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The Influence of Fear Effect to The Lotka–Volterra Predator–Prey System

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Abstract

In the presence of predator population, the prey population can change its behavior significantly. Fear for predator population increases the prey population's likelihood of survival and can significantly reduce prey population's reproduction. The purpose of this study is to investigate the fear effect to the Lotka-Volterra predator-prey system by obtain the equilibrium points of the system and analyse the stability of the equilibrium points obtained. The Jacobian matrix is use to find the eigenvalues and from the eigenvalues obtained, the conditions of the equilibrium points stability can be determined. From the analysis of the system, four equilibrium points are obtained which are the trivial point, only predator exists, only prey exists, and a coexistence of predator and prey. The points are stable when only the predator exists, and when the equilibrium points coexists. The numerical simulation are carried out to show the stable equilibrium points. The analysis also shows that as the fear effect increases the prey species may approach zero which means that the prey species is driven to extinction.
Keywords: Lotka-Volterra; predator-prey; fear effect

Introduction

The interaction of predators and prey is a critical topic in theoretical ecology. Many scientists have studied evolutionary biology in recent decades, and mathematical models have played an important role in better understanding of these complex scenarios. Thomas Malthus (18th century philosopher and economist) appears to have developed mathematical models for representing the dynamics of predator-prey interactions in the early nineteenth century. The study of predator and prey models by the ecologist Lotka and the mathematician Voltra in the first half of the 20th century can be traced back to the development of mathematical ecology (Sarkar & Khajanchi, 2020). The mathematical model was eventually improved by incorporating the prey species' linear per-capita birth rate, resulting in the well-known logistic growth model. The well-known Lotka Volterra model was later improved by including a logistic growth term for the prey, and a variety of population dependent response functions were established, allowing for realistic representation of predator and prey population interactions. In general, there are four basic of interactions between the two groups: competing type, parasitic type, predator and prey type (Sarkar & Khajanchi, 2020). Among the four types, the predator-prey model has gotten a lot of attention as well as research interest (Su & Zhang, 2022). The predator-prey interaction is one of the most fundamental biological population connections, and it is a prominent research topic in ecology and biomathematics. A simple model describing predator-prey model can be written as follows:

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta xy, \\ \frac{dy}{dt} &= \delta xy - \gamma y, \end{aligned} \tag{1}$$

where the variable x is the population density of prey, the variable y is the population density of predator, $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent the growth rates of the two populations, t represents time, the prey's parameters, $\alpha$and β describe the maximum prey per capita growth rate, and the effect of the presence of predators on the prey growth rate and the predator's parameters, δ and γ respectively describe the predator's per capita death rate, and the effect of the presence of prey on the predator's growth rate.

Predators have a dual effect on the ecological system's structure (Su & Zhang, 2022). The presence of the predator can change the behavior of the prey such it can have an impact on the prey population than in direct predation (Huat et al., 2014). Predators physically attack and kill their prey, reducing the amount of prey. Also, the presence of predators alone can put prey individuals under psychological pressure, which leads to many changes in the behavior of prey species. This is a common anti-predator response known as "fear effect" which gradually slows the prey population growth (Su & Zhang, 2022) Animals react to perceived predation dangers by exhibiting a variety of anti-predator behaviors. For example, when prey assesses the risk of predation, they may leave their original high-risk habitats and change to low-risk habitats, which may result in energy expenditure, particularly if living circumstances in the low-risk habitats are poor. In a similar way fearful prey will reduce their outings for food, and the reduction in food may reduce birth and survival rates (Sarkar & Khajanchi, 2020). The terrified prey species naturally forages less, resulting a decreases in growth rate and embraces vary survival mechanism such as hunger (Sarkar & Khajanchi, 2020]. A higher level of acute predation risk might cause prey species to leave habitats or foraging places for a short period of time, returning only when the acute risk has passed and the prey species is safe (Gresswell, 2011).

Mathematical Formulation

To investigate the dynamic behaviors of the following Lotka–Volterra predator–prey system incorporating fear effect of the prey and the predator having other food resource we have,

$$\frac{dx}{dt} = \frac{rx}{1+ky} - dx - ax^2 - pxy, \qquad (2)$$

$$\frac{dy}{dt} = cpxy + my - \beta y^2, \qquad (3)$$

where, x denotes the density of prey species, y denotes the density of predator species, r denotes the birth rate of the prey species, d denotes the death rate of the prey species, a denotes the intraspecific competition of the prey species, m denotes the intrinsic grow rate of the predator species, p denotes the benefits of predation to the predator, c denotes the birth rate of predator due to predation, k denotes the level of fear, which is due to anti-predator behaviours of the prey, β denotes the competition between predator species.

Stability Analysis

By solving $\frac{dx}{dt} = \frac{dy}{dt} = 0$, the equations (2) and (3), we will have four equilibrium points which are:

Equilibrium point 1, $E_1(0, 0)$

Equilibrium point 2, $E_2\ (\frac{r-d}{a}, 0)$

Equilibrium point 3, $E_3\ \left(0, \frac{m}{\beta}\right)$

Equilibrium point 4, $E_4\ (x^*, y^*)$

General Jacobian Matrix,

$$J = \begin{bmatrix} \dfrac{r}{1+ky} - d - 2ax - py & -\dfrac{rxk}{(1+ky)^2} - px \\ cpy & cpx + m - 2\beta y \end{bmatrix}$$

Now, we need to find eigenvalues of $\det|J - \lambda I| = 0$ for each of the equilibrium points to know its stability conditions.

Equilibrium point 1: (0, 0)

$$J_1 = \begin{bmatrix} r-d & 0 \\ 0 & m \end{bmatrix}$$

After solving $\det|J_1 - \lambda| = 0$ we will get the eigenvalues at point (0,0) are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} m \\ r-d \end{bmatrix}$$

Since the first $\lambda_1 = m$, we know that equilibirum point $(0, 0)$ is unstable since $\lambda > 0$.

Equilibrium point 2: $(\frac{r-d}{a}, 0)$

$$J_2 = \begin{bmatrix} d-r & \dfrac{dk + pd - r^2k - pr}{a} \\ 0 & \dfrac{cpr - cpd}{a} + m \end{bmatrix}$$

After solving $\det|J_1 - \lambda| = 0$ we will get the eigenvalues at point $(\frac{r-d}{a}, 0)$ are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{pmatrix} d-r \\ \dfrac{cpr + am - cdp}{a} \end{pmatrix}$$

$\lambda < 0$, if $\lambda_1 = d < r$.

For $\lambda_2 < 0$,

$$cpr + am - cdp < 0$$

$$am < cdp - cpr$$

$$am < cp(d - r)$$

From the condition for $\lambda_1 < 0, d - r < 0$, which resulted in $am <$ negative values.

But $a > 0, m > 0$, therefore $(\frac{r-d}{a}, 0)$ is unstable.

Equilibrium point 3: $(\mathbf{0}, \frac{\boldsymbol{m}}{\boldsymbol{\beta}})$

$$J_3 = \begin{bmatrix} \frac{r\beta}{\beta + km} - d - \frac{pm}{\beta} & 0 \\ \frac{cpm}{\beta} & -m \end{bmatrix}$$

After solving $\det|J_1 - \lambda| = 0$ we will get the eigenvalues at point $(0, \frac{m}{\beta})$ are

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} -m \\ \frac{\beta^2 r - \beta dkm - km^2 p - \beta^2 d - \beta mp}{\beta^2 + km\beta} \end{bmatrix}$$

$\lambda_1 = -m < 0$,.

$\lambda_2 = \frac{\beta^2 r - \beta dkm - km^2 p - \beta^2 d - \beta mp}{\beta^2 + km\beta} < 0$ if $r < \frac{\beta dkm + km^2 p + \beta^2 d + \beta mp}{\beta^2}$.

Thus, $\left(0, \frac{m}{\beta}\right)$ is stable if $r < \frac{\beta dkm + km^2 p + \beta^2 d + \beta mp}{\beta^2}$.

Equilibrium point 4: $(\boldsymbol{x}^*, \boldsymbol{y}^*)$

From equation 4.1,

$$\frac{dx}{dt} = \frac{rx}{1 + ky} - dx - ax^2 - pxy = 0$$

We will get x values are,

$$x = 0 \; and \; \frac{r}{1 + ky} - d - ax - py = 0$$

Let $x = x^*$ and $y = y^*$

$$x^* = \frac{1}{a}\left(\frac{r}{1 + ky^*} - d - py^*\right) \qquad (4)$$

Where y^*satisfies

$cp\left[\frac{1}{a}\left(\frac{r}{1+ky^*} - d - py^*\right)\right] + my^* - \beta y^{*2} = 0$ from equation (4).

Noting that (x^*, y^*) satisfies the equation

$$\frac{r}{1+ky^*} - d - ax^* - py^* = 0 \tag{5}$$

$$cpx^* + m - \beta y^* = 0 \tag{6}$$

Then the Jacobian matrix of the equation (2) and (3) at the coexistence equilibrium point $E_4\ (x^*, y^*)$ is

$$J_4 = \begin{bmatrix} \frac{r}{1+ky^*} - d - 2ax^* - py^* & -\frac{rx^*k}{(1+ky^*)^2} - px^* \\ cpy^* & cpx^* + m - 2\beta y^* \end{bmatrix}$$

By rearranging equation (4.3) and (4.4),

$$\frac{r}{1+ky^*} = d + ax^* + py^*$$

$$cpx^* = \beta y^* - m$$

The Jacobian matrix will be,

$$J_4 = \begin{bmatrix} -ax^* & -\frac{rx^*k}{(1+ky^*)^2} - px^* \\ cpy^* & -\beta y^* \end{bmatrix}$$

Then we have

$$Det(J_4 - \lambda I) = 0$$

$$\begin{vmatrix} -ax^* - \lambda & -\frac{rx^*k}{(1+ky^*)^2} - px^* \\ cpy^* & -\beta - \lambda \end{vmatrix} = 0$$

$$(ax^* + \lambda)(\beta y^* + \lambda) - (cpy^*)(-\frac{rx^*k}{(1+ky^*)^2} - px^*) = 0, \tag{7}$$

$$\lambda^2 + (ax^* + dy^*)\lambda - a\beta x^*y^* - cpy^*\left(-\frac{rx^*k}{(1+ky^*)^2} - px^*\right) = 0, \tag{8}$$

From equation (4.6) we know that the,

$$Trace, \tau = -(ax^* + \beta y^*) < 0$$

$$determinant, \Delta = -a\beta x^*y^* - cpy^*\left(-\frac{rx^*k}{(1+ky^*)^2} - px^*\right)$$

$$discriminant, \tau^2 - 4\Delta = -(a^2x^* + 2adx^*y^* + \beta^*y^*) - 4\left[-a\beta x^*y^* - cpy^*\left(-\frac{rx^*k}{(1+ky^*)^2} - px^*\right)\right] < 0$$

Since its trace,τ and discriminant, $\tau^2 - 4\Delta$ are less than zero, we can conclude that the equilibrium point, E_4 is asymptotically stable if $\Delta > 0$, and $\tau < 0$, then the linear system is asymptotically stable.

Numerical Simulation

In this section, some numerical solution are shown corresponding to the system in (**2**)

$$\frac{dx}{dt} = \frac{rx}{1+ky} - dx - ax^2 - pxy,$$

$$\frac{dy}{dt} = cpxy + my - \beta y^2,$$

using $r = 2, d = k = a = m = \beta = p = 1, c = 0.5$, then we will get the equation

$$\frac{dx}{dt} = \frac{2x}{1+y} - x - x^2 - xy,$$

$$\frac{dy}{dt} = \frac{1}{2}xy + y - y^2,$$

To find the numerical solution, we only need to use the condition for the stable equilibrium points which is E_3 and E_4.

For E_3, the condition for this point to be stable is $r < \frac{\beta dkm+km^2p+\beta^2 d+\beta mp}{\beta^2}$, by letting the variable, we will get that

$$r = 2 < 4$$

By using Matlab software the simulation are

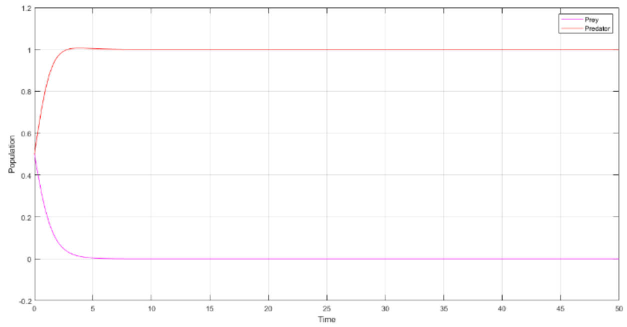

Figure 1

Graph Population vs Time when $\boldsymbol{E_3}$ is stable.

From Figure 1, for prey species, from the initial point the population decreases while the prey population increases to achieve an equilibrium point E_3 as time increases.

For E_4, we let $r = 5, d = k = a = m = \beta = p = 1, c = 0.5$, and the simulation is

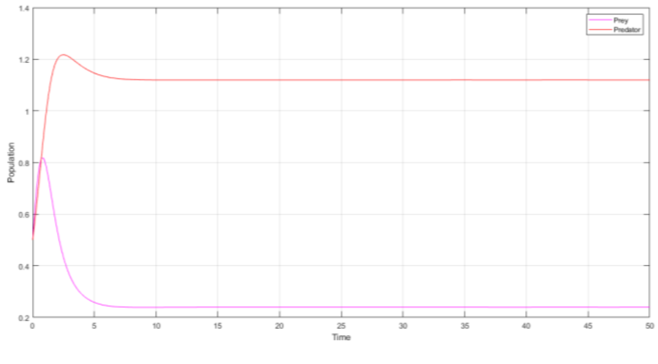

Figure 2

Graph Population vs Time when $\boldsymbol{E_4}$ is stable.

The influence of fear effect numerical simulation

By using E_4 we will analyse the influence of fear effect by variety the value of fear effect k,For the first simulation of fear factor effect, we increase the value of *k* from k=0.5 to *k=2* while the parameter values for other variables are $= 5, d = a = m = \beta = p = 1, c = 0.5$ and the simulation

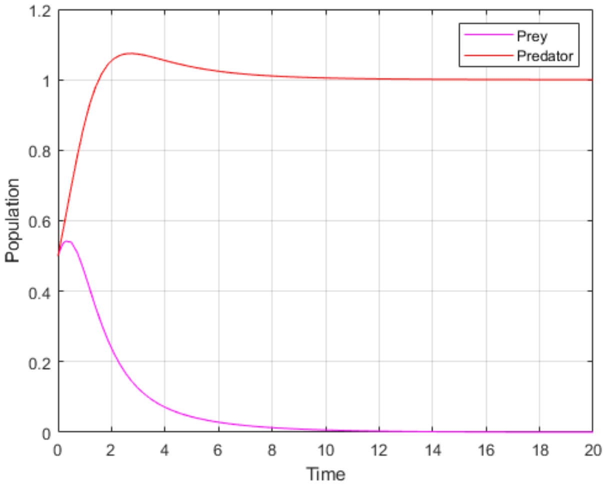

Figure 3

Population vs Time when $\boldsymbol{k} = \mathbf{2}$.

For the second simulation of fear effect influenced we decreases the value of *k* to *k=0.5* and the simulation is

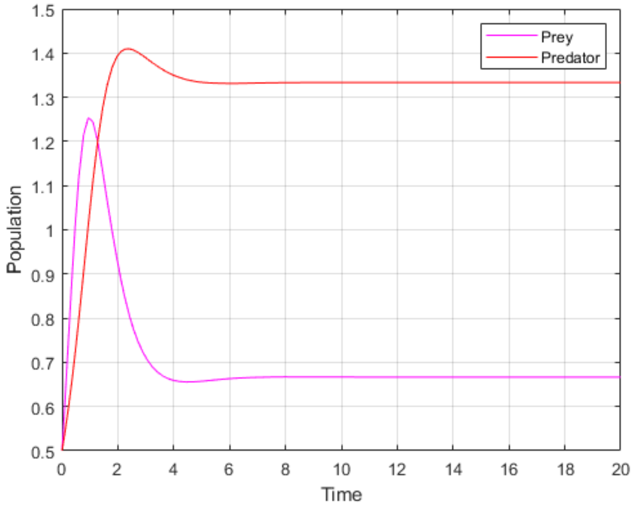

Figure 4

Populations vs Time when $\boldsymbol{k} = \mathbf{0.5}$.

By comparing Figure 3 and Figure 4, we can see that when the value of fear effect, *k* increases the population of prey species decreases, otherwise when the value of fear effect, *k* decreases the population of prey species increases.

Conclusion

From the system in equation (4.1) and (4.2), we get four equilibrium points which are $E_1(0,0), E_2\left(\frac{r-d}{a},0\right), E_3\left(\mathbf{0},\frac{\boldsymbol{m}}{\boldsymbol{\beta}}\right)$ and $E_4(x^*,y^*)$. From these four equilibrium points, E_1 and E_2 are always unstable while, E_3 and E_4 are stable under some condition(s). As for the influenced of fear effect, as the as the fear effect increases the prey species decreases. The findings of this research demonstrate that the presence of fear can significantly alter the dynamics of the predator-prey system. Fear-induced changes in the behavior and movement patterns of both predators and prey can lead to shifts in population sizes, distribution, and spatial interactions. The fear effect introduces an important dimension to predator-prey interactions that was often overlooked. It highlights how the presence and behavior of predators can influence prey populations beyond direct predation. Furthermore, the analysis of the stability of the predator-prey relationship under fear effects has revealed that fear can have both stabilizing and destabilizing effects, depending on the specific context and parameter values.

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