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Solving Mixed Convection Boundary Layer Flow of Viscoelastic Hybrid Nanofluids Past Over a Sphere Using MATLAB with BVP4C Solver

Maisarah Zainal Abidin, Wan Rukaida Wan Abdullah*
Department of Mathematics, Faculty of Science, UTM, Skudai, Johor Bahru, Malaysia
*Corresponding author: wrukaida@utm.my

Abstract
The purpose of this study is to investigate the thermal conductivity of hybrid nanofluid in mixed convection boundary layer flow past over a sphere. Hybrid nanofluid is a new class of heat transfer nanofluid engineered by dispersing two different types of nanoparticles in conventional heat transfer fluid (called the base fluid). A mathematical model has been developed for the mixed convection boundary layer flow of viscoelastic hybrid nanofluid past over a sphere using the Tiwari-Das model with copper (*Cu*) as the first nanoparticle, aluminium oxide (Al_2O_3) as the second nanoparticle, and water (H_2O) as the base fluid. The dimensionless governing equations will be taken into consideration and MATLAB BVP4C solver is used to create a simulation to analyze the flow and heat transfer characteristics of the hybrid nanofluid model. The findings obtained include velocity and temperature profiles with a variety of factors that influenced, including the viscoelastic parameter, *K*, the Prandtl number, *Pr*, the mixed convection parameter, λ, and the volume fraction of nanoparticles, φ. The results show that the hybrid nanofluids offer enhanced heat transfer performance in thermal processes, exhibiting better thermophysical properties than mono nanofluids.

Keywords: Mixed convection; viscoelastic hybrid nanofluids; boundary layer flow; sphere; BVP4C

1. Introduction

The Greek word "Nanos," which means "dwarf," is where the word "nano" first appeared. A substance is referred to be a nanoparticle when at least one dimension of its size is on the nanoscale (1–100 nanometres). The creation and use of materials at the nanoscale as well as the incorporation of the generated nanostructures into larger systems are all aspects of nanotechnology [1]. Nanofluids can be simply said a technology that uses nanoparticle in base fluid such as water, ethylene glycol, oil, and blood. This technology can be applied in medicine, healthcare, energy, biotechnology, information technology, nanoelectronics, and national security. Nanofluids have better heat transfer properties compared to traditional heat transfer fluids [2]. The fact that the suspended particles significantly improve the thermal conductivity of nanofluids is one of the causes. The volume fraction of nanoparticles had a significant impact on the thermal conductivity of nanofluid. The experiment was performed, and it proved that the particle volume fraction of the nanoparticle affects the thermal conductivity in nanofluids.

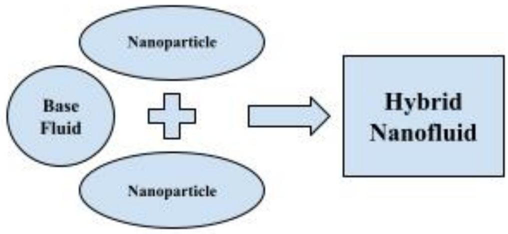

Figure 1 Combination of two nanoparticles with base fluid form hybrid nanofluid

However, Babar, H., & Ali, H. M. [3] had shown in their recent research that hybrid nanofluids gave a better thermal characteristic compared to nanofluids because its stability influenced the thermal conductivity. Hybrid nanofluids are the combination of two or more nanoparticles in a fluid, aiming to overcome the limitations of the mono nanofluid by adding a contrasting property addition to counteract the weaknesses [4]. This innovation was applied to electronic cooling, manufacturing or automotive industry, heat exchanger, and solar energy to see how this technology helped to improve performance.

2. Literature Review

Recently, Nanoparticles commonly used in the previous research paper were Titanium Dioxide (TiO_2), Aluminium Oxide (Al_2O_3), and Copper(II) Oxide (CuO) Nanofluids were used to improve the heat transfer in fluid flow as it has high thermal conductivity. The higher the thermal conductivity, the better the heat transfer process. Typical fluids used in industrial and technical applications include ethylene glycol, water, and oil. But because of poor thermal conductivity, the heat transfer rate of these fluids is limited. Therefore, it is strongly advised to use nanofluid in the applications to make up for its shortcoming. Nanofluids include nanoparticles, such as nanosized alumina particles suspended in a carrier solvent fluid with viscosity and surface tension sufficient to suspend and disseminate the particles [6]. Choi & Eastman [5] developed this term for the first time in that year. The performance of nanofluids in terms of thermodynamics, heat transfer, fluid flow, and thermo-optics is nonlinear due to a variety of enhancing mechanisms and flow conditions [7].

Based on Afifah, Y. N. [8] research, the magnetic field influences nanofluid flow, causing the boundary layer to develop a non-similar equations boundary layer. She also shows the effect of variations in nondimensional nano fluid density parameters, nondimensional nanofluid heat capacity variations, and nanoparticle volume fraction variations, on velocity profile, and temperature profile. According to Alzu'bi, O. A. S. et al. [9], when the volume fraction of catalytic nanoparticles whether Multi-Walled Carbon Nanotubes (MWCNTs) or Graphene Oxide (GO) was increased, it enhances energy transfer, raises the fluid temperature and decreases the friction drag. Instead of that, Dawar, A. et al. [10] studied, the larger the nanoparticle volume fraction and thermal Biot number, the higher the temperature profile of the spherical-shaped Cu, Al_2O_3, and TiO_2 nanoparticles.

Mahdy, A. E. N. et al. [11] stated that the addition of nanoparticles in the base fluid influences the enhancement in the thermal conductivity of the fluid. A viscous flow, incompressible, and electrically conducting nanofluid was examined in the stagnation point of a revolving sphere. In the instance of a Fe_3O_4-water nanofluid, an increase in the nanoparticle solid volume percentage raises the species concentration and velocity. The temperature of the nanofluid rises as the volume percentage of the nanoparticles increases. According to Patil, P. M. et al. [12] when the value of the mixed convection parameter is increased, the velocity profile increases as well.

The investigation of heat transfer and flow on rotary bodies of rotation in a forced flow has importance in engineering applications, such as fiber coating, projectile motion, etc. Rajasekaran & Palekar [13] addressed the impact of the forces of buoyancy on a stable forced convection flow over a rotating sphere. Hybrid nanofluids are created by suspending two classes of nanoparticles within the base fluid. Hybrid nanoparticles present an enormous benefit, except for the increase in effective thermal conductivity when nanosized particles are distributed appropriately. An experimental study of mixed convection with ($Cu - Al_2O_3/H_2O$) hybrid nanofluid for laminar flow in a sloped tube was performed by Momin [14]. The characteristics of the turbulent heat transfer and pressure decline of diluted water-based $(Cu - Al_2O_3/H_2O)$ hybrid nanofluids in a spinning sphere were investigated in a study by Suresh et al. [15]. We looked at how transient mixed Casson Nanofluid flow within the stagnation point is affected by MHDs and different wall temperatures.

The unsteady MHD radiation mixed convection flow of Casson hybrid nanofluid in the stagnation area of an impulsively spinning sphere with a magnetic field was examined in the current contribution. The partial differential equations were converted into dimensionless differential equations by utilizing an appropriate transformation, and they were then numerically solved using the hybrid linearization-differential quadrature method implemented in the MATLAB environment [16]. The effects of several parameters, including radiation, buoyancy, and rotation parameters, were studied in the flow, where the impulsive motion causes the hybrid nanofluid to be unstable and, as a result, causes the sphere to rotate impulsively. To verify the technique, the results are compared.

3. Mathematical Model

A sphere of radius a is placed in a viscoelastic hybrid nanofluid with convective boundary condition (CBC), and the flow of the mixed convection boundary layer occurs around the sphere. Assuming that the temperature of the ambient nanofluid is T_∞ and that the velocity outside the boundary is $\bar{u}_e(\bar{x})$. Figure 3.1 shows the physical model and coordinate system of the problem. T_w is taken to be the sphere's surface constant temperature. While a cooled sphere is represented by $T_w < T_\infty$. According to Merkin [17], it is assumed that $\frac{1}{2}U_\infty$ is the constant free stream's velocity.

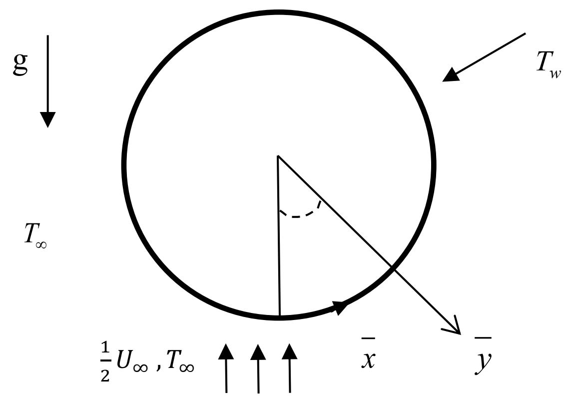

Figure 2 Coordinate System and Flow Model

Below are the dimensional governing equations obtained from continuity, momentum, and energy equations. By laws of physics, these equations were built. The continuity equation was obtained from the conservation of mass while the momentum equation was from Newton's second law. The model below were obtained from Tiwari & Das [18]:

$$\frac{\partial}{\partial \bar{x}}(\bar{r}\bar{u}) + \frac{\partial}{\partial \bar{y}}(\bar{r}\bar{v}) = 0, \tag{1}$$

$$\rho_{hnf}\left(\bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}\right) = \rho_{hnf}\left(\bar{u}_e\frac{\partial \bar{u}_e}{\partial \bar{x}}\right) + \mu_{hnf}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$
$$+k_0\left(\frac{\partial}{\partial \bar{x}}\left(\bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2}\right) + \bar{v}\frac{\partial^3 \bar{u}}{\partial \bar{y}^3} + \frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}\right)$$

$$+g\left(\rho\beta\right)_{hnf}\left(T-T_{\infty}\right)\sin\left(\frac{\bar{x}}{a}\right) \tag{2}$$

$$\bar{u}\frac{\partial T}{\partial \bar{x}}+\bar{v}\frac{\partial T}{\partial \bar{y}}=\alpha_{hnf}\frac{\partial^2 T}{\partial \bar{y}^2}, \tag{3}$$

subjected to boundary conditions

$$\bar{u}=0,\qquad \bar{v}=0,\qquad T=T_w,\qquad \text{on } \bar{y}=0,\ \bar{x}\geq 0,$$

$$\bar{u}=\bar{u}_e(\bar{x}),\qquad \frac{\partial \bar{u}}{\partial \bar{y}}=0,\qquad T=T_{\infty}, \text{as } \bar{y}\to\infty,\qquad \bar{x}\geq 0. \tag{4}$$

Below are the thermal properties used in the governing equations to dimensionless each equation that were obtained from [19], [20], and [21]:

$$\alpha_{hnf}=\frac{k_{hnf}}{(\rho C_p)_{hnf}},\qquad \alpha_{bf}=\frac{k_{bf}}{(\rho C_p)_{bf}},$$

$$\rho_{hnf}=\left(1-\varphi_{np2}\right)\left(\left(1-\varphi_{np1}\right)\rho_{bf}+\varphi_{np1}\rho_{np1}\right)+\varphi_{np2}\rho_{np2},$$

$$\mu_{hnf}=\frac{\mu_{bf}}{\left(1-\varphi_{np1}\right)^{2.5}\left(1-\varphi_{np2}\right)^{2.5}},$$

$$(\rho C_p)_{hnf}=(1-\varphi_{np2})\left((1-\varphi_{np1})(\rho C_p)_{bf}+\varphi_{np1}(\rho C_p)_{np1}\right)+\varphi_{np2}(\rho C_p)_{np2}, \tag{5}$$

$$(\rho\beta)_{hnf}=\left(1-\varphi_{np2}\right)\left(\left(1-\varphi_{np1}\right)(\rho\beta)_{bf}+\varphi_{np1}(\rho\beta)_{np1}\right)+\varphi_{np2}(\rho\beta)_{np2},$$

$$k_{hnf}=\frac{k_{np2}+(n-1)k_{bf}-(n-1)\varphi_{np2}\left(k_{bf}-k_{np2}\right)}{k_{np2}+(n-1)k_{bf}+\varphi_{np2}\left(k_{bf}-k_{np2}\right)}\times k_{bf}.$$

The $\bar{x}$ and $\bar{y}$ are referring to Cartesian Coordinates along the surface of the sphere. Moreover, $\bar{y}$ is the coordinate measured perpendicular to the sphere's surface while $\bar{u}$ and $\bar{v}$ are velocity components, g is the gravity acceleration and T is the temperature of the selected fluid-base. The viscoelastic material's constant (Walter's Liquid-B model) is $k_0 > 0$. Next, for the thermal properties of hybrid nanofluids, Al_2O_3 and *Cu* nanoparticles are denoted by the subscripts *np1* and *np2*, respectively, while base fluid and hybrid nanofluid are denoted by the subscripts *bf* and *hnf*, φ_{np1} and φ_{np2} is the volume fraction of the nanoparticle 1 and nanoparticle 2, v is the kinematic viscosity. According to [25], ρ_{hnf} is the hybrid nanofluid's density, μ_{hnf} is the viscosity, β_{hnf} is the thermal expansion coefficient, k_{hnf} and k_{bf} is the thermal conductivity of hybrid nanofluid and base fluid, and $(C_p)_{hnf}$ is the heat capacity of hybrid nanofluid. α_{hnf} and α_{bf} are thermal diffusivity of hybrid nanofluid and base fluid. The local free stream velocity outside the boundary layer is denoted by $\bar{u}_e(\bar{x})$, and the radial distance between the symmetric axis and the sphere's surface is denoted by $\bar{r}(\bar{x})$.

$$\bar{u}_e(\bar{x})=\frac{3}{2}U_{\infty}sin\left(\frac{\bar{x}}{a}\right) \text{ and } \bar{r}(\bar{x})=\ a\ sin\left(\frac{\bar{x}}{a}\right). \tag{6}$$

Below are the thermophysical properties of the base fluid (CMC-water) and nanoparticle Aluminium Oxide Al_2O_3 and Copper (*Cu*).

Table 1 Thermo-properties of base fluid and nanoparticles for hybrid nanofluids obtained from [22].

Physical properties	Cu	Al_2O_3	H_2O
$\rho(kg/m^3)$	8933	3970	997.1
$C_p(J/kgK)$	385	765	4179
$k(W/mK)$	400	40	0.613
$\beta \times 10^{-5}(mK)$	1.67	0.85	21

The governing equations will be transformed into a non-dimensional form. Below are the dimensionless variables obtained from [23]:

$$x=\frac{\bar{x}}{a},\qquad y=Re^{1/2}\left(\frac{\bar{y}}{a}\right),\qquad u=\frac{\bar{u}}{U_\infty},\qquad v=Re^{1/2}\left(\frac{\bar{v}}{U_\infty}\right),\tag{7}$$

$$r(x)=\frac{\bar{r}(\bar{x})}{a},\qquad u_e(x)=\frac{\bar{u}_e(\bar{x})}{U_\infty},\qquad \theta=\frac{(T-T_\infty)}{(T_w-T_\infty)},\qquad Re=U_\infty\left(\frac{a}{v}\right),$$

Where $Re=U_\infty\frac{a}{v}$ is the Reynolds number. Substitute equation (7) into equations (1), (2), and (3) resulted to the following dimensionless equations:

$$\frac{\partial}{\partial x}\left(r(x)u\right)+\frac{\partial}{\partial y}\left(r(x)v\right)=0,\tag{8}$$

$$\begin{aligned}
&\left(1-\varphi_{np2}\right)\left[(1-\varphi_{np1})+\varphi_{np1}\frac{\rho_{np1}}{\rho_{bf}}\right]+\varphi_{np2}\frac{\rho_{np2}}{\rho_{bf}}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)\\
&=\left(1-\varphi_{np2}\right)\left[(1-\varphi_{np1})+\varphi_{np1}\frac{\rho_{np1}}{\rho_{bf}}\right]+\varphi_{np2}\frac{\rho_{np2}}{\rho_{bf}}\left(u_e(x)\frac{\partial u_e(x)}{\partial x}\right)\\
&+\frac{1}{(1-\varphi_{np1})^{2.5}(1-\varphi_{np2})^{2.5}}\left(\frac{\partial^2 u}{\partial y^2}\right)+K\left(\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right)+v\frac{\partial^3 u}{\partial y^3}+\frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right)\\
&+\left((1-\varphi_{np2})\left[(1-\varphi_{np1})+\left(\varphi_{np1}\frac{(\rho\beta)_{np1}}{(\rho\beta)_{bf}}\right)\right]+\left(\varphi_{np2}\frac{(\rho\beta)_{np2}}{(\rho\beta)_{bf}}\right)\right)\lambda\theta\sin x,
\end{aligned}\tag{9}$$

$$\begin{aligned}
&\left[(1-\varphi_{np2})\left((1-\varphi_{np1})+\varphi_{np1}\frac{(\rho C_p)_{np1}}{(\rho C_p)_{bf}}\right)+\varphi_{np2}\frac{(\rho C_p)_{np2}}{(\rho C_p)_{bf}}\right]\left[u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}\right]\\
&=\frac{1}{\Pr}\frac{k_{np2}+(n-1)k_{bf}-(n-1)\varphi_{np2}(k_{bf}-k_{np2})}{k_{np2}+(n-1)k_{bf}+\varphi_{np2}(k_{bf}-k_{np2})}\frac{\partial^2\theta}{\partial y^2},
\end{aligned}\tag{10}$$

and the boundary conditions (4) becomes,

$$u=0,\qquad v=0,\qquad \theta'=-1,\qquad \text{on } y=0,\qquad x\geq 0\tag{11}$$

$$u=u_e(x)=\frac{3}{2}\sin x,\qquad \frac{\partial u}{\partial y}=0,\qquad \theta=0,\qquad y\to\infty,\qquad \text{as } x\geq 0.$$

Where

$$K=\frac{k_0U_\infty}{\alpha\rho_{bf}v},\ \Pr=\frac{v}{\alpha'},\ \lambda=\frac{Gr}{\mathrm{Re}^2}=\frac{g\beta_{bf}(T_w-T_\infty)a}{U_\infty{}^2},\ Gr=\frac{g\beta_{bf}(T_w-T_\infty)a^3}{v_{bf}{}^2},\tag{12}$$

K is the dimensionless viscoelastic parameter, Pr is Prandtl number, λ is constant mixed convection parameter, where Gr is Grashof number and Re is Reynold number. Declare that the forced convection flow occurs when $\lambda = 0$ and that the assisting flow (heated sphere) occurs when $\lambda > 0$ and the opposing flow (cooled sphere). It is important to note that $K = 0$ applies to viscous (Newtonian) fluids.

The following variables are taken as given in order to solve Equations (8) to (10) and the boundary conditions (11):

$$\psi = xr(x)f(x,y), \quad \theta = \theta(x,y), \tag{13}$$

where ψ is the stream function defined as

$$u = \frac{1}{r}\frac{\partial \psi}{\partial y}, \quad v = -\frac{1}{r}\frac{\partial \psi}{\partial x}, \tag{14}$$

that's already satisfies equation (8) automatically. Then, by substituting equation (14) into equation (8) until equation (11), also by considering $U_e(x) = \frac{\bar{u}_e(\bar{x})}{U_\infty} = \frac{3}{2}sin(x)$ and $r(x) = \sin x$, the dimensionless governing equation become:

$$\left(\left(1-\varphi_{np2}\right)\left[(1-\varphi_{np1})+\varphi_{np1}\frac{\rho_{np1}}{\rho_{bf}}\right]+\varphi_{np2}\frac{\rho_{np2}}{\rho_{bf}}\right)\left(x\sin^2 x\frac{\partial f}{\partial y}\frac{\partial f}{\partial y}-fx\sin^2 x\frac{\partial^2 f}{\partial y^2}-fx^2\sin x\cos x\frac{\partial^2 f}{\partial y^2}\right)$$

$$=\frac{9}{4}\sin x\cos x\left(\left(1-\varphi_{np2}\right)\left[(1-\varphi_{np1})+\varphi_{np1}\frac{\rho_{np1}}{\rho_{bf}}\right]+\varphi_{np2}\frac{\rho_{np2}}{\rho_{bf}}\right)+\frac{1}{(1-\varphi_{np1})^{2.5}(1-\varphi_{np2})^{2.5}}\left(x\frac{\partial^3 f}{\partial y^3}\right)$$

$$+K\left(\begin{array}{l}2x\sin^2 x\dfrac{\partial f}{\partial y}\dfrac{\partial^3 f}{\partial y^3}-fx\sin^2 x\dfrac{\partial^4 f}{\partial y^4}-fx^2\sin x\cos x\dfrac{\partial^4 f}{\partial y^4}\\ -x\sin^2 x\dfrac{\partial^2 f}{\partial y^2}\dfrac{\partial^2 f}{\partial y^2}-x^2\sin x\cos x\dfrac{\partial^2 f}{\partial y^2}\dfrac{\partial^2 f}{\partial y^2}\end{array}\right)$$

$$+\left((1-\varphi_{np2})\left[(1-\varphi_{np1})+\left(\varphi_{np1}\frac{(\rho\beta)_{np1}}{(\rho\beta)_{bf}}\right)\right]+\left(\varphi_{np2}\frac{(\rho\beta)_{np2}}{(\rho\beta)_{bf}}\right)\right)\lambda\theta\sin x, \tag{16}$$

$$\left[(1-\varphi_{np2})\left((1-\varphi_{np1})+\varphi_{np1}\frac{(\rho C_p)_{np1}}{(\rho C_p)_{bf}}\right)+\varphi_{np2}\frac{(\rho C_p)_{np2}}{(\rho C_p)_{bf}}\right]\left[x\frac{\partial f}{\partial y}\frac{\partial \theta}{\partial x}-x\frac{\partial f}{\partial x}\frac{\partial \theta}{\partial y}-f\frac{\partial \theta}{\partial y}-\frac{fx\cos x}{\sin x}\frac{\partial \theta}{\partial y}\right]$$

$$=\frac{1}{\Pr}\frac{k_{np2}+(n-1)k_{bf}-(n-1)\varphi_{np2}(k_{bf}-k_{np2})}{k_{np2}+(n-1)k_{bf}+\varphi_{np2}(k_{bf}-k_{np2})}\frac{\partial^2 \theta}{\partial y^2}, \tag{17}$$

and the boundary equation become,

$$f = 0, \qquad \frac{\partial f}{\partial y} = 0, \qquad \theta' = -1, \qquad \text{on } y = 0, \qquad x \geq 0$$

$$\frac{\partial f}{\partial y} \to \frac{3}{2}\frac{\sin x}{x}, \ \frac{\partial^2 f}{\partial y^2} = 0, \qquad \theta \to 0, \qquad \text{as } y \to \infty, \qquad x \geq 0. \tag{18}$$

At the lower stagnation point of the sphere, $x \approx 0$, the equation (16) and (17) are reduced to the following ordinary differential equations:

$$\left(\left(1-\varphi_{np2}\right)\left[(1-\varphi_{np1})+\varphi_{np1}\frac{\rho_{np1}}{\rho_{bf}}\right]+\varphi_{np2}\frac{\rho_{np2}}{\rho_{bf}}\right)\left(2ff''-f'^2+\frac{9}{4}\right)$$

$$+\frac{1}{(1-\varphi_{np1})^{2.5}(1-\varphi_{np2})^{2.5}}f''' + 2K\left(f'f''' - ff'''' - f''^2\right)$$
$$+\left((1-\varphi_{np2})\left[(1-\varphi_{np1})+\left(\varphi_{np1}\frac{(\rho\beta)_{np1}}{(\rho\beta)_{bf}}\right)\right]+\left(\varphi_{np2}\frac{(\rho\beta)_{np2}}{(\rho\beta)_{bf}}\right)\right)\lambda\theta = 0\,, \tag{19}$$

$$2\left[(1-\varphi_{np2})\left((1-\varphi_{np1})+\varphi_{np1}\frac{(\rho C_p)_{np1}}{(\rho C_p)_{bf}}\right)+\varphi_{np2}\frac{(\rho C_p)_{np2}}{(\rho C_p)_{bf}}\right]f\theta'$$
$$+\frac{1}{\Pr}\frac{k_{np2}+(n-1)k_{bf}-(n-1)\varphi_{np2}(k_{bf}-k_{np2})}{k_{np2}+(n-1)k_{bf}+\varphi_{np2}(k_{bf}-k_{np2})}\theta'' = 0\,, \tag{20}$$

with the boundary equations,

$$f(0)=0,\qquad f'(0)=0,\qquad \theta'(0)=-1,\qquad \text{on } y=0,\qquad x=0\,,$$
$$f'(y)\to\frac{3}{2},\ f''(y)=0,\qquad \theta(y)=0,\qquad \text{as } y\to\infty,\qquad x=0\,. \tag{21}$$

4. Research Methodology

The numerical computing method BVP4C in MATLAB will be used to solve the nonlinear ordinary differential equations with boundary conditions. Shampine et al. [24] has introduced the BVP4C solver in the MATLAB programming. It is a finite-difference code that implements the three-stage Lobatto IIIa numerical approach, which depends on an iterative structure to solve the nonlinear equations and yields consistently correct fourth-order results over the interval. The continuous solution's residual affects both the mesh choice and error handling.

Before using BVP4C solver in MATLAB, the equations (19) until (21) need to be converted into system of first order. The BVP4C solver's basic syntax is $sol = bvp4c(odefun, bcfun, solinit, options)$, which combines a system of differential equations with $odefun$ in the form of $y' = f(x, y)$ with $bcfun$, the boundary conditions, and $solinit$, the initial solution guess. An additional integration setting is referred to as $options$ in the basic syntax of the BVP4C solver and is created as an argument using the bvpset function. Hence, the Equations (19) to (21) must be rewritten as the first order differential equations as shown below:

$$y_1 = f,\qquad y_2 = f',\qquad y_3 = f'',\qquad y_4 = f''',\qquad y_4' = f'''', \tag{22}$$

$$y_5 = \theta,\qquad y_6 = \theta',\qquad y_6' = \theta'', \tag{23}$$

$$y_4' = \frac{1}{y_1}\left[\begin{array}{l}\left(y_2y_4 - y_3^2\right)\\ +\dfrac{1}{2K}\left(\begin{array}{l}\left((1-\varphi_{np2})\left[(1-\varphi_{np1})+\varphi_{np1}\dfrac{\rho_{np1}}{\rho_{bf}}\right]+\varphi_{np2}\dfrac{\rho_{np2}}{\rho_{bf}}\right)\\ \left(2y_1y_3 - y_2^2 + \dfrac{9}{4}\right)+\dfrac{1}{\left(1-\varphi_{np1}\right)^{2.5}\left(1-\varphi_{np2}\right)^{2.5}}y_4\\ +\left((1-\varphi_{np2})\left[(1-\varphi_{np1})+\varphi_{np1}\dfrac{(\rho\beta)_{np1}}{(\rho\beta)_{bf}}\right]+\varphi_{np2}\dfrac{(\rho\beta)_{np2}}{(\rho\beta)_{bf}}\right)\lambda y_5\end{array}\right)\end{array}\right] \tag{24}$$

$$y_6' = \Pr \frac{k_{np2} + (n-1)k_{bf} + \varphi_{np2}(k_{bf} - k_{np2})}{k_{np2} + (n-1)k_{bf} - (n-1)\varphi_{np2}(k_{bf} - k_{np2})}$$

$$-2\left[(1-\varphi_{np2})\left((1-\varphi_{np1}) + \varphi_{np1}\frac{(\rho C_p)_{np1}}{(\rho C_p)_{bf}}\right) + \varphi_{np2}\frac{(\rho C_p)_{np2}}{(\rho C_p)_{bf}}\right] y_1 y_6 \tag{25}$$

with the boundary conditions,

$$y_1(0) = 0, \qquad y_2(0) = 0, \qquad y_6(0) = -1,$$

$$y_2(\infty) \to \frac{3}{2}, \qquad y_3(\infty) = 0, \qquad y_5(\infty) = 0. \tag{26}$$

5. Results And Discussion

By using the first order differential Equations (24) and (25) together with the boundary conditions (26), the systems of ordinary differential Equations (19) and (20) subject to the boundary conditions (21) are solved using the BVP4C solver. A MATLAB algorithm is developed and applied to solve Equations (24) to (26) in order to simulate the numerical results obtained by the BVP4C solver. For specific values of the mixed convection parameter, λ, with the viscoelastic parameter, $K = 1$, and the Prandtl number, $Pr = 0.7$, the numerical solutions of the viscoelastic fluid (with nanoparticle volume fraction, $\varphi = 0.1$ are obtained.

The results obtained were compared with previous result of Kejing & Abdullah [21]. In order to do so, the graphs of velocity and temperature profiles for viscoelastic fluid at lower stagnation point are solved and plotted using the MATLAB algorithm. Figure 3 to 4 show the velocity profile and temperature profile obtained by Kejing & Abdullah [21] with one nanoparticle followed by Figure 5 to 6 which show the same graph but using the recent algorithm obtained.

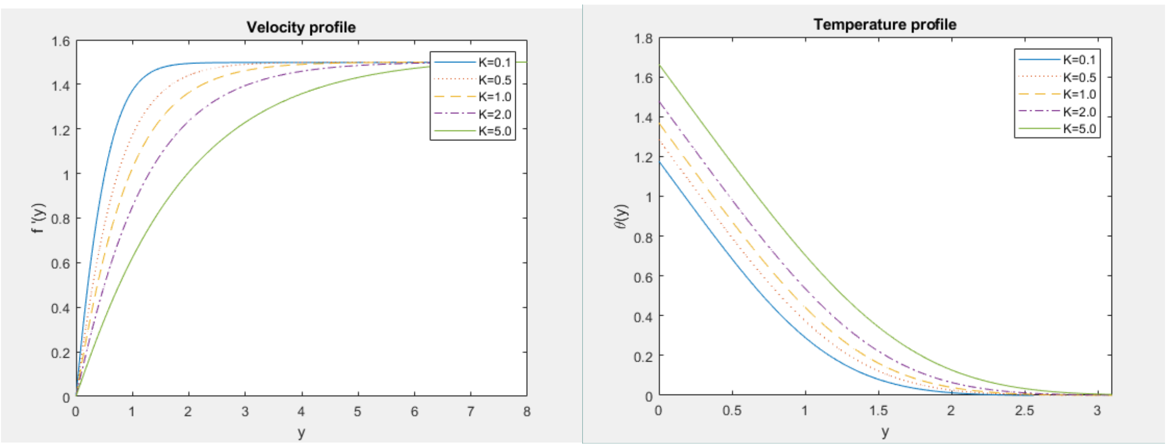

Figure 3 Velocity and temperature profile when $Pr = 1$, λ=1, and $\varphi = 0.1$

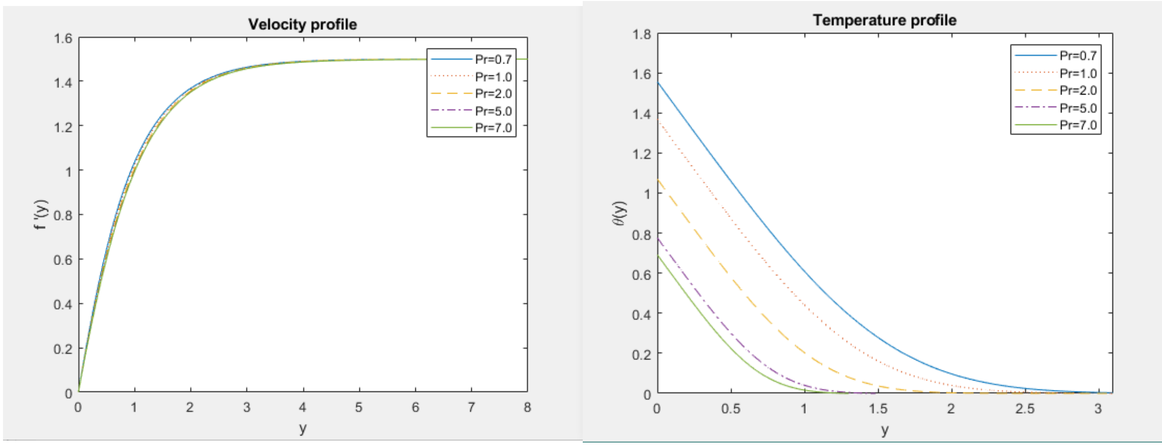

Figure 4 Velocity and temperature profile when $K = 1$, λ=1, and $\varphi = 0.1$

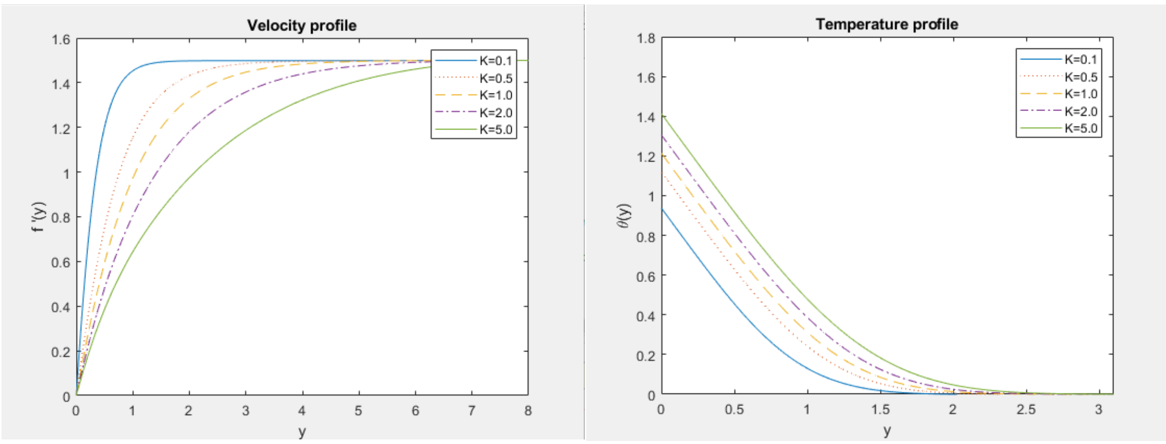

Figure 5 Velocity and temperature profile when $Pr = 1$, λ=1, and $\varphi = 0.1$

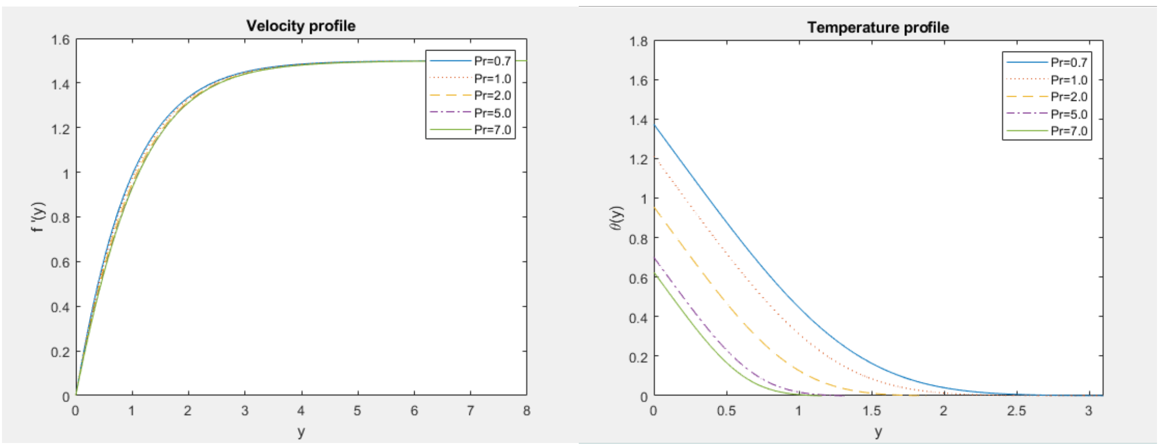

Figure 6 Velocity and temperature profile when $K = 1$, λ=1, and $\varphi = 0$

Figures 7 show the effects of viscoelastic parameter, K to the graph of hybrid nanofluid velocity profile and temperature profile. When the values of K increase, the velocity shows decreasing. However, with the existence of Al_2O_3 as the second nanoparticle, the line for $K = 1$ shows higher than Figure 5 where there exists only one nanoparticle which is copper, *Cu*. While temperature profile shows increasing when the value of K increase. However, the fluid's velocity decreases together with the velocity gradient when the Prandtl number, Pr is increased. The temperature profile in Figure 8 illustrates that when Pr values show increasing, the temperature at the sphere's surface will decrease due to a reduction in boundary layer thickness, as thermal diffusivity decreases and energy ability decreases. Besides, the values of the mixed convection parameter, λ, in Figure 9 resemble the favorable pressure gradient, which improves the friction and velocity of the hybrid nanofluid flow at the sphere's surface. When the mixed convection parameter, λ is increases due to a decrease in energy ability, the temperature profile shows a reduction in the thickness of the thermal boundary layer. Lastly, Figure 10 shows when the value of φ increases, the temperature and velocity both increased. The presence of one more nanoparticle in the fluid causes an increase in the fluid's effective thermal conductivity and an improvement in its ability to transfer heat. As a result, as increases the value of φ, the thermal conductivity also increases, and automatically increases the thickness of the thermal boundary layer.

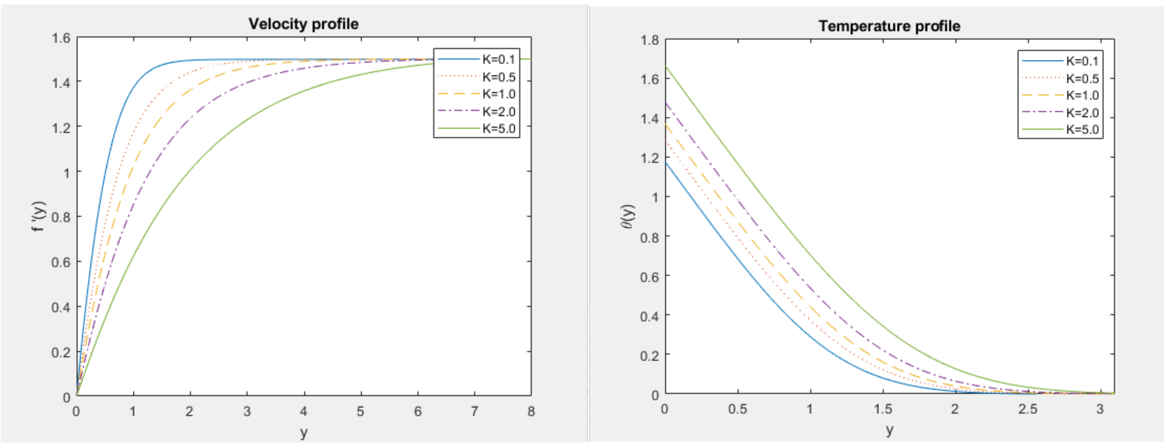

Figure 7 Velocity and temperature profile when $Pr = 1$, $\lambda = 1$, and φ_{np1} & $\varphi_{np1} = 0.1$

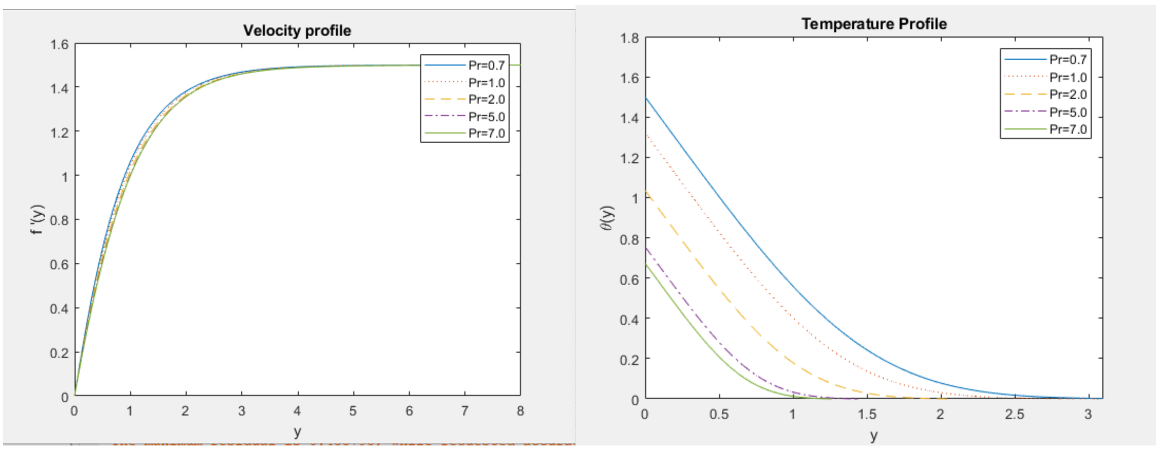

Figure 8 Velocity and temperature profile when $K = 1, \lambda = 1$, and φ_{np1} & $\varphi_{np1} = 0.1$

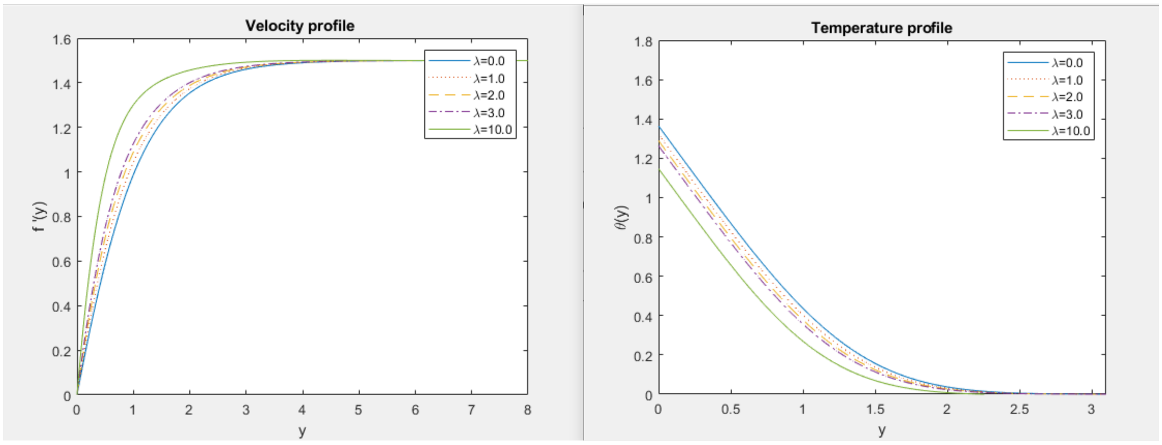

Figure 9 Velocity and temperature profile when $K = 1, Pr = 1$, and φ_{np1} & $\varphi_{np1} = 0.1$

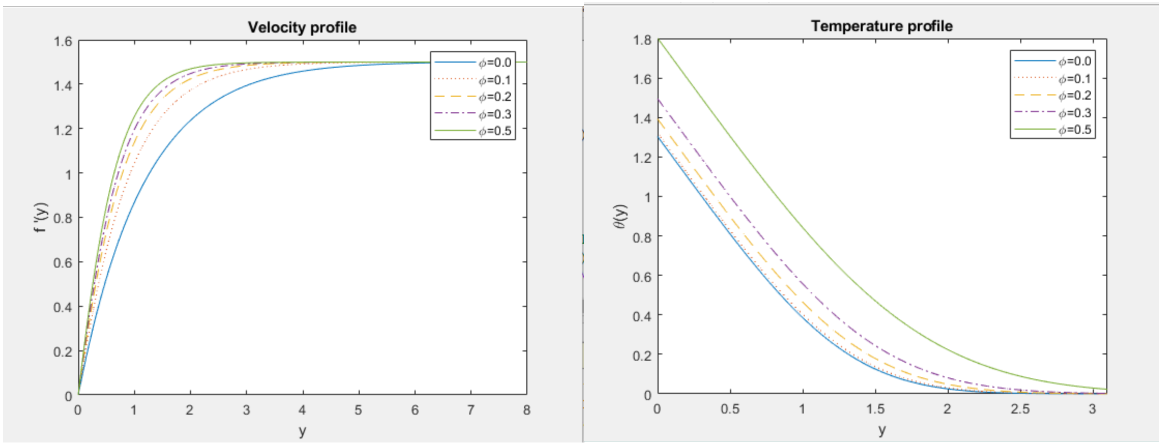

Figure 10 Velocity and temperature profile when $K = 1, Pr = 1$, and $\lambda = 1$

Conclusion

BVP4C of MATLAB has been used to solve the problem of mixed convection flow of nanofluid and hybrid nanofluids past over a sphere. It has been found that:

- when the viscoelastic parameter, K, Prandtl number, Pr, mixed convection parameter, λ and nanoparticles volume fraction, φ are changed in this problem, there are some significant effects on the nanofluid and hybrid nanofluid flow characteristics in terms of flow velocity and temperature on the surface of the sphere.
- when the hybrid nanofluid's viscoelasticity increases, the flow of the fluid will slowly decreasing and the temperature of the sphere's surface increases.
- as the Prandtl number, Pr, in the problem increases, the flow of the hybrid nanofluids and the temperature on the surface of the sphere will both decrease simultaneously.
- furthermore, as the values of the mixed convection parameter, λ increased, the velocity of the hybrid nanofluids flow increased significantly. The temperature on the sphere's surface does show a little decrease in the same condition.
- at the very least, when the volume fraction of nanoparticles, φ increases, the flow velocity and surface temperature of the sphere also increase.

In conclusion, by added aluminium oxide, Al_2O_3 as the second nanoparticle in this study, the velocity and temperature profile for a viscoelastic hybrid nanofluid shows higher than previous research which only use one nanoparticle, copper, *Cu*. It is also important to note that the heat transfer properties of a fluid are improved by the addition of nanoparticles in the fluid. Hybrid nanofluids is resulting a better when it comes to improving the thermal conductivity in a heat transfer system. Thus, hybrid nanofluids should be used a lot in industry instead of nanofluid.

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