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Barabási-Albert Model of Graph Theory in Social Media Network

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Abstract
The Barabási-Albert (BA) model, a key discovery in graph theory, offers insights into the formation and structure of complex networks. In the context of social media platforms like Facebook, understanding and identifying influential individuals is crucial for maximizing user experiences and enhance understanding of Graph Theory. The BA model and eigenvector centrality analysis are used to the social media platform Facebook in this study. The BA model replicates the formation of networks via preferred attachment, where new nodes tend to link to existing nodes with higher degrees By integrating the BA model with eigenvector and closeness centrality analysis, construct the cyclic and acyclic for interconnections among people, groups and organization. This research focuses on one particular social media network. On the many ways in which graph theory has been used to social media and how it is represented with respect to strong and weak ties. By creating the network graph using the BA model, and computing eigenvector centrality scores for each user. The eigenvector centrality scores are reflection of an individual user's significance within the network. These scores take into account not only the centrality of the user's cyclic or acyclic connections but also the centrality of the user's neighbour.
Keywords: Graph Theory, Barabási-Albert (BA) Model, Cyclic and Acyclic Graph, Eigenvector and Closeness Centrality, Adjacency Matrices.

Introduction

Social media networks, like Facebook, Twitter, and Instagram, allow users to connect with friends, family, coworkers, and consumers. Graph theory helps analyze both structured and unstructured content, providing insights into social interactions and relationships. A social network is a collection of vertices and lines representing social connections, with graph theory being the most common and adaptable description.A graph is a structure with vertices and edges connected by connections.Social network analysis uses graph theory to observe social structures, with nodes or vertices representing users and shared interests. Graphs can be directed or undirected, with a straight line representing a cyclic graph.

Social network modeling involves developing algorithms to generate graphs with essential properties similar to natural networks. or analyzing user behavior on social media platforms like Facebook, Twitter, and Instagram. Users form connections, share photos, and message friends, and each network uses a different method. To model the social network, a graph represents each action on the platform as nodes and edges. The Facebook network architecture includes users, posts, and groups. Interpersonal links, which are crucial for social network interactions, are divided into strong, weak, and absent connections.It involves identifying connections and identifying the most connected vertices. Using the Barabási-Albert model in Python, a comprehensive list of connections is essential for a complete description of a network.

Graph Theory

Graph theory is the study of graphs, which represent pairwise relationships between objects. It involves vertices and edges, with symmetric and asymmetric edges connecting vertices. Graphs are used in

real-world network systems, such as the internet, phone networks, and social media. A basic graph G consists of edges and vertices, with edges joining vertices.

Definition 1 (Kimball Martin, 2014) .A directed acyclic graph (DAG) is a weighted digraph *D* which admits a node ordering so that the adjacency matrix of D is lower triangular.

Definition 2 (Skiena, 1990) **:** A graph with at least one graph cycle is said to be cyclic. A graph is said to as acyclic if it is not cyclic. A unicyclic graph is a cyclic graph with precisely one (undirected, simple) cycle. Trees are not cyclic graphs.If all of a cyclic graph's cycles are of even.

Definition 3 (Sauras-Altuzarr W 2007) A basic labelled graph's adjacency matrix, also known as the connection matrix, is a matrix with rows and columns labelled by graph vertices and a 1 or 0 at position (v_i, v_j) depending on whether or not ($v_i, v_j)$ are adjacent. The adjacency matrix needs to contain 0s on the diagonal for a straightforward graph without self-loops. The adjacency matrix for an undirected graph is symmetric.

Barabási-Albert Model

Social networks, the Internet, and biological networks are only a few examples of complex networks that the Barabási-Albert model can describe mathematically. Hungarian physicist Albert-László Barabási and his coworker Réka Albert suggested it in the late 1990s (A.-L. Barabási, 2012).The Barabási-Albert model postulates that a network will expand gradually over time as additional nodes are added to it. When a new node is added to a network, it is more likely to be linked to other nodes of a higher degree.

Cyclic and Acyclic Graph

Cyclic graphs have at least one graph cycle, while undirected graphs contain only one cycle. An undirected cyclic graph is one where different pairs of "vertices" are considered adjacent. It has minimal structure and is not particularly interesting in itself. Cyclic graphs are specific examples of undirected graphs, which are collections of connected objects with bidirectional edges. A directed acyclic graph (DAG) represents a set of activities, with circles connected by lines to indicate flow from one action to the next.Each edge is directed, representing a specific directional flow. Topological ordering exists in directed acyclic graphs, with the starting node being higher than the ending node in value. If a directed path connects every node in a DAG, the ordering of the nodes is unique. In an acyclic graph, there are no loops or cycles, making it impossible to return to the first vertex in the network.

Adjacency List and Wieght of Edge

Based on the graph from the Barabási-Albert (BA) Model in graph theory, which is both cyclic and acyclic, this part will make an adjacency list of each linked node to figure out closeness centrality. In these two cases, you can quickly find the link between two points in the graph by storing the information. The strength of a relationship between two nodes in a graph can be quantified by assigning different values to the weights of the nodes involved. Higher-weighted nodes would represent more significant social connections, whereas lower-weighted nodes would imply less significant or infrequent interactions. Now, let's talk about weight of nodes in both acyclic and cyclic graphs with n = 9 nodes and m = 2 edges by using the random number from 0 until 10.

Table 1 data shows that a high edge weight between two nodes indicates a robust connection between them. The table shows that for edge parameter m = 1, the bond strength is highest for the adjacency list *(3, 1)*, with a value of 0.8709, and lowest for the adjacency list *(7, 2)*, with a value of 0.1844. According to the description, the number of edges utilized to calculate the example's value ranges from zero to eight.

Table 1 : Cyclic Graph

Number of Edges,m	Adjacency List,A_{ij}	Weight of Edges
2	{(0,1),(0,4),(0,5),(0,7)}	{ 0.4775, 0.2050,0.9566,0.1441}
	{(1,0),(1,3),(1,4),(1,5),(1,6),(1,7)}	{0.7396,0.1012,0.31290.7846,0.6573,0.2328}
	{(2,2),(2,3),(2,6)}	{0.1504,0.6783,0.6168}
	{(3,1),(3,2),(3,4),(3,8)}	{0.9798,0.8799,0.1125, 0.2503)
	{(4,0),(4,1),(4,5),(4,6),(4,7),(4,8)}	{0.5904,0.7180,0.7180,0.7748,0.3171,0.0221}
	{(5,0),(5,1),(5,3),(5,4),(5,5),(5,6),(5,8)}	{0.2192,0.3197,0.5540,0.8119,0.2890,0.1494, 0.4700}
	{(6,1),(6,2),(6,4),(6,5),(6,8)}	{0.2192,0.3197,0.5540,0.8119,0.2890,0.1494, 0.4700}
	{(7,1),(7,2),(7,4)}	{ 0.5593,0.4641,0.9515}
	{(8,3),(8,4),(8,5),(8,6)}	{ 0.3951,0.3972,0.5921,0.8929}

Table 2 data shows that a high edge weight between two nodes indicates a robust connection between them. The table shows that for edge parameter m = 1, the bond strength is highest for the adjacency list (1,3),(1,4),(1,6) with a value of 0.3976,0.1287, 0.7572, and lowest for the adjacency list (3*, 5*), with a value of 0.0742. According to the description, the number of edges utilized to calculate the example's value ranges from zero to eight.

Table 2 Acyclic Graph

Number of Edges,m	Adjacency List,A_{ij}	Weight of Edges
2	{(0,4),(0,7)}	{0.6742,0.8206}
	{(1,3),(1,4),(1,6)}	{0.3976,0.1287, 0.7572}
	{(2,3),(2,5),(2,6),(2,7)}	{0.9051,0.8138,0.2440,0.0742}
	{(3,5)}	{0.0742}
	{(4,6),4,7)}	{0.3061,0.9758}
	{(5,7)}	{0.9196}
	{(6,8)}	{ 0.5111}
	{}	{0}
	{}	{0}

Construction of Acyclic and Cylic Graph of Graph Theory in Facebook

Graphical models are increasingly used to provide a detailed description of a social network, allowing for better understanding of user relationships and access to resources. Facebook operates in two communication modes: single sender, multi-recipient, and group chat. The first thing users see on Facebook are contacts of others using the app. To obtain the adjacency matrix and centrality measures, analyzing every model of the Barabási-Albert (BA) Model in acyclic and cyclic graphs is necessary. A cyclic graph is a popular social media network structure with interconnections that form recursive cycles, indicating mutually reinforcing connections. In a scenario with three users (*A, B, and C*), a recursive network is created with nodes *(n = 9*) and edges *(m = 2)*.

Cyclic Graph in Facebook with *m* = 2

Consider a group of 9 people who all use Facebook as a nodes of network. As can be seen in Figure 1 , the graph has a total of 9 vertices. A number of edges *,m=1* as depicted in Cyclic Graph represents a dormant Facebook community in which all members are currently available for communication with two path. If there are two edges linking the vertices of a group, we say that the group is idle in graph theory.

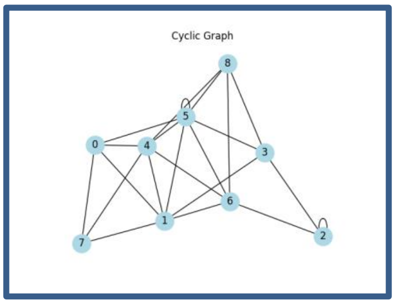

Figure 1 Cyclic Graph Model of *m* = 2

By using the parameters that are stated in n = 9 and m = 1, it is possible to identify each node that is connected 2 times to each node with other nodes. Because of this, the adjacency matrix will be constructed as shown below:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$$

It is possible to verify the eigenvector centrality for m = 2 by utilizing the matrices described above as variables in place of the adjacecny matrices described earlier.

Acyclic Graph in Facebook with *m* = 2

According to the acyclic graph in Figure 2 with the given parameters, 2 is the sender and the other is the receiver. This graph only has one direction and is capable of displaying a degree of zero.

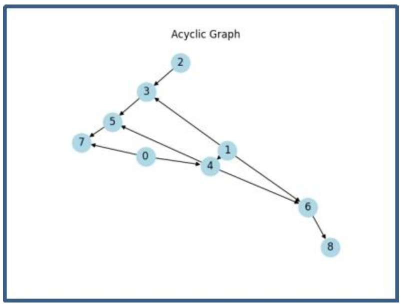

Figure 2: Acyclic with *m* = 2

The graph can be used to prove the sketch for the adjacency matrices.Each connection of the nodes will display as 1 and 0.

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Eigenvector Centrality in Cyclic Graph

The formula for calculating eigenvector centrality is typically expressed as:

$$C(x) = \frac{1}{\lambda} \sum (w * C(y),$$

Where:

- C(x) is the centrality score of node *x*
- λ is the dominant eigenvalue of the adjacency matrix
- $\sum(w * C(y))$ is the sum of the centrality scores of the neighboring nodes, weighted by their connection strengths (e.g., edge weights or binary connections)
- The eigenvector centrality algorithm iteratively computes the centrality scores until convergence is achieved, usually by using iterative methods like the power iteration method.

Eigenvector Centrality in Cyclic Graph wih *m* =2

Using the *m* = 2 cyclic graph as evidence for which of each node's high scores indicates whether the node is a transmitter or recipient. Let assume the initial guess as seen below.

$$U = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Based on the derivation on Eigenvector Centrality ,

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

General formula of Eigenvector Centrality:

$$A_{ij} = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} U_i = \begin{pmatrix} X_1 \\ X_2 \\ \bullet \\ X_t \\ v \end{pmatrix}$$

By use the expected normalization value and the adjacency matrix. Only the last three iterations leading up to determining the value of converges and eigenvector centrality are shown in the graphic below, despite the fact that this is a 8-iteration process when m = 2. With m = 2, the value of node 6 is 0.1721, making it the sending node.

0	1	0	0	1	1	0	1	0	0.1040	0.5267	5.066	0.1040
1	0	0	1	1	1	1	1	0	0.1387	0.7051		0.1392
0	0	1	1	0	0	1	0	0	0.0526	0.2656		0.0524
0	1	1	0	0	1	0	0	1	0.0926	0.4668		0.0921
1	1	0	0	0	1	1	1	1	0.1405	0.7143		0.1410
1	1	0	1	1	1	1	0	1	0.1719	0.8718		0.1721
0	1	1	0	1	1	0	0	1	0.1204	0.6073		0.1199
1	1	0	0	1	0	0	0	0	0.0756	0.3833		0.0756
0	0	0	1	1	1	1	0	0	0.1037	0.5255		0.1037
0	1	0	0	1	1	0	1	0	0.1040	0.5279	5.068	0.1042
1	0	0	1	1	1	1	1	0	0.1392	0.7047		0.1390
0	0	1	1	0	0	1	0	0	0.0524	0.2644		0.0522
0	1	1	0	0	1	0	0	1	0.0921	0.4674		0.0922
1	1	0	0	0	1	1	1	1	0.1410	0.7145		0.1410
1	1	0	1	1	1	1	0	1	0.1721	0.8719		0.1720
0	1	1	0	1	1	0	0	1	0.1199	0.6084		0.1200
1	1	0	0	1	0	0	0	0	0.0756	0.3841		0.0758
0	0	0	1	1	1	1	0	0	0.1037	0.5251		0.1036
0	1	0	0	1	1	0	1	0	0.1042	0.5278	5.068	0.1041
1	0	0	1	1	1	1	1	0	0.1390	0.7052		0.1391
0	0	1	1	0	0	1	0	0	0.0522	0.2644		0.0522
0	1	1	0	0	1	0	0	1	0.0922	0.4668		0.0921
1	1	0	0	0	1	1	1	1	0.1410	0.7146		0.1410
1	1	0	1	1	1	1	0	1	0.1720	0.8720		0.1721
0	1	1	0	1	1	0	0	1	0.1200	0.6078		0.1199
1	1	0	0	1	0	0	0	0	0.0758	0.3842		0.0758
0	0	0	1	1	1	1	0	0	0.1036	0.5252		0.1036

Closeness Centrality in Acyclic Graph

Based on Acyclic Graph, the closeness centrality of a node can be defined as the reciprocal of the sum of its shortest path distances to all other nodes in the network. Mathematically, for an undirected, connected network, the closeness centrality of a node can be calculated as follows:

$$C(v) = \frac{1}{\sum(d(u,v)}$$

Where:

- *C(v)* is the closeness centrality of node ,
- *d(u, v)* is the shortest path distance between node
- *Σ* represents the sum over all nodes.

By calculating the closeness centrality for each node in a network, researchers can identify nodes that are central in terms of information flow and communication efficiency.

Closeness Centrality in Acyclic Graph with *m* = 2

Whether a network is cyclic graph, node centrality is normally calculated using eigenvector centrality. The concept of eigenvector centrality does not hold water in the context of acyclic networks like directed acyclic graphs (DAGs). In acyclic networks, alternative centrality measures are often used to assess node importance.Table 3 show closeness centrality in acyclic graph with *m* = 2

Table 3 :Acyclic Centrality

Number of Edges,*m*	Adjacency List,A_{ij}	Weight of Edges
2	{(0,4),(0,7)}	0.111
	{(1,3),(1,4),(1,6)}	0.1
	{(2,3),(2,5),(2,6),(2,7)}	0.16667
	{(3,5)}	0.3333
	{(4,6),4,7)}	0.16667
	{(5,7)}	1.0
	{(6,8)}	1.0
	{}	0
	{}	0

Conclusion

This research aims to construct cyclic and acyclic graphs for interconnections among people, groups, and organizations using the Barabási-Albert (BA) Model in Graph Theory. The preferred attachment mechanism is responsible for the development and connection patterns of social media networks, influencing the formation of new links between popular nodes. The Barabási-Albert model can be used to simulate social media networks, providing valuable insights into the dynamics and factors contributing to the rise of key nodes. The study aims to better recognize communities, identify opinion leaders, and develop successful tactics for information transmission.

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