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Dynamics of Quasi-Volterra Quadratics Stochastics Operations on 1-Dimesional Simplex

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Abstract
The purpose of this study is to investigate the dynamics of quasi-Volterra quadratics stochastics operators (QSO) on 1-dimensional simplex. The quasi-Volterra QSO is classified into two types, that is, Type and Type 2. In both cases, the Jacobian technique is used to describe their fixed point(s). The stability of the fixed point(s) is shown. The behavior of the fixed point of an operator can be classified into three parts, namely, attracting, repelling and saddling. For Type 1, fixed points of the function $f(x) = -0.26x^2 + 1.02x + 0.24$ are x=1 and x=0.9231. Fixed points for $f(x) = 0.63x^2 - 0.36x + 0.73$ are x=1 and x=1.1587. While, $f(x) = 1.38x^2 - 1.32x + 0.94$ have fixed points x=1 and x=0.6812. Lastly, fixed points of $f(x) = -0.23x^2 + 1.08x + 0.15$ are x=1 and x=-0.6527. For Type 2, fixed points of the function $f(x) = -0.1x^2 + 0.4x$ are x=0 and x=-6. For $f(x) = -0.3x^2 + 1.2x$ have fixed points x=0 and x=0.6667. While, fixed points of $f(x) = -0.9x^2 + 1.6x$ are x=0 and x=0.6667. Furthermore, the function $f(x) = 0.3x^2 + 0.2x$ have fixed points x=0 and x=0.2667. The Maple 13 is used in order to study the dynamical behaviours of the operators. The trajectories of each point are analysed. The trajectories are shown to be converged to the attracting fixed point which describe dynamics of the quasi-Volterra QSO.

Keywords: quasi-Volterra quadratics stochastics operators; Maple

1. Introduction

Nonlinear operators can be used to explain a wide variety of systems. The quadratic example is among the simplest nonlinear instances. It has been demonstrated that quadratic dynamical systems are an excellent source of analysis for the examination of dynamical properties and modelling in a variety of fields, including population dynamics, physics, economy, and mathematics.

Lyubich et al. (1992) noted that uses of QSO on population genetics were in addition to Bernstein's work. The following scenario can be used to identify the species time development. We will define x((0))=(x 1((0)),..., x n((0))) as the probability distribution of the species in an early state of that population. Let I = 1, 2, n be the n kind of species in a population. P_ is the likelihood that an individual from the ith and jth species will cross-fertilize and give rise to an individual from the kth species (ij,k). Given x((0))=(x 1((0)),...,x n((0))), we may apply QSO as a total probability to find the probability distribution of the first generation, x((1))=(x 1((1)),...,x n((1))).

Additionally, according to Ganikhodzhaev et al. (2017), each QSO defines an algebraic structure termed genetic algebra on the vector space Rn that contains the simplex. According to Lyubich (1971), QSO in genetic algebra generally generated commutative and non-associative results. The space of all derivations is a Lie algebra with the commutator multiplication, therefore keep in mind that for any algebra. The Lie algebra of a given algebra's derivations is one of the key tools for understanding its structure in genetic algebra, along with the idea of non-associative algebras. Additionally, multiplication is clearly defined in terms of derivations, demonstrating the importance of derivations in genetic algebras based on a number of studies about genetic algebra derivations (Costa,1982; Costa,1983; Gonshor,1988). As demonstrated by Micali and Revoy in 1986, the multiplication is well-defined in terms of derivations, showing the significance of derivations in genetic algebras.

Because studying QSOs is difficult in general (unlike studying linear operators), researchers are likely to create classes of QSOs such Volterra-QSO, permutated Volterra-QSO, Quasi-Volterra QSO, Volterra-QSO, strictly non-Volterra-QSO, F-QSO, and non-Volterra (see R. Ganikhodzhaev et al., 2011). There are numerous classes of QSOs that can be defined and researched because all these classes do not encompass the entire set of QSOs.

Since the quadratic stochastic operators are specified on a 1-dimensional simplex, we will limit our analysis to n=2 and examine the dynamics of Quasi-Volterra quadratic stochastic operators in this setting. Given that S 1 is compact, the fixed-point theorem states that such a mapping has at least one fixed point. So, we'll search for every fixed point of the operators we're considering that might be a candidate for an operator's trajectory. Using the Jacobian technique, we will examine the fixed point's stability.

2. Literature Review

Let V be a mapping on the $(n-1)$ dimensional simplex

$$S^{(n-1)} = \{\mathrm{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n | x_i \geq 0, \sum_{i=1}^{n} x_i = 1\},$$

maps into itself, V: $S^{n-1} \to S^{n-1}$. V has such a form

$$V: \mathrm{x}'_k = \sum_{i,j=1}^{n} P_{ij,k} x_i x_j, \ \ k = 1,2, \dots, n$$

where $P_{ij,k}$ are coefficient of heredity and

$P_{ij,k} \geq 0, \qquad P_{ij,k}, \qquad \sum_{k=1}^{n} P_{ij,k} = 1 \qquad i,j,k = 1,2, \dots, n$

Then, V is called Quadratic Stochastic Operator (QSO).
Let V be a QSO. Say that a QSO is quasi-Volterra if there exist k_0 such that $P_{ij,k_0} > 0$ for $k_0 \neq \{i,j\}$ and $k_1$such that $P_{ij,k_1} = 0$ for $k_1 \neq \{i,j\}$. In this project, the case n=2 is considered.

When n=2 then there are several possibilities of quasi-Volterra QSO, namely:

i. Type 1, $k_0 = 1, \ k_1 = 2$
$$P_{22,1} > 0, \qquad P_{11,2} = 0$$

ii. Type 2, $k_0 = 2, \ k_1 = 1$
$$P_{11,2} > 0, \qquad P_{22,1} = 0$$

In case of *n*=2, $k_0 \neq k_1$.

3. Materials and methods

The definition of fixed point will be used consistently throughout our study. Therefore, in order to accomplish this goal, we will solve V("x") = " analytically. Additionally, iterating the function and analytical analysis are required to examine the Quasi-Volterra dynamic. Additionally, the Jacobian methodology will be used to study the stability of the fixed point using several common linear algebraic methods.

Definition 3.1 Let f be mapping from set X to set X again. If any c ∈ X and f(c) = c then c is a fixed point of f.

Definition 3.2 A fixed point x_0 for $F: \mathbb{R}^n \to \mathbb{R}^n$ is called hyperbolic if all the eigenvalues of the Jacobian matrix J of the mapping F at the point x_0 are not equal to 1.

There are three types of hyperbolic fixed points:

(a) P is an attracting fixed point if all of the eigenvalues of J(P) are less than one in absolute value.

(b) P is a repelling fixed point if all of the eigenvalues of J(P) are greater than one in absolute value.

(c) P is a saddle point otherwise. In this project, we are going to consider n = 2. Therefore, ℓ ∈ {1,2}. In what follows, we consider 1-Volterra and 2-Volterra.

Since S^1 is compact, then by fixed point theorem there exist at least one fixed point for such mapping.

4. Results and discussion

Quasi-Volterra Type 1

In this section, we are going describe the canonical for of Volterra acting on one dimensional simplex where n= 2, quasi-Volterra can form

$$V(\mathrm{x})_1 = (1 + b - 2a)x_1^2 + (2a - 2b)x_1 + b.$$

$$V(\mathrm{x})_2 = 1 - ((1 + b - 2a)x_1^2 + (2a - 2b)x_1 + b).$$

Fixed Point 4.2.1

The following proposition describe the quasi-Volterra Type 1.

Proposition 4.1 Let f be given by (4.8). Then, the following statement holds:

i) $x = 1$ is always the fixed point of f.

ii) The fixed-point $x = \frac{b}{1+b-2a}$ belongs to [0,1] if $2a - b \leq 1$ and $a \leq \frac{1}{2}$.
The following theorem describe the quasi-Volterra Type 1.

Theorem 4.1: Let V be a Quasi-Volterra Type 1. Then the points are

(i) (1,0)

(ii) $(\frac{b}{1+b-2a}, \frac{1-2a}{1+b-2a})$ if $2a - b \leq 1$ and $a \leq \frac{1}{2}$.

4.2.2 The stability of the fixed points

Proposition 4.2 Let V be a Quasi Volterra Type1. The following statement holds:

i) If $a \in (\frac{1}{2}, 1]$, then the fixed point is attracting.

ii) If $a \in [0, \frac{1}{2})$, the fixed point is repelling.

iii) If $a = \frac{1}{2}$, it called non-hyperbolic fixed point.

Proposition 4.3 The following statement holds:

i) If $0 \leq a \leq \frac{1}{2}$, then the fixed point $(\frac{b}{1+b-2a}, 1 - \frac{b}{1+b-2a})$ is attracting.

ii) On condition that $a = \frac{1}{2}$, it called non-hyperbolic fixed point.

4.2.3 The Dynamic of Quasi-Volterra Type 1

In this section, we are going to consider several examples of Quasi-Volterra QSO. We will specify concrete value of a and b. Then by using Maple 13 and RStudio Software, we compute the trajectories and the graph of the considered operators.

For Type 1, the function $f(x)$ = −0.26x 2 + 1.02x + 0.24 (a=0.75, b=0.24) was converge to x=1 in Figure 4.2 and 4.3. For $f(x)$ = 0.63x 2 − 0.36x + 0.73 (a=0.55, b=0.73) was converge to both fixed points in Figure 4.5 and 4.6. While, $f(x)$ = 1.38x 2 − 1.32x + 0.94 (a=0.28, b=0.94) was converge at x=0.6812 in Figure 4.8 and 4.9. Furthermore, function $f(x)$ = −0.23x 2 + 1.08x + 0.15 (a=0.69, b=0.15) was converge to x=1 in Figure 4.11 and 4.12.

4.3 Quasi-Volterra Type 2

In this section, we are going describe the canonical for of Volterra acting on one dimensional simplex where n= 2, quasi-Volterra can form

$$V(\mathrm{x}) = (1 - a - 2b)x_1^2 + 2bx_1$$

$$V(\mathrm{x})_2 = 1 - (1 - a - 2b)x_1^2 + 2bx_1.$$

4.3.1 Fixed Point

The following proposition describe the quasi-Volterra Type 2.

Proposition 4.4 Let f be given by (4.13). Then, the following statement holds:
- i) $x = 0$ is always the fixed point of f.
- ii) The fixed-point $x = \frac{-2b+1}{1-a-2b}$ belongs to [0,1] if $2b + a \geq 1$ and $b \geq \frac{1}{2}$.

Theorem 4.2: Let V be a Quasi-Volterra Type 2. Then the points are
- (i) (0,1)
- (ii) $(\frac{-2b+1}{1-a-2b}, \frac{-a}{1-a-2b})$ if $2b + a \geq 1$ and $b \geq \frac{1}{2}$.

4.3.2 The stability of the fixed points

The next proposition fully describes the fixed point of f given by (4.14) that is inside [0,1].

Proposition 4.5 The following statement holds:
- i) If $b \in [0, \frac{1}{2})$ then (0,1) attracting.
- ii) If $b \in (\frac{1}{2}, 1]$ then (0,1) repelling.
- iii) If $b = \frac{1}{2}$ then (0,1) non-hyperbolic fixed point.

Proposition 4.6 The following statement holds:
- i) If $\frac{1}{2} < b \leq 1$, then the fixed point $(\frac{-2b+1}{1-a-2b}, \frac{-a}{1-a-2b})$ is attracting.
- ii) If $b = \frac{1}{2}$, the fixed point $(\frac{-2b+1}{1-a-2b}, \frac{-a}{1-a-2b})$ is non-hyperbolic.

4.3.3 The Dynamic of Quasi-Volterra Type 2

In this section, we are going to consider several examples of Quasi-Volterra QSO. We will specify concrete value of a and b. Then by using Maple 13 and RStudio Software, we compute the trajectories and the graph of the considered operators.

For Type 2, the function $f(x)$ = −0.1x 2 + 0.4x(a=0.7, b=0.2) was converge at x=0 in Figure 4.14 and 4.15. For $f(x)$ = −0.3x 2 + 1.2x (a=0.1, b=0.6) was converge at x=0.6667 in Figure 4.17 and 4.18. While, $f(x)$ = −0.9x 2 + 1.6x (a=0.3, b=0.8) was 74 converge to x=0.6667 in Figure 4.20 and 4.21. Furthermore, function $f(x)$ = 0.3x 2 + 0.2 converge to x=0.6667 in Figure 4.20 and 4.21. Furthermore, function $f(x)$ = 0.3x 2 + 0.2x (a=0.5, b=0.1) was converge to x= 0 in Figure 4.23 and 4.24.

Conclusion

In this study, the Quasi-Volterra Quadratic Stochastic Operators on 1-Dimensional Simplex and its dynamical behavior is investigated. The fixed points of QSOs on one-dimensional simplex are also studied. The stability of fixed point by using Jacobian technique has been investigated.

Quasi-Volterra quadratic stochastic operators (QSOs) take into account the Type 1 and Type 2 dynamics of Quasi-Volterra QSO on 1-D simplex (2.1). The case n=2 was the only option in this investigation.

The fixed point as defined in definition see (3.2) had been proven either accepted or rejected by considering functions 4.9 and 4.14. For type 1 and type 2 Quasi-Volterra QSO, a few fixed points have been calculated (s).

For Type 1, the function $f(x) = -0.26x^2 + 1.02x + 0.24$ have fixed points x=1 and x=-0.9231. For $f(x) = 0.63x^2 - 0.36x + 0.73$ have fixed points x=1 and x=1.1587. While, $f(x) = 1.38x^2 - 1.32x + 0.94$ have fixed points x=1 and x=0.6812. Furthermore, function $f(x) = -0.23x^2 + 1.08x + 0.15$ have fixed points x=1 and x=-0.6527.

For Type 2, the function $f(x) = -0.1x^2 + 0.4x$ have fixed points x=0 and x=-6. For $f(x) = -0.3x^2 + 1.2x$ have fixed points x=0 and x=0.6667. While, $f(x) = -0.9x^2 + 1.6x$ have fixed points x=0 and x=0.6667. Furthermore, function $f(x) = 0.3x^2 + 0.2x$ have fixed points x=0 and x=2.6667.

Based on Proposition 4.2, if $a \in (\frac{1}{2}, 1]$, then the fixed point is attracting. Uncertainty $a \in [0, \frac{1}{2})$, the fixed point is repelling. On condition that $a = \frac{1}{2}$, it called non-hyperbolic fixed point. Besides, Proposition 4.3 describe that if $0 \leq a \leq \frac{1}{2}$, then the fixed point $(\frac{b}{1+b-2a}, 1 - \frac{b}{1+b-2a})$ is attracting. Furthermore, Proposition 4.5 stated that if $b \in [0, \frac{1}{2})$ then (0,1) attracting. Considering, $b \in (\frac{1}{2}, 1]$ then (0,1) repelling. Hence, if $b = \frac{1}{2}$ then (0,1) non-hyperbolic fixed point. Based on Proposition 4.6, if $\frac{1}{2} < b \leq 1$, then the fixed point $(\frac{-2b+1}{1-a-2b}, \frac{-a}{1-a-2b})$ is attracting.

In this last section, we studied the dynamic of Quasi-Volterra type 1 and type 2 by considering several examples. Value of a and b has been specified to compute the trajectories and the graph of the considered operators.

Reference

[1] Bernstein, S. N. (1924). Solution of a mathematical problem related to the theory of inheritance. Uch. Zap. n.-i. kaf. Ukrainy, 1, 83-115

[2] Camacho L M, Gmez J R,Omirov B A and Turdibaev R M (2013) Linear and Multilinear Algebra 61, no.309-322.

[3] Costa, R. (1982). On the derivations of gametic algebras for polyploidy with multiple alleles. Boletim da Sociedade Brasileira de Matemca, 13(2), 69-81. doi: 10.1007/bf02584678

[4] Costa, R. (1983). On the derivation algebra of zygotic algebras for polyploidy with multiple alleles. Boletim da Sociedade Brasileira de Matemca, 14(1), 63-80. doi: 10.1007/bf02584745

[5] Ganikhodjaev, N, Hisamuddin H. H., (2008) Associativity in inheritance or are there associative populations. Malays. J. Sci. 27(2), 131?136.

[6] Ganikhodzhaev, R. N. (1993). On the definition of bistochastic quadratic operators. Russian Mathematical Surveys, 48(4), 244-246. doi:0.1070/rm1993v048n04abeh001058

[7] Ganikhodzhaev, Rasul & Mukhamedov, Farrukh & Pirnapasov, Abror & Qaralleh, Izzat. (2017). Genetic Volterra algebras and their derivations. Communications in Algebra https://doi.org/10.1080/00927872.2017.1347663

[8] Ganikhodzhaev, R., Mukhamedov, F., Saburov, M. (2012). G-decompositions of matrices and related problems I. Linear Algebra and its Applications, 436(5), 1344-1366.

[9] Goel, N. S., Maitra, S. C., Montroll, E. W. (1971). On the Volterra and other nonlinear models of interacting populations. Reviews of modern physics, 43(2), 231.

[10] Gonshor, H. (1988). Derivations in genetic algebras. Communications in Algebra, 16(8), 1525-1542. doi: 10.1080/00927879808823643

[11] Hardy, G. H., Littlewood, J. E., & Polya, G. (1952). Inequalities cambridge univ. Press, Cambridge, (1988).

[12] Hofbauer, J., Hutson, V., Jansen, W. (1987). Coexistence for systems governed by difference equations of Lotka-Volterra type. Journal of Mathematical Biology, 25(5), 553-570.

[13] Hofbauer, J., Sigmund, K. (1998). Evolutionary games and population dynamics. Cambridge University Press. Lotka, A. J. (1920). Undamped oscillations derived from the law of mass action. Journal of the American Chemical Society, 42(8), 1595-1599.

[14] Holmgren, R. A. (2000). A first course in discrete dynamical systems. Springer Science & Business Media.

[15] Lyubich, Y. I. (1971). Basic Concepts and Theorems of the Evolutionary Genetics of Free Populations. Russian Mathematical Surveys, 26(5), 51-123. doi: 10.1070/rm1971v026n05abeh003829

[16] Lyubich, Y. I., Vulis, D., Karpov, A., Akin, E. (1992). Mathematical structures in population genetics. Biomathematics (Berlin).

[17] Marshall, AW, Olkin, I, Arnold, BC (2011). Inequalities: Theory of Majorization and Its Applications. Springer, Berlin.

[18] May, R. M. (1976). Simple mathematical models with very complicated dynamics. Nature, 261(5560), 459-467.

[19] May, R. M., Oster, G. F. (1976). Bifurcations and dynamic complexity in simple ecological models. American Naturalist, 573-599.

[20] Micali A, Revoy P (1986) Proc. Edinburgh Math. Soc. 29 (2) 187197.

[21] Mukhamedov, F., Embong, A. F. (2015). On b-bistochastic quadratic stochastic operators. Journal of Inequalities and Applications, 2015(1). doi: 10.1186/s13660-015-0744-y

[22] Mukhamedov, F., Qaralleh, I. (2014). On Derivations of Genetic Algebras. Journal of Physics: Conference Series, 553, 012004. doi:10.1088/1742- 6596/553/1/012004

[23] Narziev N. B., Ganikhodzhaev R. N., Shahidi F. (2012) Algebraic structures of Volterra type population genetics, In book: Proc. 2nd Inter. Conf. Math. Appl. in Engin. IIUM Press, pp.410?416.

[24] Parker, D. S., Ram, (1996) P: Greed and Majorization. Computer Science Department, University of California, Los Angeles

[25] Plank, M., Losert, V. (1995). Hamiltonian structures for the n-dimensional Lotka Volterra equations. Journal of Mathematical Physics, 36(7), 3520-3534.

[26] Qaralleh, I., Mukhamedov, F. (2019). Volterra evolution algebras and their graps. Linear and Multilinear algebra. doi:10.1080/03081087.2019.1664387

[27] Qaralleh, I., Ahmad, M. Z. & Alsarayreh, A. (2016). Associative and derivation genetic algebra generated from ξ(s)-QSO. AIP Conference Proceedings 1775. 030060. Doi: 10.1063/1.4965180

[28] Reed, M. (1997). Algebraic structure of genetic inheritance. Bulletin of The American Mathematical Society - BULL AMER MATH SOC. 34. 107-131. Doi: 10.1090/S0273-0979-97-00712-X.

[29] Takeuchi, Y. (1996). Global dynamical properties of Lotka-Volterra systems. Singapore: World Scientific.

[30] Ulam, S. M. (1960). A collection of mathematical problems. New York, 29. Vallander, S. S (1972). On the limit behavior of iteration sequence of certain quadratic transformations. In Soviet Math. Doklady (Vol. 13, pp. 123-126).