Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia

Vol. 17, 2023, page 1-10

Dynamical Behaviors of a Prey-Predator Model with Holling Type III Functional Response and Non-Linear Predator Harvesting

Nur Alia Dzulkifli, Tau Keong Ang
Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia
*Corresponding author: taukeong@utm.my

Abstract
Prey-predator relationship is an interaction between two different species of animal in which one of them takes the role of the predator and hunts down its prey. This study is focusing on how the dynamical behaviors of the prey-predator model change in the presence of non-linear predator harvesting with Holling Type III functional response. The model is developed using the system of Ordinary Differential Equations (ODEs). The equilibrium points in the model are trivial equilibrium, predator free equilibrium and co-existence interior equilibrium. The Routh-Hurwitz criterion is used to determine the stability of the equilibrium points. The simulations of the model are carried out in MATLAB and the bifurcations analysis is examined. Lastly, the effects of the non-linear harvesting of predator on the stability of the steady states are studied.
Keywords: Prey-predator model, Non-linear harvesting; Holling Type III functional response, Stability analysis, Bifurcation analysis

Introduction

Prey-predator relationship is an interaction between two different species of animal in which one of them takes the role of the predator and hunts down its prey. In ecological system, the prey and predator evolve together. The predator is the animal that eats another organism while the prey is the animal that is being eaten by the predator. Prey-predator interaction models have received more attention in recent years from researchers for both management and descriptive purposes. A number of biological resources are being exploited more frequently as a result of the constantly increasing need for food and resources [1]. In order to find the ways to overcome this problem, the use of harvesting efforts as the control of develop the strategies for managing the prey-predator system by using the functional response [2].

A functional response is a function of the density of prey. It displays the amount of prey that each predator consumes [3]. In 1995, Holling introduced three types of functional responses. Holling Type I is being used when the system which the prey population is available for the predator. But the Holling Type II is being used when the proportion of the prey consumed declines monotonically with prey density. Holling Type III is being used mostly when the predator encounters with a very low amount of prey due to the unavailability of prey.

The bifurcation analysis can be used in order to demonstrate the dynamical behaviors of the prey-predator system. The harvesting parameter can be considered in carrying out the bifurcation analysis [4]. The Lotka-Volterra prey-predator model is popular since it is one of the simplest model [5].

Generally, one of the main objectives of ecology is to comprehend the interactions between various species, such as prey-predator interaction. Due of its universality and significance, the dynamical relationship between predators and their prey has long been and will remain one of the main topics in both ecology and mathematical ecology [5]. Prey-predator models have received greater attention in recent years from researchers for both management and descriptive purposes. A number of biological resources are being exploited more frequently as a result of the constantly increasing need for food and resources[1].

Kar and Matsuda stated that the quantitative and qualitative understanding of the interaction of different species is crucial for the fisheries [2]. This understanding has been developed by extensive research. No species can exist independently of other species. Each species interacts with many other species since it lives in a community with them. The prey-predator model systems are primarily used to describe the population-level interactions among species that occured when looking for food and a place to live [6].

The Lotka-Volterra Model is the simplest interaction of prey-predator model. There is a few improvements that had been made by Rosenzweig and Marcarther to the Lotka-Volterra model by introducing the growth of the density dependent prey population and the prey consumption by a predator is a non-linear type [7]. The prey-predator model of fishery in which the prey is infected directly by some external toxic substance. The prey-predator model of the fishery has been modified from the Lotka-Volterra prey-predator model [8].

The functional relationship between the consumption rate of a predator and the density of its prey was first described by Holling. The same idea may be used to explain how herbivores choose their environment. For a number of big herbivores, functional responses in habitat selection at the home-range scale have been observed [9].

Holling Type I functional response is suitable for the system where the prey population is available for a predator. As food densities increase, the rate of consumption rises linearly until it reaches a maximum rate. The slope of the line corresponds to the attack rate of the predator. This type of functional response usually found at the invertebrate prey-predator interactions. The functional response of a predator to variations in food density is a crucial aspect of population control. Most experts agree that population dynamics are stabilized by the type I functional response.

Majumdar et. al stated that Holling Type II functional response is a monotonic increasing function and indicates a saturation effect as there are many prey. The rate of prey intake by a predator grows as prey density rises in the Type II functional response, but finally levels out at a peak where the rate of consumption is independent of changes in prey density [10]. Nosrati and Shafiee had studied about the dynamic analysis of the fractional-order singular Holling Type-II prey-predator system which to investigate the impacts of the fractional derivative on the dynamical behaviors of the ecosystem [11].

Because of the shortage of prey, Holling Type III functional response is typically used when the predators contact the prey population with a very low amount. However, when prey is accessible, the response behaves more like Holling Type II functional response. Since the presence of more prey in the environment benefits the predator, it is assumed in the majority of prey-predator models that the functional response is a monotonically increasing function of prey [1].

Few researchers that have investigated the different types of prey-predator model using Holling Type III functional response. Jiang et. al [10] studied fish species, phytoplankton species, and prey-predator models with Holling Type III functional responses. He and Li [3] used the Holling Type III functional response in order to investigate the dynamics of discrete time of prey-predator. Majumdar et. al [1] studied the dynamical behaviors of prey-predator model with Holling Type III functional response and also with non-linear harvesting as the predator population encounter the prey population with a very lower quantity because of unavailability of prey.

The harvesting of the prey or predator population in a prey-predator paradigm is quite interesting from an economic and biological aspect [12]. There are two types of harvesting which are linear and non-linear. In this research, we focus on non-linear predator harvesting. Numerous researchers have examined prey-predator models by employing various methods of harvesting while considering species-specific growth and its interactions. Datta et al. [8] studied about non-linear age-selective prey harvesting with maturation delay, where the target prey for harvesting is above a specific age. This harvesting goes through a Hopf-bifurcation with the respect to the maturation delay. Li and Xiao [13] investigated the bifurcations of a predator and prey system of Holling Type IV and Leslie type predator numerical response. Mortuja et. al [3] analyzed on predator-prey system with the non-linear prey harvesting with square root functional response. They determined that the system with non-linear harvesting is rich in dynamics compared to system without the harvesting and they also observed that prey-predator population will extinct if there is no limitation in harvesting. Majumdar et. al investigated about the complex dynamical behaviour of the prey and predator model with the Holling Type III functional response and the non-linear predator harvesting. The effects of the toxin on a harvested fishery model is studied and the result of the study is the harvesting seems more influential to the system rather than the toxins [13].

In this study, the ordinary differential equation is used to propose prey-predator model with Holling Type III functional response. The stability of the equilibrium points of the prey-predator model is determined by using Routh-Hurwitz criterion. This research aims to study the dynamical behaviors of the prey-predator system and the effect of the predator harvesting to the system.

Model Formulation
The Prey-predator model system is given as

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{k}\right) - \frac{aX^2Y}{b^2 + X^2}, \tag{1}$$

$$\frac{dY}{dt} = \frac{eaX^2Y}{b^2 + X^2} - mY, \tag{2}$$

where $X(t)$ is the prey density while $Y(t)$ is the predator density at time, t. The term $\frac{aX^2Y}{b^2+X^2}$ refers to the functional response of the predator which is the Holling Type III functional response. We apply a non-linear harvesting for the predator population. The model system becomes

$$\frac{dX}{dt} = rX\left(1 - \frac{X}{k}\right) - \frac{aX^2Y}{b^2 + X^2}, \tag{3}$$

$$\frac{dY}{dt} = \frac{eaX^2Y}{b^2 + X^2} - mY - \frac{qEY}{n_1E + n_2Y}. \tag{4}$$

Table 1: The description of the parameters.

Symbol	Definition
X	Prey density
Y	Predator density
t	Time
r	Prey growth rate
k	Prey carrying capacity
a	Maximum predation rate of predator
b	Environmental protection for the prey
e	Conversion efficiency of predator
m	Predator mortality rate
q	Catch ability coefficient
E	Effort applied to harvest individuals
n_1	Constant
n_2	Constant

All the parameters above are assumed to be positive.

Non-dimensionalization
In the harvested prey-predator model, non-dimensionalization is applied to reduce the number of parameters.

$$x = \frac{X}{A}, y = \frac{Y}{B}, \tau = \frac{t}{D}, A = k, B = \frac{rk}{a}, D = \frac{1}{r},$$
$$\beta = \frac{b}{k}, \delta = \frac{m}{r}, \alpha = \frac{ea}{r}, g = \frac{n_1Ea}{n_2rk}, h = \frac{aqE}{n_2r^2k},$$

By using the above non-dimensionalization variable, the dimensional system becomes,

$$\frac{dx}{d\tau} = x(1-x) - \frac{x^2 y}{\beta^2 + x^2},$$
$$\frac{dy}{d\tau} = \frac{\alpha x^2 y}{\beta^2 + x^2} - \delta y - \frac{hy}{g+y}. \qquad (5)$$

Stability Analysis

1 Equilibrium Points

The equilibrium points for prey-predator model can be determined by equating equations (3.5) and (3.6) to zero,

$$x(1-x) - \frac{x^2 y}{\beta^2 + x^2} = 0, \qquad (6)$$

$$\frac{\alpha x^2 y}{\beta^2 + x^2} - \delta y - \frac{hy}{g+y} = 0, \qquad (7)$$

By solving above equations, there are three different types of equilibrium points which are,

i. $P_0(0,0)$: The trivial equilibrium point.
ii. $P_1(1,0)$: The predator free equilibrium point.
iii. $P_2(0.5229, 0.4776)$: The co-existence interior equilibrium point.

2 Jacobian Matrix

To determine the stability of the equilibrium points, the system can be generalized by using Jacobian matrix. The Jacobian matrix can be expressed as

$$J = \begin{bmatrix} 1 - 2x - \frac{2xy}{\beta^2 + x^2} + \frac{2x^3 y}{\left(\beta^2 + x^2\right)^2} & -\frac{x^2}{\beta^2 + x^2} \\ \frac{2\alpha xy}{\beta^2 + x^2} - \frac{2\alpha x^3 y}{\left(\beta^2 + x^2\right)^2} & \frac{\alpha x^2}{\beta^2 + x^2} - \delta - \frac{h}{g+y} + \frac{hy}{\left(g+y\right)^2} \end{bmatrix}. \qquad (8)$$

Stability Analysis at Trivial Equilibrium point

Given that the trivial equilibrium point is (0,0). From equation (8), let the Jacobian matrix as below

$$J(P_0) = \begin{bmatrix} 1 & 0 \\ 0 & -\delta - \frac{h}{g} \end{bmatrix}, \qquad (9)$$

We get the characteristic polynomial as

$$\lambda^2 + \frac{(\delta g - g + h)}{g}\lambda + \frac{\delta g + h}{g} = 0. \qquad (10)$$

The eigenvalues of the matrix are 1 and $-\delta - \frac{h}{g}$. Since one of the eigenvalues is negative and the other is positive. Thus, the trivial equilibrium $P_0(0,0)$ is unstable.

Stability Analysis at Predator Free Equilibrium point

Given that the trivial equilibrium point is (1,0). From equation (8), let the Jacobian matrix as below

$$J(P_1) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{11}$$

where
$a_{11} = -1,$
$a_{12} = -\frac{1}{\beta^2+1},$
$a_{21} = 0,$
$a_{22} = \frac{(-\beta^2\delta + \alpha - \delta)g - \beta^2 h - h}{(\beta^2+1)g},$

and the determinant matrix is

$$\begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0. \tag{12}$$

We get the characteristic polynomial as

$$\lambda^2 + c_1\lambda + c_2 = 0, \tag{13}$$

where
$c_1 = -a_{11} - a_{22},$
$c_2 = a_{11}a_{22} - a_{12}a_{21}.$

Thus, by Routh-Hurwitz criterion, we can prove the stability of equilibrium point when

$$c_1 > 0 \text{ and } c_2 > 0, \tag{14}$$

Since $c_1 = 1.3966\ and\ c_2 = 0.3966$, the predator free equilibrium point is stable.

Stability Analysis at Co-Existence Equilibrium point
Given that the trivial equilibrium point is (0.5229,0.4776). From equation (8), let the Jacobian matrix as below

$$J(P_2) = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \tag{15}$$

where
$b_{11} = 1 - 2x - \frac{2xy}{\beta^2+x^2} + \frac{2x^3y}{(\beta^2+x^2)^2},$
$b_{12} = -\frac{x^2}{\beta^2+x^2},$
$b_{21} = \frac{2\alpha xy}{\beta^2+x^2} - \frac{2\alpha x^3 y}{(\beta^2+x^2)^2},$
$b_{22} = \frac{\alpha x^2}{\beta^2+x^2} - \delta - \frac{h}{g+y} + \frac{hy}{(g+y)^2},$

and the determinant matrix is

$$\begin{vmatrix} b_{11} - \lambda & b_{12} \\ b_{21} & b_{22} - \lambda \end{vmatrix} = 0. \tag{16}$$

We get the characteristic polynomial as

$$\lambda^2 + c_1\lambda + c_2 = 0, \tag{17}$$

where
$c_1 = -b_{11} - b_{22},$
$c_2 = b_{11}b_{22} - b_{12}b_{21}.$

Thus, by Routh-Hurwitz criterion, we can prove the stability of equilibrium point when

$$c_1 > 0 \text{ and } c_2 > 0, \tag{18}$$

Since $c_1 = 0.2080\ and\ c_2 = 0.0638$, the co-existence equilibrium point is stable.

Results and discussion

Bifurcation Analysis

The bifurcation analysis had been performed in order to study the dynamical behaviors of the prey-predator model. The model is analyzed by using numerical software XPPAUT and MATCONT to examine the steady states by using the initial value of $x = 0.5, y = 0.5$ with parameters value of $\alpha = 0.7, \beta = 0.5, \delta = 0.1, h = 0.18$ and $g = 0$.

Figure below shows on how the harvesting will affect the prey and predator system. Figure 5.1 and 5.2 illustrate the steady state diagrams with respect to the harvesting parameter, h. The red solid lines represent the stable steady states while the blue solid lines represent the unstable steady state.

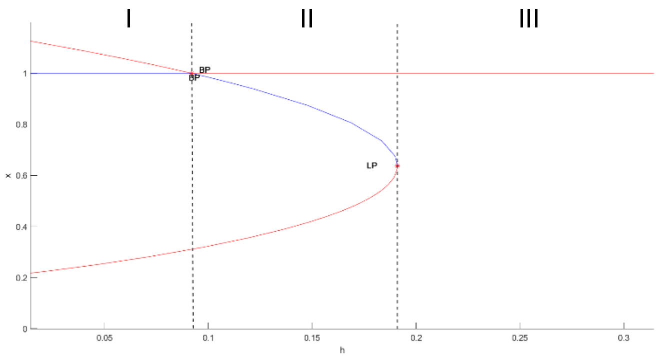

Figure 1 Bifurcation diagram of predator, x with respect to predator harvesting, h

Figure 1 shows transcritical bifurcation (BP) occurs when two branches interchange with each other and their stability is changing. From the diagram, the population of prey decreases as the harvesting parameter increases. This happens because of the harvesting to the predator population. The coexistence equilibrium point is stable but after go through the transcritical bifurcation point ($h = 0.092$) it becomes unstable. The saddle-node bifurcation (LP) occurs only at the coexistence equilibrium point at $h = 0.1911$. The bifurcation happens after there are change of the stability at the bifurcation point.

Region I represent the low predator harvesting rate which the harvesting value can be up until $h = 0.092$. The predator free equilibrium is unstable as the predator popular remains unchanged since the rate of harvesting is low. They can continue to survive by hunting on the prey. While in region II which is the intermediate predator harvesting rate, the predator free equilibrium point becomes stable as the predator harvesting begins to effect on the predator population. The predator population decrease while the prey population increase.

However, the bistability happens between predator-free equilibrium point and co-existence equilibrium point at the intermediate predator harvesting. At region III, only predator free equilibrium point remains stable as the predator harvesting at a high level. The reproduction rate of predator cannot overcome the high predator harvesting rate and lead predator to extinct.

I II III

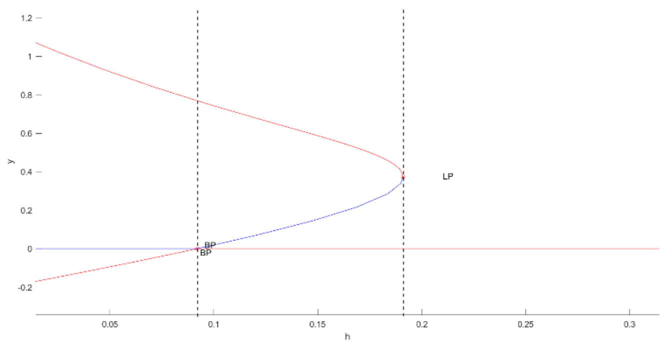

Figure 2 Bifurcation diagram of predator, y with respect to predator harvesting, h

Referring to the Figure 2 there are two bifurcations occur which are saddle node bifurcation and transcritical bifurcation. The transcritical bifurcation (BP) occurs at $h = 0.092$ between the predator free equilibrium point and the co-existence equilibrium point. The interchange of two branches between two equilibrium point at the transcritical bifurcation point occurs while the stability of the equilibrium point is changed. The predator free equilibrium point is changing from unstable to the stable state however the co-existence equilibrium is changing form the stable to the unstable state. The saddle-node bifurcation (LP) occurs when the pair of hyperbolic equilibrium which are the stable and the unstable unite at the bifurcation point which is at $h = 0.1911$. The harvested predator affected the population of the predator as we can see in the figure.

Region I shows the predator free and co-existence equilibrium point at the low predator harvesting rate. The predator free equilibrium is unstable as the predator population is still alive due to the low rate of predator harvesting. Moreover, the prey and predator coexist in this state as the prey population can survive itself and provide sufficient food for the predators. Region II shows the bistability occurs between the predator-free equilibrium point and the co-existence equilibrium at the intermediate predator harvesting rate. The predator can hunt down the prey as the prey population is not drastically decreased. However, at this state the predator population also can be affected by the harvesting rate and resulting to predator death. At region III, the predator free equilibrium point remains stable. This is because of the high predator harvesting and resulting the predator to extinct due to the ability of the predator to reproduce cannot overtake the predator harvest.

Time Series Analysis

Time series plots are obtained in order to analyze the behaviors of the prey-predator population in the system over the time by using MATLAB with same values of parameter as above but with three difference values of h which are h=0.05, 0.18 and 0.3.

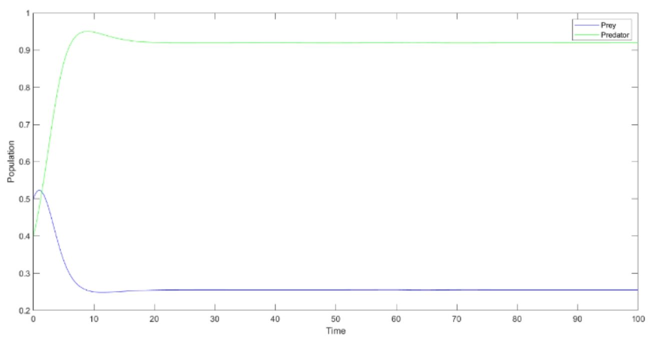

Figure 3 Time plots of Model (5) at $h = 0.05$ with initial value $(x_0, y_0) = (0.5, 0.4)$.

Figure 3 shows that the co-existence equilibrium point stable at h=0.05. The co-existence between prey and predator happen at the low harvesting rate. A low predator harvest rate can let prey and predator populations coexist by enabling prey to grow and provide a sufficient supply of food for predators. This can lead to improved survival rates, increased reproductive success, and population growth among predators. The predator population rises rapidly than the prey population because of the predation pressure. The predation pressure on the prey species grows as predator populations increase and prey population decrease.

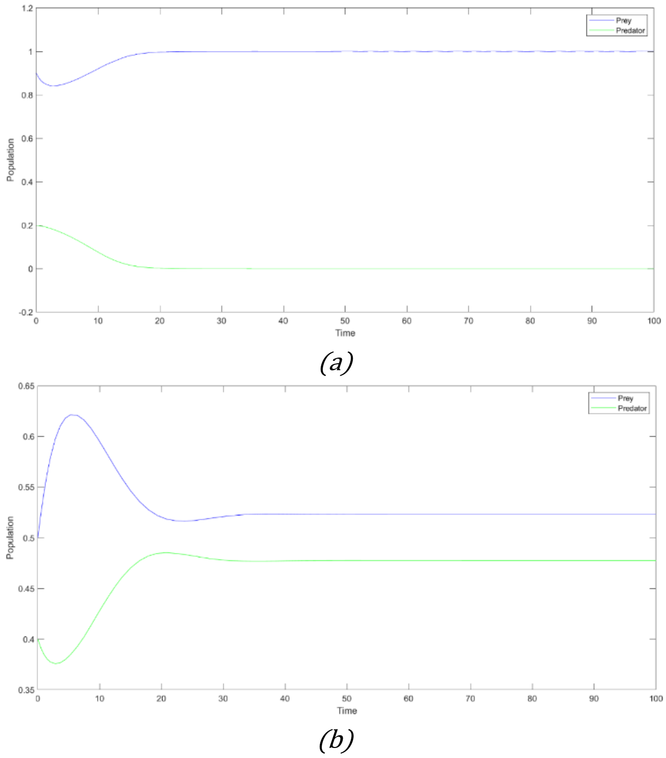

Figure 4 Time plots of Model (5) at (a) $h = 0.18$ with initial value $(x_0, y_0) = (0.9, 0.2)$ and (b) with initial value $(x_0, y_0) = (0.5, 0.4)$ at $h = 0.18$.

Figure 4(a) and 4(b) show that prey-predator model exhibit bistability behaviors at an intermediate predator harvesting rate which is $0.0920 < h < 0.1911$. This means that system has two equilibrium stable states at a particular instant. The result is depending on the initial condition for the prey and predator population.

In the Figure 4(a), it shows that the predator-free equilibrium point is stable at the initial value of $(x_0, y_0) = (0.9, 0.2)$. The predator population is driven to extinction as the predator reproduction rate is insufficient to overcome the predator harvesting rate. However, in the Figure 4(b), the co-existence equilibrium point is stable at the initial value of $(x_0, y_0) = (0.5, 0.4)$. At the intermediate predator harvesting, the predator population can sustain itself while also controlling the prey population, resulting in a stable coexistence equilibrium in which both species can survive.

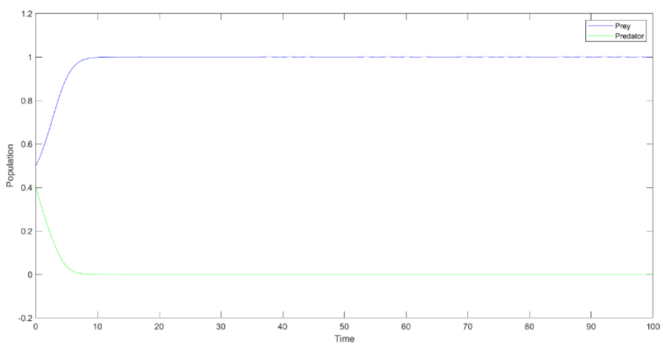

Figure 5 Time plots of Model (5) with initial value $(x_0, y_0) = (0.5, 0.4)$ at $h = 0.3$.

Figure 5 shows the predator free equilibrium stable at high predator harvesting rate, $h = 0.3$. The predator population is decreasing due to the because of predator population is being harvested at a high rate. The reproduction rate of predator is unable to keep up with the higher predator harvesting rate, resulting in the extinction of the predator population. The prey population is increasing as the reduced of predation pressure at high predation harvesting rate. This allows the prey population to grow since there is only few numbers of preys are being consumed by predators.

Conclusion

This study discussed about the dynamical behaviors of prey-predator model with with Holling Type-III functional response and non-linear predator harvesting. The stability analysis and the bifurcation analysis of the prey-predator model is analyzed. The stability properties of the prey-predator model are studied at the trivial equilibrium point, predator free equilibrium point and co-existence equilibrium point by using the Routh-Hurwitz criterion. There are bistability occurs in this system which is between the predator free equilibrium point and the coexistence equilibrium point. Lastly, the bifurcation analysis is carried out by using the XPPAUT and also MATCONT in the MATLAB for a better result of the bifurcation analysis. The stability of the equilibrium point is changing after go through the bifurcation point shows that the harvesting of predator is affecting the system.

At low rate of predator harvesting, the predator remains unchangeable and resulting in the predator free equilibrium unstable while the co-existence equilibrium stable. Due to the low rate of predator harvesting which did not affecting the prey and predator population. The predator can hunt down the prey as usual. At intermediate rate of predator harvesting, the predator free equilibrium point becomes stable after going through certain value of harvesting. This due to the rate of predator harvesting starts to affect the predator population. The predator will begin to extinct. Other than that, the co-existence equilibrium points also stable. This is because of few predator population who still can go hunts the prey as usual. This phenomena where there are two equilibrium points are stable is called bistability.At high rate of predator harvesting, only predator free remains stable. The predator population decrease and may lead to the population extinction. However, the prey population increase due to the reduce of the predation pressure.

Acknowledgement

I would like to express my sincere gratitude to my research supervisor, Dr Ang Tau Keong for the guidance and support. I would also like to thank everyone who has contributed directly or indirectly to this research.

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