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The Laplacian Spectrum of The Composite Order Cayley Graph of The Dihedral Groups of Order Eight

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Abstract
The Laplacian spectrum of a graph is a set of multiplicities of eigenvalues of the graph's Laplacian matrix. The Laplacian matrix of a graph depends on the adjacency matrix and the diagonal matrix of vertex degrees for the graph. In this paper, the main objective is to compute the Laplacian spectrum of the composite order Cayley graph of the dihedral group of order eight. A Cayley graph is a structure comprised of vertices and edges that characterizes the group's presentation and is determined by a certain set of generators in which two vertices are connected by an edge under particular circumstances. In order to compute the Laplacian spectrum of the composite order Cayley graph of the dihedral group of order eight, the structures of the graphs are first examined by constructing the graphs. Then, based on the adjacency of the vertices in graph, the adjacency matrix, diagonal matrix of vertex degrees and Laplacian matrix are determined. With the aid of Maple software, the characteristic polynomial and the eigenvalues of the Laplacian matrix are calculated. As a result, the Laplacian spectrum of the composite order Cayley graph of dihedral group of order eight is determined.
Keywords: Laplacian spectrum; Composite order Cayley graph; Dihedral groups

1. Introduction

Graphs are made up of a collection of dots called *vertices* and lines connecting those dots called *edges*. When two vertices are connected by an edge, it can be said that they are adjacent. A vertex or node is an intersection point of a graph. An edge is a connection between two nodes. A link represents the movement of vertices. The graph is called undirected graph if the edges connect two vertices symmetrically and a directed graph if the edges in the graph are pointing in only one direction. The edges are typically drawn as arrows indicating the direction.

Let Γ be a graph that consists a set of edges $E(\Gamma) = \{e_1, e_2, \ldots, e_n\}$ and a set of vertices, $V(\Gamma) = \{v_1, v_2, \ldots, v_n\}$. Then the adjacency matrix of the graph Γ, $A(\Gamma) = [a_{ij}]$ is the $m \times m$ matrix where for a directed graph, if there is an edge exists between v_i to v_j, then the value of $a_{ij} = 1$, otherwise the value is 0. Besides, for an undirected graph, if there is an edge that exists between v_i to v_j, then the value of $a_{ij} = 1$ and $a_{ji} = 1$, otherwise the value is 0 [1]. Let $D(\Gamma)$ be the diagonal matrix of vertex degree which is the number of graph edges which touch the graph's vertex.

The Laplacian matrix of Γ, $L(\Gamma)$ is defined by $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $A(\Gamma)$ is the adjacency matrix and $D(\Gamma)$ is the diagonal matrix of vertex degrees for the graph Γ [2]. Roots of the characteristic equation of the Laplacian matrix are called the Laplacian eigenvalues or sometimes just eigenvalues of Γ. The Laplacian spectrum, $L - Spec(\Gamma)$ is the set of eigenvalues with multiplicities [3].

Group is a system that consists of a set of elements and a binary operation that can be applied to two elements of the set, which together satisfy the properties such as closure, associativity, identity and inverse property [4].

A dihedral group of order $2n$ denoted by D_{2n} is represented by the following group presentation, with e denoting the identity element:

$$D_{2n} = < a, b \mid a^n = b^2 = e, bab^{-1} = a^{-1} > \text{ for } n \geq 3.$$

In addition, a group is a collection of objects that follow a combination rule. Given any two group elements, the rule produces another group element that is dependent on the two elements chosen. A graph, which is a collection of points called vertices and lines connecting them, can be used to represent the information in a group. In the case of a graph encoding a group, the vertices are group elements and the edges are determined by the combination rule is called as a Cayley graph of the group. A Cayley graph is denoted by $Cay(G, S)$, where G is a finite group and S is a non-empty subset of G. It can be concluded that Cayley graph is dependent on a specific set of generators of the group [5].

Meanwhile, a composite order Cayley graph is a graph that contain a set of vertices of the elements of G and two distinct vertices x and y are connected by an edge whenever $xy^{-1} \in S$, that is $x = sy$ for some $s \in S$, where s is the element of G with composite order. A composite order Cayley graph is denoted as $Cay_c(G, S)$. In 2019, Tolue introduced a new graph which was composite order Cayley graph [6]. Pahil Muhidin [7] has constructed the composite order Cayley graphs of quaternion groups of order 2^n where $n \geq 3$. Extending the idea in [6], in this paper, the composite order Cayley graphs are reconstructed for the dihedral groups of order eight.

2. Literature Review

2.1 Basic Concepts Related to the Dihedral Group

In order to reconstruct the composite order Cayley graph of the dihedral groups, some basic concepts that are related to the dihedral groups are presented in this section.

Definition 2.1 [8] Group

Let G be a set together with a binary operation which is denoted as " • " that combines any two elements a and b to form an element denoted as $a \bullet b$. The following group axioms which known as three requirements are satisfied:

1) Associativity : $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for all a, b and $c \in G$.
2) Identity element : There exists an element e such that for every $a, e \bullet a \equiv a$ and $a \bullet e \equiv a \in G$
3) Inverse element : There exists an element $b \in G$ such that $a \in G$, $a \bullet b \equiv e$ and $b \bullet a \equiv e \in G$, where e is the identity element. The element b is unique for each a where it is called the inverse of a and is commonly denoted as a^{-1}.

Definition 2.2 [9] Dihedral Group

For any $n \geq 3$, the dihedral group D_{2n} of order $2n$ is defined by

$$D_{2n} = < a, b \mid a^n = b^2 = e, bab^{-1} = a^{-1} >.$$

2.2 Graph Related to Groups

A graph is a collection of vertices or nodes and edges, where an edge can be drawn between two vertices to show that these vertices are somehow related to each other. A graph is defined as follows:

Definition 2.3 [10] Graph

A graph $\Gamma = (V, E)$ is a structure consisting of a set of objects called vertices V and a set of objects called edges E. A function f assigns to each edge a subset $\{v_0, v_1\}$, where v_0 and v_1 are vertices, shown as follows:

$$\Gamma = (V, E, f) \text{ or } \Gamma = (V, E).$$

Definition 2.4 [11] Graph of Group

A graph of a group G, Γ_G, is an object made up of a collection of two vertices V and two edges E, that is labelled as $\Gamma_G = (V, E)$ according to geometric group theory. The elements of G are the vertices of Γ_G and the elements of E_G are connecting the two components of $V(G)$.

In 1878, Cayley graphs has been introduced by Cayley to describe the idea of abstract groups [12]. The Cayley graph of a group is defined as follows:

Definition 2.5 [13] Cayley Graph of a Group
Let G be a group and S is a subset of $G\backslash\{e\}$. A Cayley graph of G relative to S is a graph such that the vertex of the graph is the elements of G and a vertex x is connected by a directed edge from x to y whenever $y = xs$, for some $s \in S$. It is denoted by, $Cay(G, S)$.

A Cayley graph is a connected graph if $\boldsymbol{S}$ is a generating set of $\boldsymbol{G}$, particularly every element in $\boldsymbol{G}$ can be written as a product of elements of $\boldsymbol{S}$ [13]. In 2002, Alspach deals with the numeration of various families of Cayley graphs and digraphs [14]. In 2015, the prime order Cayley graph has been introduced and discussed by the Tolue in [15].

Four years later, Tolue extended the research in [6] by introducing the composite order Cayley graph. The only difference of this graph is that the elements of the subset $\boldsymbol{S}$ is of composite order, which is the complement of the primes. The definition of the composite order Cayley graph is given as follows:

Definition 2.6 [6] Composite Order Cayley Graph
Let G be a group and S be a set of composite order elements of G. A composite order Cayley graph is a graph where the vertex of the graph is the elements of G and for two distinct vertices in G, x and y are connected by an edge whenever $xy^{-1} \in S$, that is $x = sy$, for some $s \in S$.. The notation of this graph is $Cay_c(G, S)$.

Finally, in 2019, Shojaee, Tolue and Erfanian [16] explored some approaches on prime order and composite order Cayley graphs. At that time, they computed the topological indices of prime order and composite order Cayley graphs of elementary abelian p-group of prime power order.

2.3 The Laplacian Spectrum of Graph

Babai [17] was the first mathematician who considered the spectrum of Cayley graphs by determining the eigenvalues of Cayley graphs. In the past decades, more consideration has been given to the Laplacian spectrum, since it has been applied to several fields, such as randomized algorithms, combinatorial optimization problems and machine learning [18].

There are few researchers have extensively studied the Laplacian spectral radius of graphs and obtained various bounds for it with respect to varying graph parameters and whenever possible, the corresponding extremal graphs have been characterized [19].

In order to compute the Laplacian spectrum of a graph, it is necessary to obtain the Laplacian matrix of the graph first. The terms adjacency matrix, degree of vertex, diagonal matrix of vertex degrees and Laplacian matrix are defined as follows:

Definition 2.7 [20] Adjacency Matrix
The adjacency matrix of a graph Γ, which is indicated by $A(\Gamma)$, is defined as in the following:

$$A(\Gamma) = \begin{cases} x_{ij} = 1, & \text{if} \quad v_i \sim v_j \\ x_{ij} = 0, & \text{if} \quad otherwise \end{cases}$$

where $v_i \sim v_j$ denotes that the vertex v_i is adjacent to vertex v_j.

Definition 2.8 [21] Degree of Vertex
The degree of a graph's vertex, $deg(v)$ is the number of graph edges which touch v.

Definition 2.9 [22] Diagonal Matrix of Vertex Degrees
The diagonal matrix of vertex degrees,$D(\Gamma)$, is

$$D(\Gamma) = \begin{cases} d(i), & \text{if} \quad i = j \\ 0, & otherwise \end{cases}$$

where $d(i)$ is the degree of vertex i.

Definition 2.10 [23] Laplacian Matrix
The Laplacian matrix of a graph Γ is $L(\Gamma) = D(\Gamma) - A(\Gamma)$, where $A(\Gamma)$ is the adjacency matrix and $D(\Gamma)$ is the diagonal matrix of vertex degrees for the graph Γ.

Definition 2.11 [20] The Characteristic Polynomial of a Matrix
Let A be an $n \times n$ matrix. The characteristic polynomial of A is the function $f(\lambda)$ given by $f(\lambda) = det(\lambda I - A)$

Definition 2.12 [23] Laplacian Spectrum
Laplacian spectrum of a graph Γ denoted by $L - Spec(\Gamma)$, is the set $\{{\lambda_1}^{k_1}, {\lambda_2}^{k_2}, \dots, {\lambda_n}^{k_n}\}$, where $\lambda_1, \lambda_2, \dots, \lambda_n$ are the eigenvalues of the Laplacian matrix of Γ with multiplicities $k_1, k_2, \dots, k_n$ respectively.

3. Results and discussion

The composite order Cayley graph of the dihedral group of order $\mathbf{2_n}$ has been constructed by Tolue in [6]. In this paper, the composite order Cayley graph of the dihedral group of order eight, $\boldsymbol{D_8}$ is reconstructed.

3.1 Reconstruction of The Composite Order Cayley Graph of Dihedral Group of Order Eight

The group presentation of $\boldsymbol{D_8}$ as follows:

$$\boldsymbol{D_8} = < \boldsymbol{a, b} \mid \boldsymbol{a^4 = b^2 = e, bab^{-1} = a^{-1}} >.$$

Hence, the elements of $\boldsymbol{D_8}$ are:

$$\boldsymbol{D_8} = \{\boldsymbol{e, a, a^2, a^3, b, ab, a^2b, a^3b}\}.$$

The order of each element of $\boldsymbol{D_8}$ is calculated to determine which element has composite order. The order of the eight elements in $\boldsymbol{D_8}$ is shown as follows,

$g \in D_8$	e	a	a^2	a^3	b	ab	a^2b	a^3b
$\lvert g \rvert$	1	4	2	4	2	2	2	2

Then, by the definition of composite order Cayley graph given in Definition 2.6, the set of vertices of $Cay_c(D_8, S_1)$ is:

$$V(Cay_c(D_8, S_1)) = D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$

The subset of D_8 with the elements of composite order is $S_1 = \{a, a^3\}$. Let $x, y \in D_8$. Vertex x is connected to vertex y, denoted as $x \sim y$ if $xy^{-1} \in S$ which implies that exists $s \in S$ such that $xy^{-1} = S$ or $x = sy$. Therefore, the set of edges of $Cay_c(D_8, S_1)$ is given as follows:

$$E(Cay_c(D_8, S_1)) = \{\{e, a^3\}, \{e, a\}, \{a, e\}, \{a, a^2\}, \{a^2, a\}, \{a^2, a^3\}, \{a^3, a^2\}, \{a^3, e\},$$
$$\{b, a^3b\}, \{b, ab\}, \{ab, b\}, \{ab, a^2b\}, \{a^2b, ab\}, \{a^2b, a^3b\}, \{a^3b, a^2b\}, \{a^3b, b\}\}.$$

3.2 The Laplacian Matrix of the Composite Order Cayley Graph of the Dihedral Group of Order Eight

Based on Definition 2.10, the Laplacian matrix of $Cay_c(D_8, S_1)$, $L(Cay_c(D_8, S_1))$ is given by

$$L(Cay_c(D_8, S_1)) = D(Cay_c(D_8, S_1)) - A(Cay_c(D_8, S_1)).$$

Thus, firstly, the adjacency matrix of $Cay_c(D_8, S_1)$, is determined based on the set of edges of $Cay_c(D_8, S_1)$ and the structure of the graph. Based on Definition 2.7, the entry for the adjacency matrix, $a_{ij} = 1$ if the pair of elements are connected by an edge and $a_{ij} = 0$ if the elements are not connected by an edge where i denoted as rows and j denoted as columns.

Therefore, the adjacency matrix of $Cay_c(D_8, S_1)$ is

$$A(Cay_c(D_8,S_1)) = \begin{array}{c} \begin{matrix} e & a & a^2 & a^3 & b & ab & a^2b & a^3b \end{matrix} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{matrix} e \\ a \\ a^2 \\ a^3 \\ b \\ ab \\ a^2b \\ a^3b \end{matrix}$$

In addition, the diagonal matrix of vertex degrees of $Cay_c(D_8, S_1)$ is determined as follows:

$$D(Cay_c(D_8,S_1)) = \begin{cases} d(i), & if\ i = j \\ 0, & otherwise \end{cases}$$

where $d(i)$ is the degree of vertex i.

Based on the composite order Cayley graph of D_8, $Cay_c(D_8, S_1)$, the degree of all vertices is 2 since there are two edges connected to each vertex. Therefore, the diagonal matrix of vertex of $Cay_c(D_8, S_1)$ is

$$D(Cay_c(D_8,S_1)) = \begin{array}{c} \begin{matrix} e & a & a^2 & a^3 & b & ab & a^2b & a^3b \end{matrix} \\ \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \end{array} \begin{matrix} e \\ a \\ a^2 \\ a^3 \\ b \\ ab \\ a^2b \\ a^3b \end{matrix}$$

Based on Definition 2.10, the Laplacian matrix of composite order Cayley graph of D_8 is given by

$$L(Cay_c(D_8,S_1)) = D(Cay_c(D_8,S_1)) - A(Cay_c(D_8,S_1)).$$

Therefore, the Laplacian matrix of $Cay_c(D_8, S_1)$ is

$$L(Cay_c(D_8,S_1)) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

3.3 The Laplacian Spectrum of $Cay_c(D_8, S_1)$

In order to compute the Laplacian spectrum of $Cay_c(D_8, S_1)$, the eigenvalue of Laplacian matrix is determined that assists by Maple software. Hence, the characteristic polynomial of $L(Cay_c(D_8, S_1))$ is

$$f(\lambda) = \lambda^8 - 16\lambda^7 + 104\lambda^6 - 352\lambda^5 + 656\lambda^4 - 640\lambda^3 + 256\lambda^2$$

Thus, the eigenvalues are $\lambda = 0$ with multiplicity 2, $\lambda = 4$ with multiplicity 2 and $\lambda = 2$ with multiplicity 4 as well. Based on Definition 2.12, $L - Spec(\Gamma) = \{{\lambda_1}^{k_1}, {\lambda_2}^{k_2}, \dots, {\lambda_n}^{k_n}$, the Laplacian spectrum of $Cay_c(D_8, S_1)$ is $\{0, 0, 2, 2, 2, 2, 4, 4\} = \{0^2, 2^4, 4^2\}$.

Therefore, the Laplacian spectrum $Cay_c(D_8, S_1)$ is determined to be the set of the eigenvalues of 0 and 4 with multiplicities two and the eigenvalues of 2 with multiplicities of four.

Conclusion

sixteen are reconstructed based on the groups' presentation and the definition of the composite order Cayley graph. From the graph, the adjacency matrix, diagonal matrix of vertex degrees and Laplacian matrix is determined. Then, the Laplacian spectrum of the composite order of the dihedral group of order eight is computed. It can be observed that every vertex in the graph has degree two.

Furthermore, the Laplacian spectrum of the composite order of the dihedral group of order eight is determined to be the set of the eigenvalues of 0 and 4 with multiplicities two and the eigenvalues of 2 with multiplicities of four.

References

[1] Abdussakir, A., Elvierayani, R. R., & Nafisah, M. On the spectra of commuting and non-commuting graph on dihedral group. *Jurnal Matematika Murni dan Aplikasi*. 2017. 4(4):176-182.

[2] Cvetković, D., Rowlinson, P., & Simić, S. K. Signless Laplacians of finite graphs. *Linear Algebra and its Applications*. 2007. 423(1):155-171.

[3] Abreu, N. M., Cardoso, D. M., Martins, E. A., Robbiano, M., & San Martı, B. On the Laplacian and signless Laplacian spectrum of a graph with k pairwise co-neighbor vertices. *Linear algebra and its applications*. 2012. 437(9):2308-2316.

[4] Joseph A. Gallian, *Contemporary Abstract Algebra*, 7th ed. 2010.

[5] Magnus, Wilhelm; Karrass, Abraham; Solitar Donald. *Combinatorial Group Theory:Presentations of Groups in Terms of Generators and Relations*. 2004.

[6] Tolue, B. Some graph parameters on the composite order Cayley graph. *Caspian Journal of Mathematical Sciences (CJMS)*. 2019. 8(1):10–17.

[7] Pahil Muhidin, O. *Prime order and composite order Cayley graph of generalized quaternion group and quasi-dihedral group*. Master' Dissertation. Universiti Teknologi Malaysia. 2020.10–17.

[8] Dummit, D. S. and Foote, R. M. *Abstract Algebra*. 3rd ed. John Willey & Sons Inc. 2004.

[9] Derek J. S. Robinson. *A Course in the Theory of Groups*.2017. 978-1-4684-0128-8.

[10] Wilson, R. J. *Introduction to graph theory*. Harlow, United Kingdom:Prentice Hall. 2015.

[11] Diestel, R. *Graph Theory*. 3rd ed. Germany: Springer. 2000.

[12] Caucal, D. Cayley graphs of basic algebraic structures. *Discrete Mathematics & Theoretical Computer Science*, 21(1), 1-19. 2020.

[13] Sabidussi, G. Vertex-transitive graphs. *Monatshefte für Mathematik*, 1964.68(5): 426–438.

[14] Alspach, B., & Mishna, M. Enumeration of Cayley graphs and digraphs. *Discrete mathematics*. 2002. 256(3):527-539.

[15] B. Tolue, *The prime order Cayley graph*, U. P. B Sci. Bull., Series A. 2015. 77,(3): 207 – 218.

[16] Shojaee, I., Tolue, B. and Erfanian, A. Some new approaches on prime and composite order Cayley graphs. *Quasigroup and Related System*. 2019. 27: 147 –156.

[17] Babai, L. *Spectra of Cayley Graphs*. J. Comb. Theor. Series B. 27. 1979. 180 –189.

[18] H. Taheri and G. H. Fath-Tabar, "New upper bound on the largest laplacian eigenvalue of graphs," *Facta Universitatis, Series: Mathematics and Informatics*, vol. 35, pp. 533–540, 2020.

[19] K. L. Patra, B. K. Sahoo, and B. K. Sahoo, "Bounds for the Laplacian spectral radius of graphs,"

Electronic Journal of Graph Theory and Applications. 2017. 5(2): 276–303.

[20] Prasad, L. A survey on energy of graphs. *Annals of Pure and Applied Mathematics*. 2014. 8(2):183-191.

[21] Jones, O. *Spectra of simple graphs.* Whitman College, Walla-Walla. 2013.

[22] Arfken, G. *Mathematical Methods for Physicists, 3rd ed.* Orlando, FL: Academic Press, pp. 181-184 and 217-229, 1985.

[23] Dutta, J., & Nath, R. K. Spectrum of commuting graphs of some classes of finites groups. 2016:1604.07133.