solute transport in porous media. Under various initial and boundary conditions, analytical solutions of the proposed conformable advection-dispersion equation (CADE) are produced. The analytical solutions of CADE in three different form which are stretched Gaussian, error function and asymmetrical are reduced to classical ADE with the conformable derivative order $\alpha = 1$. The analytical solutions are obtained using Fourier transformation technique. Furthermore, CADE models is also validated on the basis of empirical data in literatures, as opposed to the standard ADE (power-law) breakthrough curves of fluid flow and solute transport in porous media. As the result, it indicates that the presented CADE models is applicable for each case. Therefore, the CADE models described here can be utilized to accurately describe fluid flow and solute transport in porous media.

Keywords: Advection-dispersion; Conformable derivative; Analytical solutions; Fluid flow; Solute transport

1. Introduction

Fluid flow and solute transport modelling in porous media is significant in a wide range of applications including hydrogeology, groundwater management, environmental protection, soil physics, petroleum engineering, chemical engineering and energy extraction. Many applications in the environment and the chemical process sector require transport in porous media [1]. Porous materials can be found in almost every aspect of existence including technology and nature. Almost all solid and semi-solid materials are "porous" to variable degrees with the exception of metals, some hard rocks and some polymers. To be classified as a porous medium, a substance or structure must possess these two characteristics. It must have holes or pores implanted in the solid or semi-solid matrix that are devoid of substances. A fluid such as air, water, oil or a mixture of fluids is frequently present in the pores. It must be permeable to a wide range of fluids which means fluids must be able to pass through one face of a material sample and emerge on the other.

Advection and dispersion in natural pore-fracture networks are the fundamental fluid flow and solute transport mechanisms in porous media [2]. According to Philips and Castro [3], advection is mechanical transport of solutes along with the bulk flux of the water, where the water flux is driven by the gradient in the total mechanical energy of the solution. Dispersion refers to the spreading of the contaminant plume from highly concentrated areas to less concentrated areas.

The solution to ADE with certain initial and boundary conditions frequently necessitates the use of numerical methods. In other cases, when an analytical approach is feasible, the solutions frequently involve constant velocities. There are various analytical solutions for constant velocity and dispersion coefficients, as well as various boundary conditions [4]. Its analytical solutions help to understand the contaminant or pollutant concentration distribution behaviour through an open medium like rivers, lakes, air and porous medium like groundwater. The classical advection–dispersion equations (ADE) for modelling advection–dispersion transport processes have become widely used. It is mostly applied in the area of transportation modelling [5]. Specifically, the advection-dispersion equation is commonly used as governing equation for movement of contaminants or more widely solutes in saturated porous media. There are numerous analytical solutions for classical ADE. However, analytical solutions for the fractional advection–dispersion equation with hybrid circumstances tend to be relatively complicated, requiring tedious mathematical derivations and numerical simulations [6]. Therefore, a new perspective to time-dependent modelling is necessary with the conformable derivative technique arising.

In this study, the conformable derivative is used to generalise the classical ADE, resulting in the conformable advection–dispersion equation (CADE). Three cases of CADE will be considered, which are stretched Gaussian solution, error function solution and asymmetrical solution. CADE models' prospective applicability have been validated through analytical derivations and experimental evaluations.

2. Literature Review

2.1. Preliminary Concepts

Definition 1. Given a function *f(t)*: [0, ∞) → R, then the conformable derivative of *f* with order α ∈ (0, 1] is defined by

$$T_\alpha f(t) = \lim_{\varepsilon \to 0} \frac{f\left(t+\varepsilon t^{1-\alpha}\right)-f(t)}{\varepsilon}, \tag{1}$$

for all t > 0. If *f* is α-differentiable, i.e., $T_\alpha f(t)$ exists and $\lim_{t\to 0}(T_\alpha f)(t)$ also exists, then the conformable derivative at 0 is given by

$(T_\alpha f)(0) = \lim_{t\to 0}(T_\alpha f)(t).$

Lemma 2. If *f* is differentiable and α-differentiable for *t* > 0, then $T_\alpha f(t) = t^{1-\alpha}\frac{df}{dt}$ is valid [7].

2.2. Advection-Dispersion Equation

The one-dimensional ADE for the description of concentration distribution of fluid flow and solute transport in porous media is represented as

$$\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial x^2} - u\frac{\partial C(x,t)}{\partial x}, \tag{2}$$

2.3. Conformable Advection-Dispersion Equation

The conformable advection-dispersion equation is expressed as

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = D_\alpha \frac{\partial^2 C}{\partial x^2} - u\frac{\partial C}{\partial x} \tag{3}$$

2.4. Fourier Transform Method

Fourier transform (FT) is a mathematical transform that dissolve functions depending on space or time into functions depending on spatial or temporal frequency. The Fourier transform is not limited to functions of time but the domain of the original function is commonly referred to as the time domain. There is also an inverse Fourier transform that mathematically synthesizes the original function from its frequency domain representation, as proven by the Fourier inversion theorem.

3. Mathematical Formulation

3.1. Stretched Gaussian Solution

For an instantaneous point source, the conformable advection-dispersion equation is

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = D_\alpha \frac{\partial^2 C(x,t)}{\partial x^2} - u\frac{\partial C(x,t)}{\partial x}, \tag{4}$$

$$C(x,0) = \delta(x)$$

Employing the Fourier transform to both sides of the CADE in equation (4) and taking Lemma 2 will get

$$\frac{d\hat{C}\ (\omega,t)}{dt} + (D_\alpha \omega^2 + ui\omega)t^{\alpha-1}\hat{C}(\omega,t) = 0, \tag{5}$$

with the initial condition $\hat{C}(\omega,0) = \frac{1}{\sqrt{2\pi}}$. The analytical solution of the equation (5) can be expressed as

$$\hat{C}(\omega,t) = \frac{1}{\sqrt{2\pi}} exp\left\{-\frac{D_\alpha\omega^2 + ui\omega}{\alpha}\ t^\alpha\right\} \tag{6}$$

The inverse Fourier transform will deduce the analytical solution of the CADE represents as a stretched Gaussian distribution as the following

$$C(x,t) = \sqrt{\frac{\alpha}{4\pi D_\alpha t^\alpha}}\ \exp\left\{\frac{-\alpha}{4D_\alpha t^\alpha}(x - \frac{ut^\alpha}{\alpha})^2\right\}, \tag{7}$$

and when $\alpha = 1$,

$$C(x,t) = \frac{1}{\sqrt{4\pi Dt}}\ \exp\left\{\frac{-(x-ut)^2}{4Dt}\right\} \tag{8}$$

3.2. Error Function Solution

Consider the conformable advection–dispersion equation in semi-infinite media

$$\frac{\partial^\alpha C(x,t)}{\partial t^\alpha} = D_\alpha \frac{\partial^2 C(x,t)}{\partial x^2} - u\frac{\partial C(x,t)}{\partial x} \tag{9}$$

with the conditions

$$C(x,0) = 0, C(\infty,t) = 0, C(0,t) = C_0 \tag{10}$$

The CADE in equation (9) reduces to the following conformable diffusion equation by using transformation

$$\frac{\partial^\alpha C(x',t)}{\partial t^\alpha} = D_\alpha \frac{\partial^2 C}{\partial x'^2} \tag{11}$$

Applying a dimensionless Boltzmann transform, ξ and rearranging equation (11) based on Lemma 2, we can obtain an ordinary differential equation,

$$\frac{d^2C}{d\xi^2} + 2\alpha^3\xi\frac{\partial C}{\partial \xi} = 0 \tag{12}$$

The equation (12) has the following analytical solution

$$\frac{C_0 \operatorname{erfc}\left(\alpha\sqrt{\alpha}\xi\right)}{\operatorname{erfc}\left(\frac{-ut^\alpha}{\sqrt{4\alpha D_\alpha t^\alpha}}\right)} \tag{13}$$

Substituting ξ and x' into equation (13) yields (14)

$$C_0 erfc\left(\frac{x - \frac{ut^\alpha}{\alpha}}{\sqrt{4\alpha D_\alpha t^\alpha/\alpha}}\right)\left[erfc\left(\frac{-ut^\alpha}{\sqrt{4\alpha D_\alpha t^\alpha}}\right)\right]^{-1}$$

It is important to note that the CADE degenerates to the classical ADE because the conformable fractional derivative decreases to the classical integer derivative with differential order $\alpha = 1$. As a result, the analytical CADE solution in equation (14) yields the conventional ADE solution, i.e.

$$C(x,t) = C_0 erfc\left(\frac{x-ut}{\sqrt{4Dt}}\right)\left[erfc\left(\frac{-ut}{\sqrt{4Dt}}\right)\right]^{-1} \tag{15}$$

When the average velocity, $u = 0$, equation (14) turns into

$$C(x,t) = C_0 erfc\left(\frac{\sqrt{\alpha}x}{\sqrt{4D_\alpha t^\alpha}}\right) \tag{16}$$

Meanwhile, the traditional ADE is reduced to the normal diffusion equation, which has the following well-known analytical solution in terms of error function:

$$C(x,t) = C_0 erfc\left(\frac{x}{\sqrt{4Dt}}\right) \tag{17}$$

3.2. Asymmetrical Solution

Ogata and Bank [8] proposed an asymmetrical solution to classical ADE with acceptable boundary conditions, i.e., the CADE in equation (9) with $\alpha = 1$, as a solution to classical ADE.

$$C(x,t) = \frac{C_0}{2}\left[\operatorname{erfc}\left(\frac{x-ut}{\sqrt{4Dt}}\right) + \exp\left(\frac{ux}{D}\right)\operatorname{erfc}\left(\frac{x+ut}{\sqrt{4Dt}}\right)\right] \tag{18}$$

We aim to propose an asymmetrical solution of the presented CADE in equation (9) using analogous reasons, as motivated by the aforementioned publications. Using a transformation similar to the one described in [8], that is

$$C(x,t) = K(x,T)\exp\left(\frac{ux}{2D_\alpha} - \frac{u^2T}{4D_\alpha}\right) \tag{19}$$

where $T = t^\alpha/\alpha$. The following boundary value problem is deduced by substituting equation (19) into equation (9).

$$\frac{\partial K(x,T)}{\partial t} = D_\alpha \frac{\partial^2 K(x,T)}{\partial x^2}$$
$$C(0,T) = C_0\; exp\left(\frac{u^2T}{4D_\alpha}\right), C(x,0) = 0, C(\infty,T) = 0 \tag{20}$$

which is similar to the problem presented by Ogata and Bank [8].

As a result, the desired CADE solution in equation (9) can be written as

$$C(x,t) = \frac{C_0}{2}\left[erfc\left(\frac{x - ut^{\alpha}/\alpha}{\sqrt{4D_{\alpha}t^{\alpha}/\alpha}}\right) + exp\left(\frac{ux}{D_{\alpha}}\right) erfc\left(\frac{x + ut^{\alpha}/\alpha}{\sqrt{4D_{\alpha}t^{\alpha}/\alpha}}\right)\right] \tag{21}$$

4. Results and discussion

4.1. Stretched Gaussian Solution

Figure 1 illustrates the outcomes of equation (7) formed by the process of Fourier transformation technique. The concentration values calculated from the analytical solution of equation (7) for under point source problem as stretched Gaussian solution are shown. The parameter values used here are based on the experimental study done for CO_2-CH_4 dispersion in Estaillades carbonate conducted by Honari et. al., (2015). The solid curves in Figure 1 represent equation (7) in which concentration values are evaluated in a finite domain $0 \leq x \leq 20$ at times $t(second) = 86400$ is chosen for various value of α which are 0.2, 0.5, 0.7 and 1.0, for the input values $u = 0.009\ mms^{-1}$, $D = 0.152\ mm^2/s$ are taken.

Based on Figure 1, the four solid curves at $t(second) = 86400$ which is same as 1 day show that the concentration values at $\alpha = 0.2, 0.5, 0.7$ and 1.0 decrease starting from the origin, $(x = 0)$ to the other end of finite domain $(x = 20)$. Also, as parameter of α becomes higher, concentration value at respective position is lower or it can also be said that as parameter of α rises from 0.2 to 1.0, the value of concentration decreased but for $x < 6$ only. After that, there is an area which the concentration for the smallest α (0.2) is lower than the higher α. In addition, from the same figure, it can also be seen at the smallest value of $\alpha = 0.2,$ the concentration value decreases the fastest and for $\alpha = 1.0$, the line is where almost no changes compared to the others as the other end of the medium is approached. This means that the value of concentration is so small where it is almost zero when the parameter $\alpha = 1.0$.

Apart from that, in Figure 2, three different curves with different velocities are drawn for the value $u = 0.009\ mm\ s^{-1}, 0.23\ mm\ s^{-1}$ and $1.149\ mm\ s^{-1}$ at $\alpha = 0.2$. The time taken for this process is 86400 second and the dispersion coefficient, *D* is $0.152\ mm^2/s$. When the velocity, $u = 0.009\ mm\ s^{-1}$ and $0.23\ mm\ s^{-1}$, the highest value of concentration obtained for this two curves is almost at the same level. This condition indicates that the curves and path of this two concentration are quite consistent. When at $u = 1.149\ mm\ s^{-1}$, the concentration is almost at zero starting from the origin $(x = 0)$ to the end of range x which is at $x = 20$ mm. This shown that the value of concentration is very small at this velocity.

Figure 3 illustrates the concentration values plotted with dispersion coefficient $D = 0.152\ mm^2s^{-1}$ and $0.123\ mm^2s^{-1}$ for velocity, $u = 0.009\ mm\ s^{-1}$ and $t = 86400\ s$. The parameter of α used in this case is 0.5. These two lines have the same shape but different in concentrations. From the figures, it is observed that when the dispersion coefficient is higher, the concentration becomes lower. The concentration of each line increases from the origin until one point where around $x = 7\ mm$. After that, the concentration value decreases until they have the same value of concentration at around $x = 17mm$. Overall, Figures 1- 3, it can be seen that the solution satisfies the Gaussian distribution shape (plotted on the side only here).

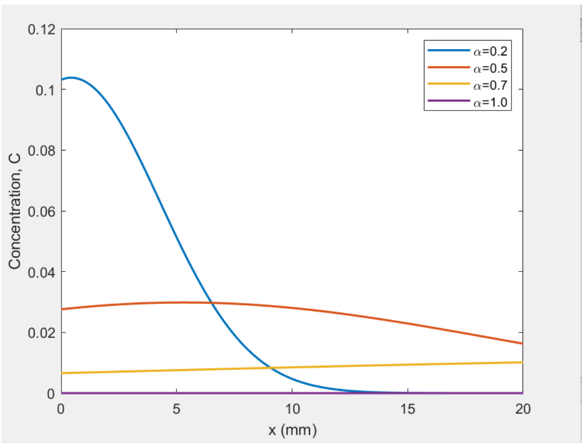

Figure 1 Concentration profiles of the CADE model in equation (7) with the ordinary ADE ($\alpha = 1$)

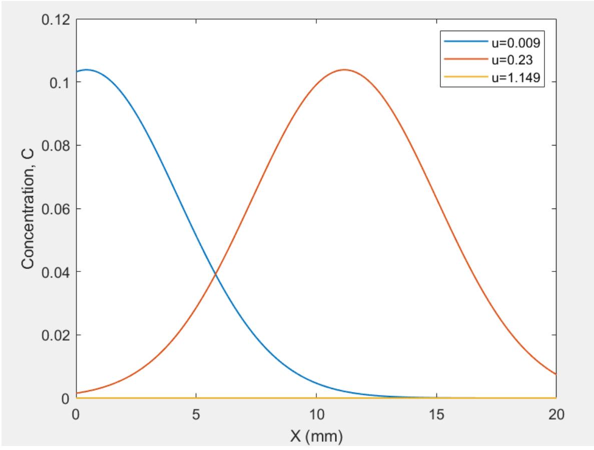

Figure 2 The CADE model in equation (7) for different velocity, u at $\alpha = 2$

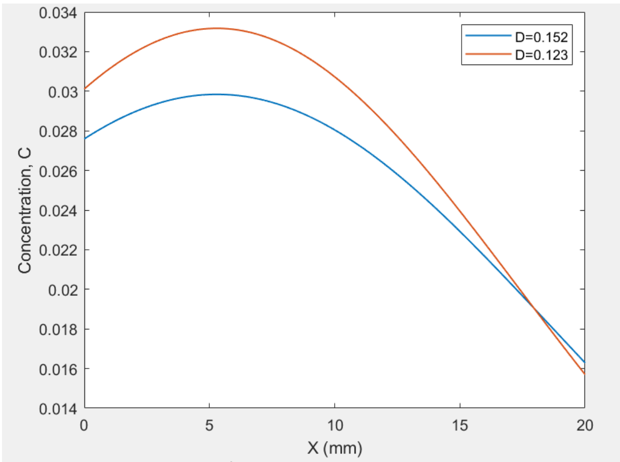

Figure 3 The CADE model in equation (7) with the breakthrough profile for different dispersion coefficient, D

4.2. Error Function Solution

Figure 4 shows the CO_2 concentration values evaluated from analytical solution of equation (14) for Dirichlet problem as error function solution. The parameters are taken from the experiment of CO_2-CH_4 displacement and dispersion in sandpacks in enhanced gas recovery conducted by Liu et. al., (2015). The curves in figure 4 represent concentration values in a finite domain $0 \leq x \leq 1$ for various value of α which are 0.2, 0.5, 0.7 and 1.0 , for input values $C_0 = 1.0, u = 0.114$ (10^{-3}m/s), $t = 86400\ s$ and $D = 1.890$ (10^{-7} m^2/s) are taken.

Based on Figure 4, the smaller the values of α, the faster the concentration values decrease. When $\alpha = 1$, there is almost no changes along the path compared to the others where the concentration value maintain at $C = 1$ from the beginning until the end of range of x which is at $x = 1$ m.

Apart from that, in Figure 5, three different curves are drawn for the different value of dispersion coefficient which are $D = 2.517$ (10^{-7} m^2/s), 4.517 (10^{-7} m^2/s) and 6.517 (10^{-7} m^2/s) for velocity, $u = 0.057\ (10^{-3} mm\ s^{-1})$ and $t = 86400\ s$. The parameter of α used in this case is 0.7. The graphic shows that the values of concentration decrease slightly more quickly as the dispersion coefficient value rises, from $x = 0$ to $x = 0.25$. However, the concentration behavior act differently after that.

In Figure 6, three different curves are drawn for the value $u = 0.057 \times 10^{-3}\ m/s$, 0.085 $\times\ 10^{-3}\ m/s$ and 0.142 $\times\ 10^{-3}\ m/s$. The time taken for this process is 86400 second and the dispersion coefficient, *D* is $1.890 \times 10^{-7} m^2/s$. The initial condition for three curves are the same since the initial condition, $C_0 = 1$. For $u = 0.057\ \times 10^{-3}\ m/s$ where the smallest value of velocity among them, the concentration was dropped fastest followed by $u = 0.085 \times 10^{-3}\ m/s$ and 0.142 $\times\ 10^{-3}\ m/s$.

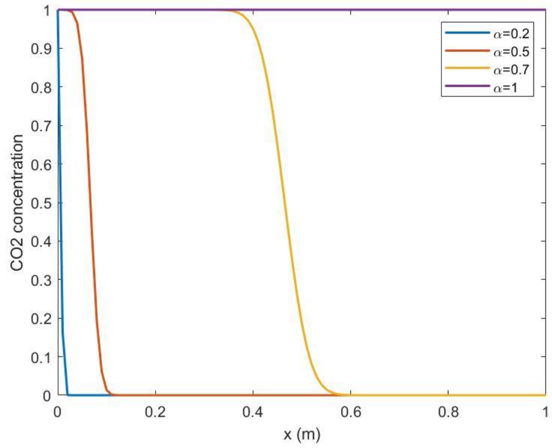

Figure 4 Concentration profile of the CADE model in equation (14) with breakthrough profiles for CO_2-CH_4 in sandpacks at different α

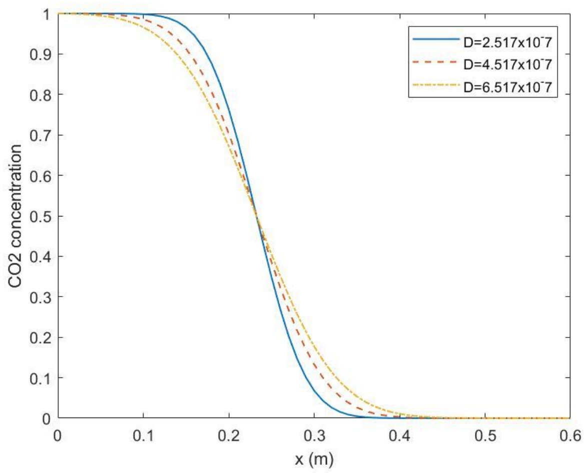

Figure 5 The CADE model in Equation (3.26) with breakthrough profiles for D at $\alpha = 0.7$

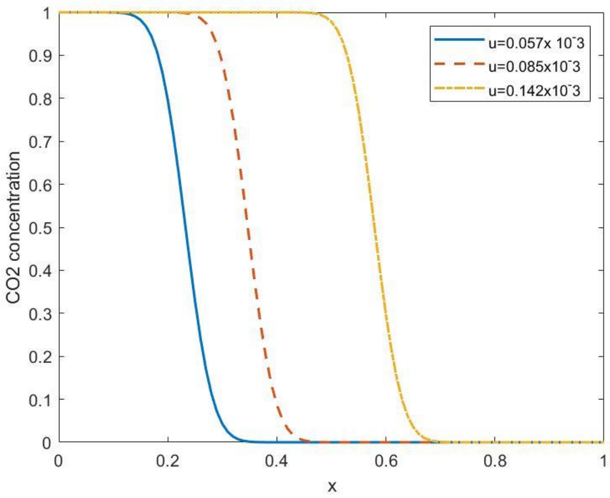

Figure 6 The CADE model in equation (14) for different velocity, u.

4.3. Asymmetrical Solution

Figure 7 shows the concentration values evaluated from analytical solution of equation (21) for asymmetrical solution. The parameter values used here are based on the experimental data from Honari et. al., (2015). The curves in Figure 7 represent concentration values in a finite domain $0 \leq x \leq 1$ with $t(second) = 86400\ s$ for various value of α which are 0.2, 0.5, 0.7 and 1.0 , for input values $C_0 = 1.0, u = 0.0001$ (m/s) and $D = 16.4\ (10^{-8}$ m^2/s) are taken.

Based on Figure 7, the smaller the values of α, the faster the concentration values decrease. When $\alpha = 1$, there is almost no changes along the path compared to the others where the concentration value maintain at $C = 1$ from the beginning until the end of range of *x* which is at $x = 1$ m.

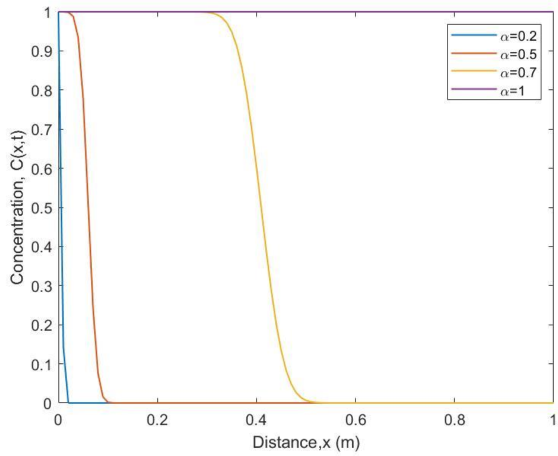

Figure 7 Concentration profiles of the CADE model in equation (21) for different α.

Apart from that, in Figure 8, three different curves are drawn for the value $u = 0.23\ mm\ s^{-1}, 0.387\ mm\ s^{-1}$ and $0.696\ mm\ s^{-1}$. The curves in Figure 8 represent concentration values for value of $\alpha = 0.2$, input values $C_0 = 1.0,\ t = 86400\ s$ and $D = 0.152$ mm^2/s are taken. When the velocity is smaller, the values of concentration will drop faster. For $u = 0.23\ mm\ s^{-1}$, the concentration was dropped fastest followed by $u = 0.387\ mm\ s^{-1}$. When $u = 0.696\ mm\ s^{-1}$, the line changes almost nothing along the path in comparison to the others since the concentration value is 1 from the start of the range of x to the end, which is at x=20 mm.

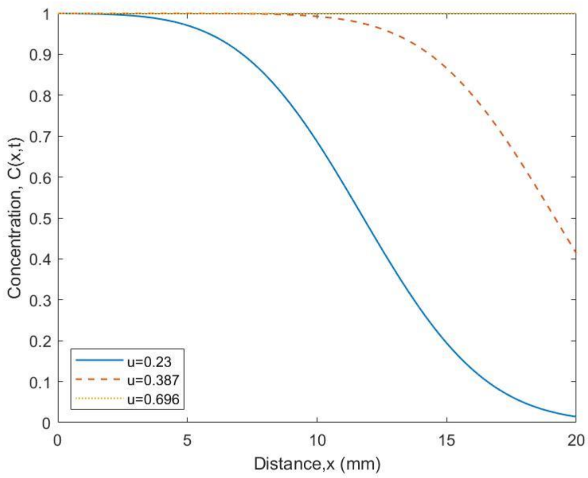

Figure 8 The CADE model in equation (21) with different velocity, u

Conclusion

The main scope of this study is to produce analytical solutions for the conformable derivative advection–dispersion equations. As a result, by illustrating the behavior of fluid flow and solute transport in porous media, the CADE models represented by these solutions are validated. All of the CADE models provided are in great agreement with actual data in the literature, indicating that they may be used to accurately describe fluid flow and solute transport in porous media. It is indicated that the conformable derivative can be associated with the concept of derivative with respect to another function. The variation or different of conformable derivative orders have been analysed through the plotting graphs in this study. Such analytical solutions could be used to validate the conformable derivative model.

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