Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia

Vol. 17, 2023, page 210-222

Commuting Order Product Prime Graph for Nonabelian Metabelian Groups of Order at Most 30

Zurin Nurdiana Eisya Zamri, Nor Muhainiah Mohd Ali*
Department of Mathematics,
Faculty of Science, UTM, Skudai,
Johor Bahru, Malaysdia
*Corresponding author: normuhainiah@utm.my

Abstract
The study of geometric features of finite groups has become a focus of group theory research, leading to the definition of several graphs of groups and the investigation of graphical features of finite groups. G is metabelian if and only if its quotient group is abelian, where A is an abelian normal subgroup of G. The order product prime graph of finite groups is defined as a graph in which the vertices are the elements of the group and any two vertices are adjacent if and only if they commute and the product of their orders is a prime power. The purpose of this research is to conduct research into nonabelian metabelian groups up to order 30. The order product prime graphs for groups of order greater than 24 and up to 30 are determined, and the commuting order product prime graph for those groups is determined using definitions, theorems, and related findings from previous research.

Keywords: Commuting; order product prime; graph; nonabelian metabelian groups;

1 Introduction

A metabelian group is a group *G* that has at least a normal subgroup A such that A and the factor group G/A are both abelian. On the other side, a graph is a mathematical structure composed of both vertices and edges. In 2012, Abdul Rahman [1] has determined that all groups of order at most 24 are metabelian. Devi [2] asserted that studying the relationships between a group's elements or subgroups can reveal information about its characteristics. This relationship is equivalent to the corresponding defined graph's vertex adjacency. Cayley [3] defined a graph that explains the abstract structure of a group generated by a set of generators in 1878,

Sattanathan and Kala [4] defined an order prime graph of groups as a graph with group elements as the vertices and two vertices a and b are adjacent if and only if gcd (*a*,*b*) = 1. Rajendra and Reddy [5] extended on this concept by defining a general order prime graph of finite groups as a graph with group elements as vertices and two vertices a and b are adjacent if and only if the greatest common divisors of $|a|$ and $|b|$ is equal to 1 or p, where p is a prime number.

Newmann [6] defined a non-commuting graph of groups as the commuting graph's complement. Bertram [6] afterwards used the combinatorial properties of the commuting graph to prove three basic and nontrivial theorems on finite groups. Aschbacher [7] defined the commuting graph on subgroups of groups, which resulted in the definition of several graphs with subgroups of groups as vertices.

Then, Bello [8] introduces new graph of groups known as the order product prime graph. The vertices of the order product prime graph are the elements of the group and two vertices x and y in the graph are adjacent if and only if $|x||y| = p^s$ where p is a prime number and s is any positive integer. Also, Simon [9] has carried out more research on those kind of groups for order 25 until 31. Mohd Zulkefli [10] has determined the order product prime graph of all nonabelian metabelian groups of order at most 24. Hence, the commuting order product prime graph for all nonabelian groups of order at most 30 are constructed in this research.

2 Preliminaries

Some preliminary and basic definitions associated with this research are provided in this section.

Definition 2.1 [10] **(Metabelian)** A group *G* is metabelian if there exist a normal subgroup such that both A and ${}^{G}/_{A}$ are abelian.

Definition 2.2 [11] **(Graph)** A graph, Γ, is a mathematical structure consisting of two sets namely vertices and edges which are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The graph is called directed if its edges are identified with ordered pair of vertices, otherwise, Γ is called undirected. The adjacency of any two vertices $v_1, v_2 \in V(\Gamma)$ is denoted by $v_1 \sim v_2$.

Definition 2.3 [11] **(Connected Graph)** A graph Γ is connected if it has (u, v)-path whenever $u, v \in (\Gamma)$, otherwise Γ is disconnected.

Definition 2.4 [11] **(Complete Graph)** If all vertices in a graph are adjacent to each other, then the graph is called a complete graph. The symbol K_n denote a complete graph with n vertices.

Definition 2.5 [8] **(Order Product Prime Graph)** Let G be a finite group, then the order product prime graph of G, $\Gamma^{opp}(G)$, is a graph whose vertices are the elements of G and two vertices x, y are adjacent if and only if $|x||y| = p^s, s \in N$ for some prime p.

Definition 2.6 [9] (Commuting Order Product Prime Graph)
Let G be a finite group, then the commuting order product prime graph of G, $\Gamma^{copp}(G)$, is a graph whose vertices are the elements of G and two vertices x, y are adjacent if and only if $|x||y| = p^s, s \in \mathbb{N}$ and p is a prime number.

3 Results and discussion

In this section, the commuting order product prime graph for all nonabelian metabelian group other than dihedral groups, quaternion groups and quasi-dihedral groups of order at most 30 are presented and proved by using Definition 2.20. There are 15 groups of nonabelian metabelian groups which consist of dihedral groups, quaternion groups and quasi-dihedral groups.

3.1 Nonabelian Metabelian Groups of order at most 30

There are 43 groups of nonabelian metabelian groups which consist of dihedral groups, quaternion groups and quasi-dihedral groups are given in the following table.

Table 3.1 All Nonabelian Metabelian Groups of Order at Most 30

No.	Groups	$\|G\|$	Group Presentation
1.	D_3	6	$\langle a, b \| a^3 = b^2 = 1, bab = a^{-1} \rangle$

2.	D_4	8	$\langle a,b \mid a^4 = b^2 = 1, bab = a^{-1}\rangle$
3.	Q_4	8	$\langle a,b \mid a^4 = 1, a^2 = b^2, aba = b\rangle$
4.	D_5	10	$\langle a,b \mid a^5 = b^2 = 1, bab = a^{-1}\rangle$
5.	$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$	12	$\langle a,b \mid a^4 = b^3 = 1, aba = a\rangle$
6.	A_4	12	$\langle a,b,c \mid a^2 = b^2 = c^3 = 1, ba = ab, ca = abc, cb = ac\rangle$
7.	D_6	12	$\langle a,b \mid a^6 = b^2 = 1, bab = a^{-1}\rangle$
8.	D_7	14	$\langle a,b \mid a^7 = b^2 = 1, bab = a^{-1}\rangle$
9.	D_8	16	$\langle a,b \mid a^8 = b^2 = 1, bab = a^{-1}\rangle$
10.	Quasihedral-16	16	$\langle a,b \mid a^8 = b^2 = 1, bab = a^3\rangle$
11.	Q_8	16	$\langle a,b \mid a^8 = 1, a^4 = b^2, aba = b\rangle$
12.	$D_4 \times \mathbb{Z}_2$	16	$\langle a,b,c \mid a^4 = b^2 = c^2 = 1, ac = ca. bc = cb, bab = a^{-1}\rangle$
13.	$Q \times \mathbb{Z}_2$	16	$\left\langle a,b,c \middle\vert \begin{matrix} a^4 = b^4 = c^2 = 1, b^2 = a^2, ba = a^3b, ac = ca, \\ bc = cb \end{matrix} \right\rangle$
14.	Modular-16	16	$\langle a,b \mid a^8 = b^2 = 1, ab = ba^5\rangle$
15.	B	16	$\langle a,b \mid a^4 = b^4 = 1, ab = ba^3\rangle$
16.	K	16	$\langle a,b,c \mid a^4 = b^2 = c^2 = 1, cbc = ba^2, bab = a, ac = ca\rangle$
17.	$G_{4,4}$	16	$\langle a,b \mid a^4 = b^4 = 1, abab = 1, ba^3 = ab^3\rangle$
18.	D_9	18	$\langle a,b \mid a^9 = b^2 = 1, bab = a^{-1}\rangle$
19.	$S_3 \times \mathbb{Z}_3$	18	$\langle a,b,c \mid a^3 = b^2 = c^3 = 1, bab = a^{-1}, ac = ca, bc = cb\rangle$

20.	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	18	$\langle a,b,c \| a^2 = b^3 = c^3 = 1, bc = cb, bab = a, cac = a\rangle$
21.	D_{10}	20	$\langle a,b \| a^{10} = b^2 = 1, bab = a^{-1}\rangle$
22.	$Fr_{20} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$	20	$\langle a,b \| a^4 = b^5 = 1, ba = ab^2\rangle$
23.	$\mathbb{Z}_4 \times \mathbb{Z}_5$	20	$\langle a,b \| a^4 = b^5 = 1, bab = a\rangle$
24.	$Fr_{21} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$	21	$\langle a,b \| a^3 = b^7 = 1, ba = ab^2\rangle$
25.	D_{11}	22	$\langle a,b \| a^{11} = b^2 = 1, bab = a^{-1}\rangle$
26.	$S_3 \times \mathbb{Z}_4$	24	$\langle a,b,c \| a^3 = b^2 = c^4 = abab = aca^{-1}c^{-1} = bcb^{-1}c^{-1} = 1\rangle$
27.	$S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	24	$\langle a,b,c \| a^6 = b^2 = c^2 = abab, aca^{-1}c^{-1} = bcb^{-1}c^{-1} = 1\rangle$
28.	$D_4 \times \mathbb{Z}_3$	24	$\langle a,b,c \| a^4 = b^2 = c^3 = 1, baba = 1, ac = ca.bc = cb\rangle$
29.	$Q \times \mathbb{Z}_3$	24	$\langle a,b,c \| a^3 = b^4 = c^2 = 1, b^3c = cb, ab = ba, ac = ca\rangle$
30.	$A_4 \times \mathbb{Z}_2$	24	$\langle a,b,c \| a^2 = b^3 = c^2 = 1, ab = ba, ac = ca, c = bcbcb\rangle$
31.	$(\mathbb{Z}_6 \times \mathbb{Z}_2 \rtimes \mathbb{Z}_2)$	24	$\langle a,b,c \| a^3 = b^2 = c^2 = (cb)^4 = 1, ab = ba, aca = c\rangle$
32.	D_{12}	24	$\langle a,b \| a^{12} = b^2 = 1, bab = a^{-1}\rangle$
33.	$\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$	24	$\langle a,b \| a^4 = b^6 = 1, bab = a\rangle$
34.	$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	24	$\langle a,b \| a^8 = b^3 = 1, bab = a\rangle$
35.	$\mathbb{Z}_3 \rtimes Q$	24	$\langle a,b \| a^{12} = b^4 = abab^{-1} = a^6b^2\rangle$
36.	D_{13}	26	$\langle a,b \| a^{13} = b^2 = 1, bab = a^{-1}\rangle$
37.	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$	27	$\langle a,b,c \| a^3 = b^3 = c^3 = 1, bc = cba, ab = ba, ac = ca\rangle$

38.	$\mathbb{Z}_9 \rtimes \mathbb{Z}_3$	27	$\langle a,b \mid a^9 = b^3 = 1, ab = ba^4 \rangle$
39.	D_{14}	28	$\langle a,b \mid a^{14} = b^2 = 1, bab = a^{-1} \rangle$
40.	$\mathbb{Z}_7 \times \mathbb{Z}_4$	28	$\langle a,b,c \mid a^7 = b^2 = c^2 = 1, ac = ca, bc = cb, baba = 1 \rangle$
41.	D_{15}	30	$\langle a,b \mid a^{15} = b^2 = 1, bab = a^{-1} \rangle$
42.	$S_3 \times \mathbb{Z}_5$	30	$\langle a,b,c \mid a^3 = b^2 = c^5 = 1, ac = ca. bc = cb, bab = a^{-1} \rangle$
43.	$D_5 \times \mathbb{Z}_3$	30	$\langle a,b,c \mid a^5 = b^2 = c^3 = 1, ac = ca, bc = cb, baba = 1 \rangle$

3.2 Commuting Order Product Prime Graph of Nonabelian Metabelian Groups for Order at Most 30

In this section, the Commuting Order Product Prime Graph of Nonabelian Metabelian Groups for Order at Most 30 are given by using Definition 2.20[10].

Let $\mathbb{Z}_7 \times \mathbb{Z}_4$ be a nonabelian metabelian group of order 28. Then, the commuting order product prime graph of $\mathbb{Z}_7 \times \mathbb{Z}_4$ is

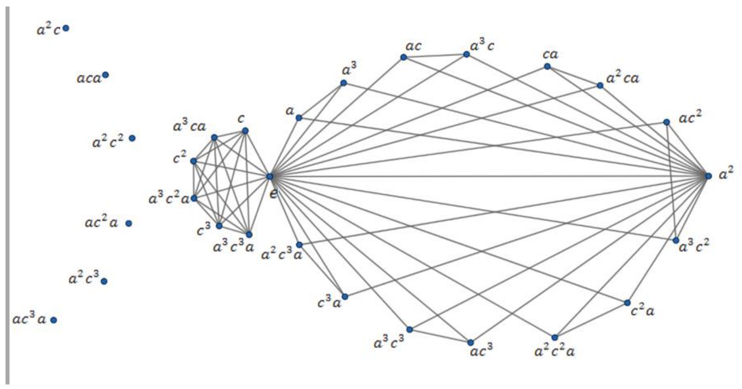


Figure 1 **COPP Graph of $\mathbb{Z}_7 \times \mathbb{Z}_4$**

Proof Suppose $\mathbb{Z}_7 \times \mathbb{Z}_4$ be a nonabelian metabelian group of order 28. Given the group presentation of $\mathbb{Z}_7 \times \mathbb{Z}_4$ as $\langle a,b,c \mid a^7 = b^2 = c^2 = 1, ac = ca, bc = cb, baba = 1 \rangle$ and the elements in this group are $e, a, a^2, a^3, c, ac, a^2c, a^3c, ca, aca, a^2ca, a^3ca, c^2, ac^2, a^2c^2, a^3c^2, c^2a, ac^2a, a^2c^2a, a^3c^2a, c^3$

, $ac^3, a^2c^3, a^3c^3, c^3a, ac^3a, a^2c^3a$,and a^3c^3a. By referring to the result of Proposition 3.4, the commuting order product prime graph is given in the Table 4.25 based on their connectivity with the elements x and y by satisfying the condition for $xy = yx$ for all $x, y \in G$ where $x \neq y$.

Table 4.25 The connectivity between x and y for $\mathbb{Z}_7 \times \mathbb{Z}_4$

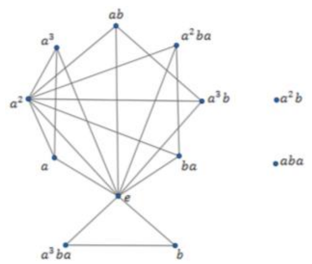
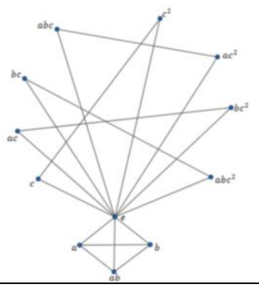
$x \in G$	Elements that commute with y	Total of edges
e	All commute except $e, a^2c, aca, a^2c^2, ac^2a, a^2c^3, ac^3a$	21
a	e, a^2, a^3	3
a^2	$e, a, a^3, ac, a^3c, ca, a^2ca, ac^2, a^3c^2, c^2a, a^2c^2a, ac^3, a^3c^3, c^3a, a^2c^3a$	15
a^3	e, a, a^2	3
c	$e, a^2, a^3ca, c^2, a^3c^2a, c^3, a^3c^3a$	7
ac	e, a^2, a^3c	3
a^3c	e, a^2, ac	3
ca	e, a^2, a^2ca	3
a^2ca	e, a^2, ca	3
a^3ca	$e, a^2, c, c^2, a^3c^2a, c^3, a^3c^3a$	7
c^2	$e, a^2, a^3ca, c, a^3c^2a, c^3, a^3c^3a$	7
ac^2	e, a^2, a^3c^2	3
a^3c^2	e, a^2, ac^2	3
c^2a	e, a^2, a^2c^2a	3
a^2c^2a	e, a^2, c^2a	3
a^3c^2a	$e, a^2, a^3ca, c, c^2, c^3, a^3c^3a$	7
c^3	$e, a^2, c, a^3ca, c^2, a^3c^2a, a^3c^3a$	7

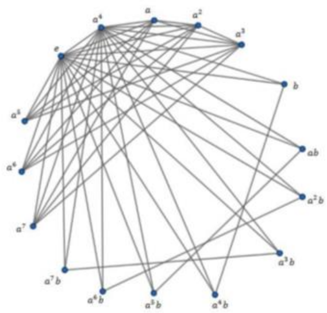
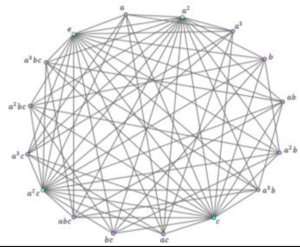
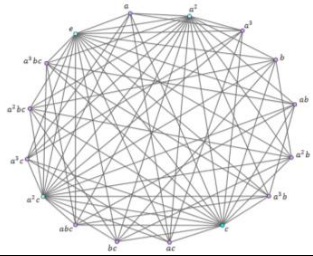
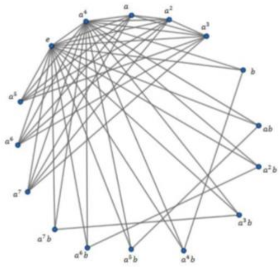
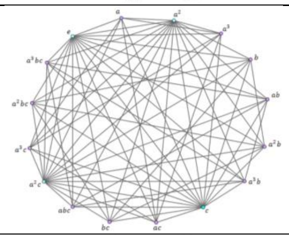
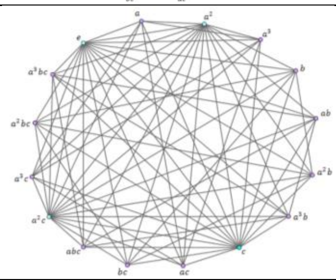
ac^3	e, a^2, a^3c^3	3
a^3c^3	e, a^2, ac^3	3
c^3a	e, a^2, a^2c^3a	3
a^2c^3a	e, a^2, c^3a	3
a^3c^3a	$e, a^2, c, a^3ca, c^2, a^3c^2a, c^3$	7

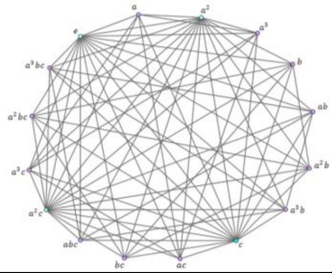
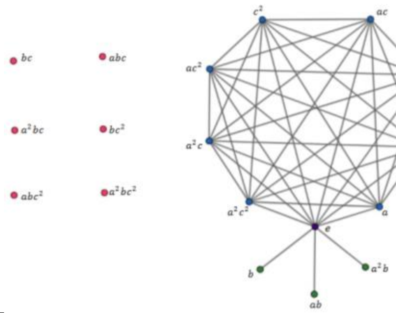
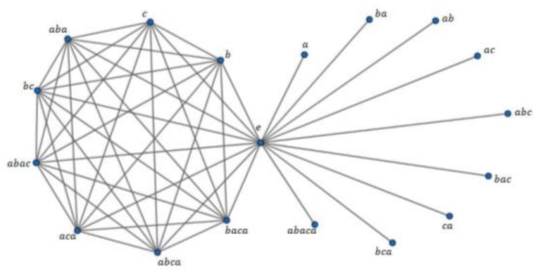
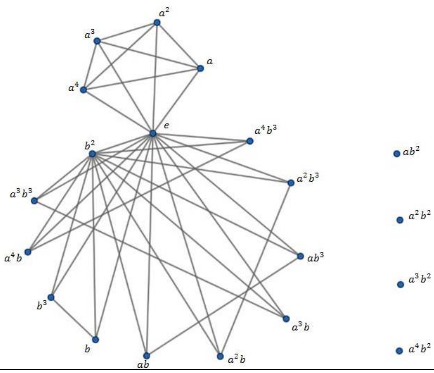
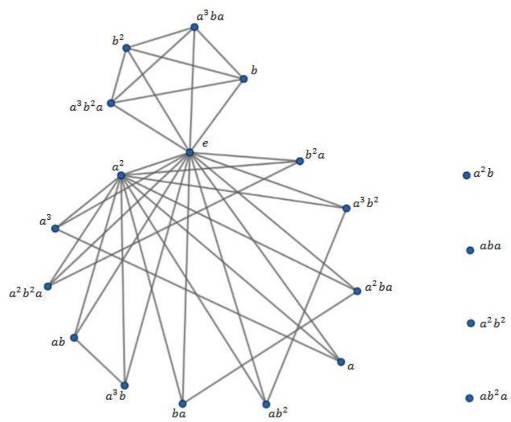
The other six elements that are not mentioned in the table are isolated vertices such as $a^2c, aca, a^2c^2, ac^2a, a^2c^3$, and ac^3a. Thus, the graph, $\Gamma^{copp}(\mathbb{Z}_7 \times \mathbb{Z}_4)$ is shown in Figure 4.1 above.

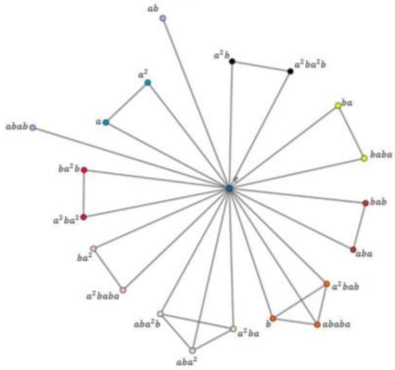
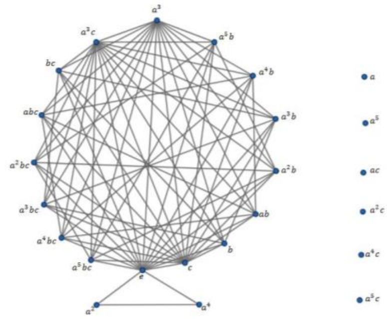
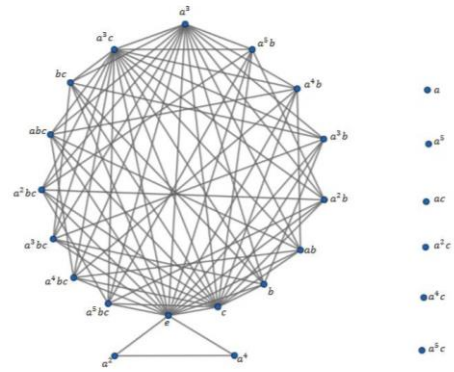
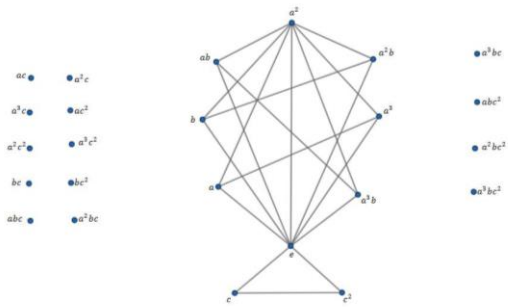
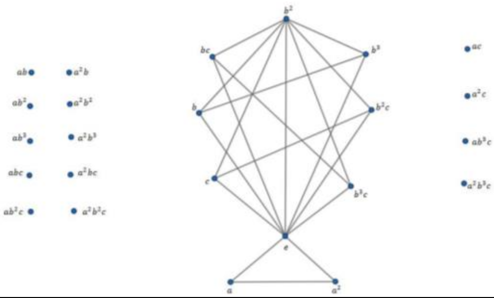
By using theorem and definition, the results of commuting order product prime graph for nonabelian metabelian groups of order at most 30 are given as follows.

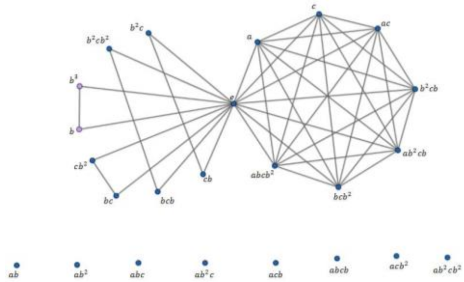
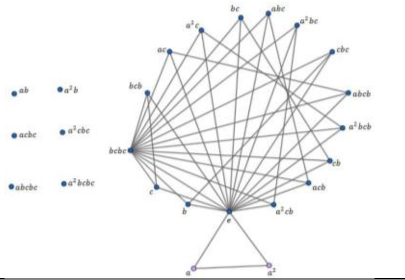
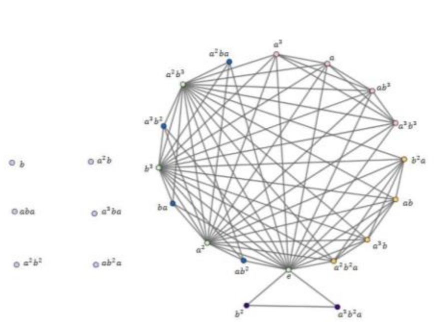
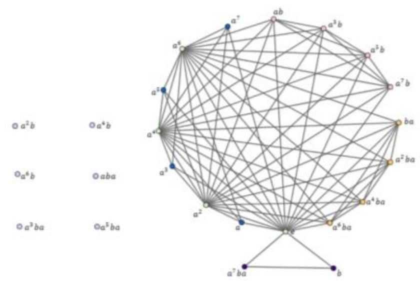
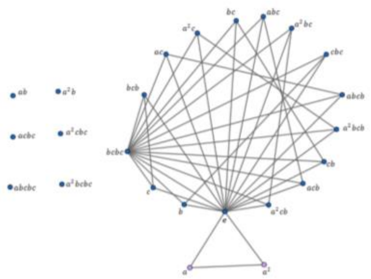
Table 3.2: Commuting Order Product Prime Graph for Nonabelian Metabelian Groups of Order At Most 30

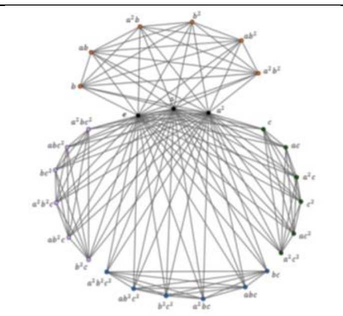
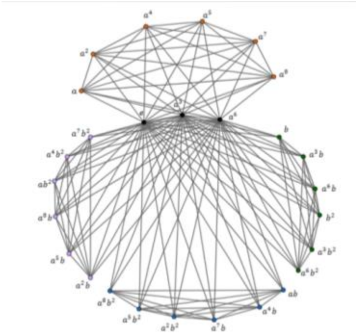
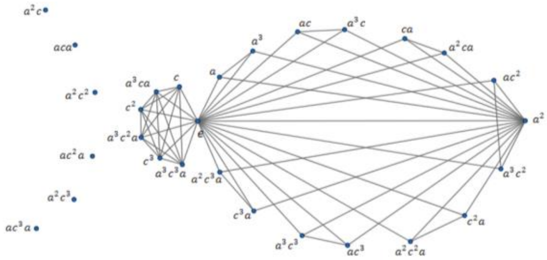
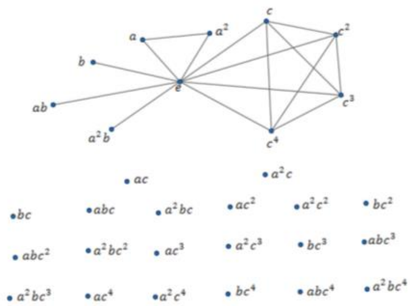
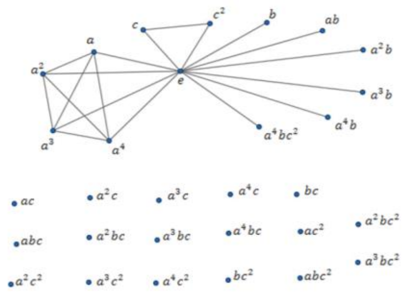
NO.	STRUCTURE	$\|G\|$	$\Gamma^{copp}(G)$
1.	D_3	6	$K_1 + (K_2 \cup \overline{K}_3)$
2.	D_4	8	$K_2 + (K_2 \cup 2K_2)$
3.	Q_4	8	$K_2 + (K_6 \cup 4K_2)$
4.	D_5	10	$K_1 + (K_4 \cup \overline{K}_5)$
5.	$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$	12	
6.	A_4	12	
7.	D_6	12	$K_1 + [K_1 + (K_0 \cup 3K_2) \cup K_2] \cup \overline{K}_2$
8.	D_7	14	$K_1 + (K_6 \cup \overline{K}_7)$
9.	D_8	16	$K_2 + (K_6 \cup 4K_2)$

10.	**Quasihedral-16**	**16**	
11.	$\boldsymbol{Q_8}$	**16**	$K_2 + (K_{14} \cup 8K_2)$
12.	$\boldsymbol{D_4 \times \mathbb{Z}_2}$	**16**	
13.	$\boldsymbol{Q \times \mathbb{Z}_2}$	**16**	
14.	$\boldsymbol{M_{16}}$ $\boldsymbol{(\mathbb{Z}_8 \rtimes \mathbb{Z}_2)}$	**16**	
15.	$\boldsymbol{B}$ $\boldsymbol{(\mathbb{Z}_4 \rtimes \mathbb{Z}_4)}$	**16**	
16.	$\boldsymbol{K}$	**16**	

17.	$\boldsymbol{G_{4,4}}$	**16**	
18.	$\boldsymbol{D_9}$	**18**	$K_1 + (K_8 \cup \overline{K}_9)$
19.	$\boldsymbol{S_3} \times \mathbb{Z}_3$	**18**	
20.	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$	**18**	
21.	$\boldsymbol{D_{10}}$	**20**	$K_1 + [K_1 + (K_0 \cup 5K_2) \cup K_4] \cup \overline{K}_4$
22.	$\boldsymbol{Fr_{20}} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$	**20**	
23.	$\mathbb{Z}_4 \times \mathbb{Z}_5$	**20**	
24.	$\boldsymbol{Fr_{21}} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_3$	**21**	

			
25.	$\boldsymbol{D_{11}}$	**22**	$K_1 + (K_{10} \cup \overline{K}_{11})$
26.	$\boldsymbol{S_3} \times \mathbb{Z}_4$	**24**	
27.	$\boldsymbol{S_3} \times \mathbb{Z}_2 \times \mathbb{Z}_2$	**24**	
28.	$\boldsymbol{D_4} \times \mathbb{Z}_3$	**24**	
29.	$\boldsymbol{Q_4} \times \mathbb{Z}_3$	**24**	
30.	$\boldsymbol{A_4} \times \mathbb{Z}_2$	**24**	

			
31.	$(\mathbb{Z}_6 \times \mathbb{Z}_2 \rtimes \mathbb{Z}_2)$	**24**	
32.	$\boldsymbol{D_{12}}$	**24**	$K_1 + [K_1 + (K_0 \cup 6K_2) \cup K_2] \cup \overline{K}_6$
33.	$\mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4)$	**24**	
34.	$\mathbb{Z}_3 \rtimes \mathbb{Z}_8$	**24**	
35.	$\mathbb{Z}_3 \rtimes \boldsymbol{Q_4}$	**24**	
36.	$\boldsymbol{D_{13}}$	**26**	$K_1 + (K_{12} \cup \overline{K}_{13})$
37.	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3$	**27**	

			
38.	$\mathbb{Z}_9 \rtimes \mathbb{Z}_3$	**27**	
39.	$\boldsymbol{D_{14}}$	**28**	$K_1 + [K_1 + (K_0 \cup 7K_2) \cup K_6] \cup \overline{K}_6$
40.	$\mathbb{Z}_7 \times \mathbb{Z}_4$	**28**	
41.	$\boldsymbol{D_{15}}$	**30**	$K_1 + [K_2 \cup K_4 \cup \overline{K}_{15}] \cup \overline{K}_8$
42.	$\boldsymbol{S_3} \times \mathbb{Z}_5$	**30**	
43.	$\boldsymbol{D_5} \times \mathbb{Z}_3$	**30**	

4 Conclusion

As a result of this research, the commuting order product prime graph of finite groups of order at most 30 and the catergorization of the graph based on the type is described.

References

[1] Abdul Rahman, S.F., *Metabelian Groups of Order at Most 24*. MSc. Dissertation, Universiti Teknologi Malaysia, November 2010.

[2] Devi, P. *Studies on Graphs Associated with Subgroups of Groups*. Ph.D. Report. Gandhigram Rural Institut-Demmed University, Tamil Nadu; 2016.

[3] Cayley, A. *The Theory of groups: Graphical Representation.* American Journal of Mathematics. 1878. 1(2): 174-176.

[4] Sattanathan, M. and Kala , R. An *Introduction to Order Prime Graph.* Int. J. Contemp. Math. Sciences. 2009. 4(10): 467-474.

[5] Rajendra, R. and Reddy, K. S. P. *On General Order Prime Graph of Finite Group*. Proceedings of Jangjeon Mathematical Society. 2014. 17(4): 641-644.

[6] Newmann, B. H. *A Problem of Paul Erdos on Groups.* J. Austral. Math. Soc. 21 (Series A). 1976. 21(4): 467-472.

[7] Aschbacher, M. Simple *Connectivity of p-Group Complexes*. Israel Journal of Mathematics. 1993. 82: 1-43.

[8] Muhammed Bello. *Order Product Prime Graph and Its Variations of Some Finite Groups*. Ph.D. Report, Universiti Teknologi Malaysia, 2021.

[9] Simon, N. L. Metabelian Groups of Order at Most 32. Undergraduate Project Report. Universiti Teknologi Malaysia. 2017.

[10] Mohd Zulkefli, M. R. *The Order Product Prime Graph of Nonabelian Metabelian Groups of Order at Most 24.* Universiti Teknologi Malaysia: Undergraduate Project Report. 2022.

[11] Sur, E. and Srimani, P. K. *Topological properties of star graphs*. Computers math. Applic. 1993. 25(12). 87-98.