Proceedings of Science and Mathematics
Faculty of Science, Universiti Teknologi Malaysia
Vol. 16, 2023, page 263 – 273

The Solution of Fuzzy Matrix Equation System

Rosimah Rosli, Amidora Idris*
Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia
*Corresponding author: amidora@utm.my

Abstract
The solution of the fuzzy matrix equation system is investigated in this paper and requires further research in matrix theory and fuzzy analysis areas such as fuzzy set theory, fuzzy numbers, fuzzy matrix, and fuzzy linear system. This paper considers a square fuzzy matrix equation system with a crisp coefficients matrix, arbitrary triangular fuzzy numbers on the right side, and unknown fuzzy numbers that are solved using the embedded method and Friedman's proposed technique. These methods are used to present the solutions of fuzzy linear system and fuzzy matrix equation system and determine the types of solutions based on definitions.

Keywords: Fuzzy numbers; Fuzzy system of linear equation; Fuzzy linear system; Fuzzy matrix equation system

1. Introduction

The idea of uncertainty is one of the issues raised by the numerous paradigms shifts in science, mathematics, and engineering in the twenty-first century. In the late 20th century, the first phase of the shift from a traditional to a modern perspective on uncertainty started. In 1965, Lotfi A. Zadeh made a significant contribution to the modern understanding of uncertainty by establishing a theory known as the fuzzy set which is a set with ambiguous bounds [1]. Instead of affirmation or denial, the degree of the membership function determines membership in a fuzzy collection.

Zadeh [2, 3] in 1975, Dubois with Parade and Nahmias [4] in 1978 were the first to explore the concept of fuzzy numbers and the arithmetic operations that can be performed on them. Puri and Ralescu [5] as well as Goetschell [6] and Wu and Ming [7, 8] presented a novel approach for coping with fuzzy numbers and the structure of fuzzy numbers spaces.

At least some of the system's parameters are often portrayed as fuzzy than crisp numbers in numerous applications. Therefore, developing a numerical approach that can treat and solve general fuzzy linear systems is crucial. Light of Friedman [9] in 1998 proposed an embedded method general model for solving fuzzy linear systems with a crisp coefficients matrix and an arbitrary fuzzy numbers vector on the right side. Numerous studies have been conducted on how to deal with more advanced forms of fuzzy linear systems including dual fuzzy linear systems [10, 11, 12], general fuzzy linear systems [13, 14, 15, 16], fully fuzzy linear systems [17, 18, 19], dual fully fuzzy linear systems [20, 21] and general dual fuzzy linear systems [22].

However, few studies have been conducted in decades for a fuzzy matrix equation which is frequently utilized in control theory and control engineering. Zengtai and Xiaobin in 2010 used the block Gaussian elimination method and the undertermined coefficients method to investigate a classification of fuzzy matrix equations which is a crisp matrix and the right-side matrix is a fuzzy numbers matrix.

Zengtai and Xiaobin also adopted the generalized inverses to study least squares solutions of the inconsistent fuzzy matrix equation in 2011 [23].

In this paper represents the solution to solving $n \times n$ fuzzy matrix equation system with crisp coefficient matrix, A, the unknown fuzzy numbers matrix, $\tilde{X}$ and the right-side arbitrary triangular fuzzy numbers matrix, $\tilde{Y}$. This system can be denoted by $A\tilde{X} = \tilde{Y}$ and solve the system by using embedded method and Friedman's proposed technique in [10] by replacing the $n \times n$ fuzzy matrix equation into $2n \times 2n$ crisp matrix equation. Lastly, investigates the types of solution of fuzzy matrix equation system.

Preliminaries
In the following represents some preliminaries that can be used in this paper.

Definition 1: Semi-continuous[24]
Let $x \subseteq \mathbb{R}$. The function $f: X \to \mathbb{R}$ is upper semi-continuous if for every $a \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - a| < \delta \text{ for } a, x \in X$$

then

$$f(x) < f(a) + \varepsilon$$

Definition 2: Fuzzy set[25]
Let U be universal set, then the fuzzy subset $\tilde{A}$ in U is defined by its membership function $\mu_{\tilde{A}}(x): U \longrightarrow [0,1]$ which assign a real number $\mu_{\tilde{A}}(x)$ in the interval $[0, 1]$ to each member element $x \in U$, where the value of $\mu_{\tilde{A}}(x)$ at x shows the grade of membership of x in $\tilde{A}$. A fuzzy set $\tilde{A}$ in U may be represented as a set of ordered pairs of a generic element x and its membership value,

$$\tilde{A}(x) = \{x, \mu_{\tilde{A}}(x) \mid x \in U\}$$

Definition 3: $\boldsymbol{\alpha}$ −cut[25]
Given a fuzzy set $\tilde{A} \subseteq U$ and any real number $a \in [0,1]$, then the α −cut also can call as α −level or cut worthy set of $\tilde{A}$ denoted by $A_\alpha \subseteq U$ is the crisp set

$$A_\alpha = \{x \in U \mid \mu_{\tilde{A}}(x) \geq \alpha\}$$

Definition 4: Core of fuzzy set[25]
The core of fuzzy set $\tilde{A}$ is the of all points $x \in U$ such that

$$core(\tilde{A}) = \{x \in U \mid \mu_{\tilde{A}}(x) = 1\}$$

Definition 5: Convex of fuzzy set[24]
A fuzzy set $\tilde{A}$ is convex if and only if for $x_1, x_2 \in U,\ \lambda \in [0,1]$

$$\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq \min[\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)]$$

Alternatively, $\tilde{A}$ is convex if all its α −cut level set are convex.

Definition 6: Normal of fuzzy set[24]
A fuzzy set is normal if its core is non-empty. In other words, there exists at least one point $x \in U$ such that

$$\mu_{\tilde{A}}(x) = 1$$

Definition 7: Support of fuzzy set[25]
The support of a fuzzy set $\tilde{A}\ of\ U, supp(\tilde{A})$ is the crisp of all $x \in U$ defined as

$$supp(\tilde{A}) = \{x \in U \mid \mu_{\tilde{A}}(x) > 0\}$$

Definition 8: Fuzzy Numbers[26]

A fuzzy number is a function such as $u: \mathbb{R} \to [0,1]$ satisfying the following properties

i. u is normal that is there exists $x_0 \in \mathbb{R}$ such that $u(x_0) = 1$
ii. u is fuzzy convex which $u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}$ for any $x, y \in \mathbb{R}$ for $\lambda \in [0,1]$
iii. u is semicontinuous
iv. $supp\ (u) = \{x \in \mathbb{R} \mid u(x) > 0\}$ is the support of u and its closure $cl(supp\ u)$ is compact

Definition 9: Parametric form on fuzzy numbers[27]

An arbitrary fuzzy numbers in parametric form is represented by ordered pairs of functions $\left(\underline{v}(r), \overline{v}(r)\right), r \in [0,1]$ which satisfy the following requirements

i. $\underline{v}(r)$ is a bounded monotonic increasing left continuous function over [0,1]
ii. $\overline{v}(r)$is bounded monotonic decreasing left continuous function over [0, 1]
iii. $\underline{v}(r) \leq \overline{v}(r), r \in [0,1]$

The set of all these fuzzy numbers is denoted by E^1.

Definition 10: Triangular fuzzy numbers[27]

A fuzzy number represented with three points, $A = (a_1, a_2, a_3)$ is interpreted as membership function as follows

$$\mu_A(x) = \begin{cases} 0, & x \leq a_1 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2}, & a_2 \leq x \leq a_3 \\ 0, & x \geq a_3 \end{cases}$$

The conversion triangular fuzzy number to interval using r –cut by letting $\tilde{b} = (t_1, t_2, t_3)$ and gives formula

$$\tilde{b} = (t_1, t_2, t_3) = [t_1 + (t_2 - t_1)r, t_3 - (t_3 - t_2)r]$$

2. Methodology

In the following represents the existing methods that will be used to solve fuzzy matrix equation system.

Definition 11: Fuzzy matrix equation system[28]

The $n \times n$ fuzzy matrix equation is called fuzzy matrix equation system denoted by $A\tilde{X}=\tilde{Y}$ as follows

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & & \tilde{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{x}_{n1} & \tilde{x}_{n2} & \cdots & \tilde{x}_{nn} \end{pmatrix} = \begin{pmatrix} \tilde{y}_{11} & \tilde{y}_{12} & \cdots & \tilde{y}_{1m} \\ \tilde{y}_{21} & \tilde{y}_{22} & & \tilde{y}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{y}_{n1} & \tilde{y}_{n2} & \cdots & \tilde{y}_{nm} \end{pmatrix} \quad (1)$$

$$(a_{ij})_{n\times n}(\tilde{x}_{ij})_{n\times n} = (\tilde{y}_{ij})_{n\times n}$$

where $A = (a_{ij}) = n \times n$ crisp matrix, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$

$\tilde{X} =$ Unknown $n \times n$ fuzzy numbers matrix $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$

$\tilde{Y} = \tilde{y}_{ij} = n \times n$ fuzzy numbers matrix, $i = 1, 2, \dots, n, j = 1, 2, \dots, n$

Definition 12: Embedded method[29]

For an arbitrary fuzzy number $\tilde{x}$in parametric form the embedding $\pi: \mathbb{R}^2 \to \mathbb{R}^2$ is defined as follows

$$\pi\left(\underline{x}(r),\overline{x}(r)\right)=\left(\overline{x}(r)-\underline{x}(r),\overline{x}(r)+\underline{x}(r)\right)$$

Lemma[29]

Let $\tilde{x}=\left(\underline{x}(r),\ \overline{x}(r)\right)$ and $\tilde{y}=\left(\underline{y}(r),\ \overline{y}(r)\right)$ are arbitrary fuzzy numbers for and let k is a real number. Then,

i. $\tilde{x}=\tilde{y}$ if and only if $\underline{x}(r)=\underline{y}(r)$ and $\overline{x}(r)=\overline{y}(r)$
ii. $(\tilde{x}+\tilde{y})=\left(\underline{x}(r)+\underline{y}(r),\ \overline{x}(r)+\overline{y}(r)\right)$
iii. $k\tilde{x}=\begin{cases}(k\underline{x}(r),k\overline{x}(r)) & \geq 0\\ (k\overline{x}(r),k\underline{x}(r)) & <0\end{cases}$

Proof:

i. If $\tilde{x}=\tilde{y}$, then becomes $\pi(\tilde{x})=\pi(\tilde{y})$ by multiply both side with π.
then, $\pi(\underline{x}(r),\ \overline{x}(r))=\pi(\underline{y}(r),\ \overline{y}(r))$
Thus, $\underline{x}(r)=\underline{y}(r)$ and $\overline{x}(r)=\overline{y}(r)$

If $\underline{x}(r)=\underline{y}(r)$ and $\overline{x}(r)=\overline{y}(r)$, then also can be written as $\left(\underline{x}(r),\ \overline{x}(r)\right)=\left(\underline{y}(r),\ \overline{y}(r)\right)$
Therefore, $\tilde{x}=\tilde{y}$,

ii. $(\tilde{x}+\tilde{y})=\left(\left(\underline{x}(r),\ \overline{x}(r)\right)+\left(\underline{y}(r),\ \overline{y}(r)\right)\right)=\left(\underline{x}(r)+\underline{y}(r),\overline{x}(r)+\overline{y}(r)\right)$

iii. Let $k\geq 0,\ k(\tilde{x})$
Since $\tilde{x}=\left(\underline{x}(r),\ \overline{x}(r)\right)$, then $k\left(\underline{x}(r),\ \overline{x}(r)\right)=\left(k\underline{x}(r),\ k\overline{x}(r)\right)$

Let $k<0,-k(\tilde{x})$
Since $\tilde{x}=\left(\underline{x}(r),\ \overline{x}(r)\right)$, then $-k\left(\underline{x}(r),\ \overline{x}(r)\right)=\left(-k\underline{x}(r),-k\overline{x}(r)\right)=\left(k\overline{x}(r),\ k\underline{x}(r)\right)$

Friedman's proposed technique[10]

In 1998, Menahem Friedman, Ma Ming and Abraham Kandel proposed a technique for solving $n\times n$ fuzzy linear system with a crisp coefficients matrix and arbitrary fuzzy number vector in right side by using embedded method and extend the $n\times n$ fuzzy linear system to $2n\times 2n$ crisp linear system. Applied the Lemma and Friedman's proposed method to solve fuzzy matrix equation system by replacing $n\times n$ fuzzy matrix equation system, $A\tilde{X}=\tilde{Y}$ which is equation (1) in Definition 11 into the $2n\times 2n$ crisp matrix system, $S\tilde{X}=\tilde{Y}$ as follows

$$S\tilde{X}=\tilde{Y} \tag{2}$$

$$\begin{pmatrix} S_{1,1} & S_{1,2} & \cdots & S_{1,n} & S_{1,n+1} & S_{1,n+2} & \cdots & S_{1,2n} \\ S_{2,1} & S_{2,2} & \cdots & S_{2,n} & S_{2,n+1} & S_{2,n+2} & \cdots & S_{2,2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ S_{n,1} & S_{n,2} & \cdots & S_{n,n} & S_{n,n+1} & S_{n,n+2} & \cdots & S_{n,2n} \\ S_{n+1,1} & S_{n+1,2} & \cdots & S_{n+1,n} & S_{n+1,n+1} & S_{n+1,n+2} & \cdots & S_{n+1,2n} \\ S_{n+2,1} & S_{n+2,2} & \cdots & S_{n+2,n} & S_{n+2,n+1} & S_{n+2,n+2} & \cdots & S_{n+2,2n} \\ \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \vdots \\ S_{2n,1} & S_{2n,2} & \cdots & S_{2n,n} & S_{2n,n+1} & S_{2n,n+2} & \cdots & S_{2n,2n} \end{pmatrix} \begin{pmatrix} \underline{x}_{11} & \underline{x}_{12} & \cdots & \underline{x}_{1n} \\ \underline{x}_{21} & \underline{x}_{22} & \cdots & \underline{x}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \underline{x}_{n1} & \underline{x}_{n2} & \cdots & \underline{x}_{nn} \\ -\overline{x}_{11} & -\overline{x}_{12} & \cdots & -\overline{x}_{2n} \\ -\overline{x}_{21} & -\overline{x}_{22} & \cdots & -\overline{x}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{x}_{n1} & -\overline{x}_{n2} & \cdots & -\overline{x}_{2n} \end{pmatrix}$$

$$= \begin{pmatrix} \underline{y}_{11} & \underline{y}_{12} & \cdots & \underline{y}_{1n} \\ \underline{y}_{21} & \underline{y}_{22} & \cdots & \underline{y}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \underline{y}_{n1} & \underline{y}_{n2} & \cdots & \underline{y}_{nn} \\ -\overline{y}_{11} & -\overline{y}_{12} & \cdots & -\overline{y}_{2n} \\ -\overline{y}_{21} & -\overline{y}_{22} & \cdots & -\overline{y}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -\overline{y}_{n1} & -\overline{y}_{n2} & \cdots & -\overline{y}_{2n} \end{pmatrix}$$

where $s_{i,j} = \begin{cases} a_{i,j} \geq 0, & s_{i,j} = a_{i,j}, s_{i+n,j+n} = a_{ij} \\ a_{i,j} \leq 0, & s_{i,j+n} = -a_{i,j}, s_{i+n,j} = -a_{ij} \\ elsewhere & 0 \end{cases}$

$S = (s_{i,j}),\ i = 1, 2, \dots, n, n+1, n+2, \dots, 2n$ and $j = 1, 2, \dots, n, n+1, n+2, \dots, 2n$

Matrix S also can be written as $S = \begin{pmatrix} B & C \\ C & B \end{pmatrix}$
where $B =$ positive entries of matrix A
$C =$ absolute value for negative entries of matrix A

Now, $S\tilde{X} = \tilde{Y}$ can be uniquely solve to get solution for $\tilde{X}$ if and only if matrix S is nonsingular as mention in the following theorem

Theorem 1[10]
The matrix S is nonsingular if and only if the matrices $A = B - C$ and $B + C$ are both nonsingular matrix.

Since matrix S is nonsingular based on Theorem 1, then matrix S is invertible. Thus, matrix S have an inverse matrix,S^{-1}.

Theorem 2[10]
If S^{-1} exists it must have the same structure of S.

According to Theorem 2, matrix S^{-1} also can be written as $S^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix}$

where $D = \frac{1}{2}\left[(B + C)^{-1} + (B - C)^{-1}\right]$ (3)

$E = \frac{1}{2}\left[(B + C)^{-1} - (B - C)^{-1}\right]$ (4)

Theorem 3[30]
Suppose that the inverse of matrix A exist and $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$ is a solution of this fuzzy matrix equation system. Then,

$$\tilde{X} = \left(\underline{x}_j(r), \overline{x}_j(r)\right) = (\underline{x}_1(r),\ \overline{x}_1(r), \underline{x}_2(r),\ \overline{x}_2(r), \dots, \underline{x}_n(r),\ \overline{x}_n(r))^T$$

By Definition 12, the system can be written as

$$\tilde{X} = (\overline{x}_1(r) - \underline{x}_1(r), \overline{x}_1(r) + \underline{x}_1(r), \overline{x}_2(r) - \underline{x}_2(r), \overline{x}_2(r) + \underline{x}_2(r), \dots, \overline{x}_n(r) - \underline{x}_n(r), \overline{x}_n(r) + \underline{x}_n(r))^T$$

is the solution of the following system $A\tilde{X} = \tilde{Y}$

$$A\tilde{x}_j = \tilde{y}_j$$
$$A\left(\underline{x}_j(r), \overline{x}_j(r)\right) = \left(\underline{y}_j(r), \overline{y}_j(r)\right)$$

where $\tilde{Y} = \left(\underline{y}_j(r), \overline{y}_j(r)\right) = (\underline{y}_1(r),\ \overline{y}_1(r), \underline{y}_2(r),\ \overline{y}_2(r), \dots, \underline{y}_n(r),\ \overline{y}_n(r))^T$ and by Definition 12, the system can be written as

$$\tilde{Y} = (\overline{y}_1(r) - \underline{y}_1(r), \overline{y}_1(r) + \underline{y}_1(r), \overline{y}_2(r) - \underline{y}_2(r), \overline{y}_2(r) + \underline{y}_2(r), \dots, \overline{y}_n(r) - \underline{y}_n(r), \overline{y}_n(r) + \underline{y}_n(r))^T$$

Definition 13: Fuzzy solution[10]
Let $\tilde{X} = \left(\underline{x}_j(r),\ \overline{x}_j(r)\right),\ j = 1, 2, \dots, n, r \in [0,1]$.
The fuzzy number vector $U = \{\left(\underline{u}_i(r),\ \overline{u}_i(r)\right), i = 1, 2, \dots, n, r \in [0,1]\}$ defined by

$$\underline{u}_i(r) = min\{\ \underline{x}_j(r), \overline{x}_j(r), \underline{x}_j(1), \overline{x}_j(1)\ \}$$

$$\overline{u}_i(r) = max\ \{\ \underline{x}_j(r), \overline{x}_i(r), \underline{x}_j(1), \overline{x}_j(1)\}$$

is called the fuzzy solution of equation (1). Then, if $\tilde{X} = \left(\underline{x}_j(r),\ \overline{x}_j(r)\right)$ are all fuzzy numbers which

$$\underline{u}_i(r) = \underline{x}_j(r),\ \overline{u}_i(r) = \overline{x}_j(r),$$

then U is called a strong fuzzy solution. Otherwise, U is called a weak fuzzy solution.

Definition 14: Strong fuzzy solution[31]
If $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$ is a solution of equation (1) and for each $j = 1, 2, \dots, n$ which hold the inequalities $\underline{x}_j(r) \leq \overline{x}_j(r)$, then the solution, $\tilde{X}$ is called a strong fuzzy solution.

Definition 15: Weak fuzzy solution[31]
If $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)^T$ is a solution of equation (1) and for each $j = 1, 2, \dots, n$ which hold the inequalities $\underline{x}_j(r) \geq \overline{x}_j(r)$, then the solution, $\tilde{X}$ is called a weak fuzzy solution.

3. Result and discussion

In the following gives some example for solving fuzzy matrix equation system that solve by using embedded method and Friedman's proposed technique as follows

Example 1:
For this example, consider 2×2 fuzzy matrix equation system as follows

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} = \begin{pmatrix} (-3, 0, 3) & (-4, 0, 6) \\ (1,\ 3,\ 5) & (0, 3, 7) \end{pmatrix}$$

Solution:
Matrix notation, $A\,\tilde{X} = \tilde{Y}$

$$\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \underline{x}_{11}, \overline{x}_{11} & \underline{x}_{12}, \overline{x}_{12} \\ \underline{x}_{21}, \overline{x}_{21} & \underline{x}_{22}, \overline{x}_{22} \end{pmatrix} = \begin{pmatrix} (3r - 3, 3 - 3r) & (4r - 4, 6 - 6r) \\ (2r+1, 5-2r) & (3r, 7 - 4r) \end{pmatrix}$$

Therefore, the extended 4×4 system can be written in matrix form as below

Matrix notation, $S\tilde{X} = \tilde{Y}$

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \underline{x}_{11} & \underline{x}_{12} \\ \underline{x}_{21} & \underline{x}_{22} \\ -\overline{x}_{11} & -\overline{x}_{12} \\ -\overline{x}_{21} & -\overline{x}_{22} \end{pmatrix} = \begin{pmatrix} 3r-3 & 4r-4 \\ 2r+1 & 3r \\ 3r-3 & 6r-6 \\ 2r-5 & 4r-7 \end{pmatrix}$$

By Theorem 1, find the determinant of S

$$|S| = \begin{vmatrix} 2 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 3 \neq 0 \text{ (nonsingular matrix)}$$

Since S is nonsingular matrix, therefore we have S^{-1}.

$$S^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{4}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} & \frac{4}{3} \end{pmatrix}$$

Check whether $SS^{-1} = I$. After that use S^{-1} to solve fuzzy matrix equation system so that can obtain its solution.

Matrix notation, $\tilde{X} = S^{-1}\,\tilde{Y}$

$$\begin{pmatrix} \underline{x}_{11} & \underline{x}_{12} \\ \underline{x}_{21} & \underline{x}_{22} \\ -\overline{x}_{11} & -\overline{x}_{12} \\ -\overline{x}_{21} & -\overline{x}_{22} \end{pmatrix} = \begin{pmatrix} r & r \\ 1+r & 2r \\ -(2-r) & -(3-2r) \\ -(3-r) & -(4-2r) \end{pmatrix}$$

Here, the conclusion can be made for this example based on Definition.14.

Let $r = 0$, since $\underline{x}_{11} \leq \overline{x}_{11} = 0 \leq 2$, $\underline{x}_{12} \leq \overline{x}_{12} = 1 \leq 3$, $\underline{x}_{21} \leq \overline{x}_{21} = 0 \leq 3$ and $\underline{x}_{22} \leq \overline{x}_{22} = 0 \leq 4$ and satisfy the condition for solution of strong fuzzy.

Thus, $\tilde{x}_{11} = (\underline{x}_{11}, \overline{x}_{11})$, $\tilde{x}_{12} = (\underline{x}_{12}, \overline{x}_{12})$, $\tilde{x}_{21} = (\underline{x}_{21}, \overline{x}_{21})$ and $\tilde{x}_{22} = (\underline{x}_{22}, \overline{x}_{22})$ are a strong fuzzy solution.

Example 2:
For this example, consider 3×3 fuzzy matrix equation system as follows

$$\begin{pmatrix} 2 & 1 & -3 \\ 1 & 6 & 2 \\ 6 & 5 & 0 \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23} \\ \tilde{x}_{31} & \tilde{x}_{32} & \tilde{x}_{33} \end{pmatrix} = \begin{pmatrix} (-92, 25, 113) & (-126, 38, 126) & (-177, 4, 117) \\ (-130, 5, 258) & (60, 264, 515) & (-13, 394, 562) \\ (-121, 80, 304) & (45, 319, 535) & (-48, 425, 604) \end{pmatrix}$$

Solution:
Matrix notation, $A\tilde{X} = \tilde{Y}$

$$\begin{pmatrix} 2 & 1 & -3 \\ 1 & 6 & 2 \\ 6 & 5 & 0 \end{pmatrix} \begin{pmatrix} \underline{x}_{11} & \underline{x}_{12} & \underline{x}_{13} \\ \underline{x}_{21} & \underline{x}_{22} & \underline{x}_{23} \\ \underline{x}_{31} & \underline{x}_{32} & \underline{x}_{33} \\ -\overline{x}_{11} & -\overline{x}_{12} & -\overline{x}_{13} \\ -\overline{x}_{21} & -\overline{x}_{22} & -\overline{x}_{23} \\ -\overline{x}_{31} & -\overline{x}_{32} & -\overline{x}_{33} \end{pmatrix} = \begin{pmatrix} -92+117r & -126+164r & -177+181r \\ -130+135r & 60+204r & -13+407r \\ -121+201r & 45+274r & -48+473r \\ -133+88r & -126+88r & -117+113r \\ -258+253r & -515+251r & -562+168r \\ -304+224r & -535+216r & -604+179r \end{pmatrix}$$

Therefore, the extended 6×6 system can be written in matrix form as below

Matrix notation, $S\tilde{X} = \tilde{Y}$

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 3 \\ 1 & 6 & 2 & 0 & 0 & 0 \\ 6 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 6 & 2 \\ 0 & 0 & 0 & 6 & 5 & 0 \end{pmatrix} \begin{pmatrix} \underline{x}_{11} & \underline{x}_{12} & \underline{x}_{13} \\ \underline{x}_{21} & \underline{x}_{22} & \underline{x}_{23} \\ \underline{x}_{31} & \underline{x}_{32} & \underline{x}_{33} \\ -\overline{x}_{11} & -\overline{x}_{12} & -\overline{x}_{13} \\ -\overline{x}_{21} & -\overline{x}_{22} & -\overline{x}_{23} \\ -\overline{x}_{31} & -\overline{x}_{32} & -\overline{x}_{33} \end{pmatrix} = \begin{pmatrix} -92+117r & -126+164r & -177+181r \\ -130+135r & 60+204r & -13+407r \\ -121+201r & 45+274r & -48+473r \\ -133+88r & -126+88r & -117+113r \\ -258+253r & -515+251r & -562+168r \\ -304+224r & -535+216r & -604+179r \end{pmatrix}$$

By Theorem 1, find the determinant of S

$$|S| = \begin{vmatrix} 2 & 1 & 0 & 0 & 0 & 3 \\ 1 & 6 & 2 & 0 & 0 & 0 \\ 6 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 6 & 2 \\ 0 & 0 & 0 & 6 & 5 & 0 \end{vmatrix} = -8585 \neq 0 \text{ (nonsingular matrix)}$$

Since S is nonsingular matrix, therefore we have S^{-1}.

$$S^{-1} = \begin{pmatrix} -\frac{80}{8585} & -\frac{1395}{8585} & \frac{1690}{8585} & \frac{930}{8585} & \frac{120}{8585} & -\frac{330}{8585} \\ \frac{96}{8585} & \frac{1674}{8585} & -\frac{311}{8585} & -\frac{1116}{8585} & -\frac{144}{8585} & \frac{396}{8585} \\ -\frac{248}{8585} & -\frac{32}{8585} & \frac{88}{8585} & \frac{2883}{8585} & \frac{372}{8585} & -\frac{1023}{8585} \\ \frac{930}{8585} & \frac{120}{8585} & -\frac{330}{8585} & -\frac{80}{8585} & -\frac{1395}{8585} & \frac{1690}{8585} \\ -\frac{1116}{8585} & -\frac{144}{8585} & \frac{396}{8585} & \frac{96}{8585} & \frac{1674}{8585} & -\frac{311}{8585} \\ \frac{2883}{8585} & \frac{372}{8585} & -\frac{1023}{8585} & -\frac{248}{8585} & -\frac{32}{8585} & \frac{88}{8585} \end{pmatrix}$$

Check whether $SS^{-1} = I$. After that use S^{-1} to solve fuzzy matrix equation system so that can obtain its solution.

Matrix notation, $\tilde{X} = S^{-1}\,\tilde{Y}$

$$\begin{pmatrix} \underline{x}_{11} & \underline{x}_{12} & \underline{x}_{13} \\ \underline{x}_{21} & \underline{x}_{22} & \underline{x}_{23} \\ \underline{x}_{31} & \underline{x}_{32} & \underline{x}_{33} \\ -\overline{x}_{11} & -\overline{x}_{12} & -\overline{x}_{13} \\ -\overline{x}_{21} & -\overline{x}_{22} & -\overline{x}_{23} \\ -\overline{x}_{31} & -\overline{x}_{32} & -\overline{x}_{33} \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{11646}{1717} + 21r & 24r & -3 + 33r \\ -\frac{137881}{8585} + 15r & 9 + 26r & -6 + 55r \\ -\frac{11526}{8585} + 12r & 3 + 12r & 13 + 22r \\ -\left(\frac{53056}{1717} - 9r\right) & -(35 - 11r) & -(44 - 14r) \\ -\left(\frac{265444}{8585} - 34r\right) & -(65 - 30r) & -(68 - 19r) \\ -\left(\frac{178372}{8585} - \frac{171499}{8585}r\right) & -\left(\frac{77274}{1717} - \frac{257276}{8585}r\right) & -\left(\frac{472127}{8585} - \frac{171227}{8585}r\right) \end{pmatrix}$$

Here, the conclusion can be made based on Definition 14.

Let $r = 0$, therefore, $\underline{x}_{11} \le \overline{x}_{11} = -\frac{11646}{1717} \le \frac{53056}{1717}, \underline{x}_{12} \ge \overline{x}_{12} = 0 \ge 35, \underline{x}_{13} \ge \overline{x}_{13} = -3 \ge 44,$
$\underline{x}_{21} \le \overline{x}_{21} = -\frac{137881}{8585} \le \frac{265444}{8585}, \underline{x}_{22} \ge \overline{x}_{22} = 9 \ge 65, \underline{x}_{23} \ge \overline{x}_{23} = -6 \ge 68,$
$\underline{x}_{31} \le \overline{x}_{31} = -\frac{11526}{8585} \le \frac{178372}{8585}, \underline{x}_{32} \ge \overline{x}_{32} = 3 \ge \frac{77274}{1717}$ and $\underline{x}_{33} \ge \overline{x}_{33} = 13 \ge \frac{472127}{8585}$.

Since, $\tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{13}, \tilde{x}_{21}, \tilde{x}_{22}, \tilde{x}_{23}, \tilde{x}_{31}, \tilde{x}_{32}$ and $\tilde{x}_{33}$ satisfy the condition for solution of strong fuzzy, then $\tilde{x}_{11} = (\underline{x}_{11}, \overline{x}_{11}), \tilde{x}_{12} = (\underline{x}_{12}, \overline{x}_{12}), \tilde{x}_{13} = (\underline{x}_{13}, \overline{x}_{13}), \tilde{x}_{21} = (\underline{x}_{21}, \overline{x}_{21}), \tilde{x}_{22} = (\underline{x}_{22}, \overline{x}_{22}),$ $\tilde{x}_{23} = (\underline{x}_{23}, \overline{x}_{23}), \tilde{x}_{31} = (\underline{x}_{31}, \overline{x}_{31}), \tilde{x}_{32} = (\underline{x}_{32}, \overline{x}_{32})$ and $\tilde{x}_{33} = (\underline{x}_{33}, \overline{x}_{33})$ are a strong fuzzy solution.

Conclusion

This paper solves a square fuzzy matrix equation system with a crisp coefficients matrix, unknown fuzzy numbers, and arbitrary triangular fuzzy numbers on the right side which are solving by using the embedded method and Friedman's proposed technique. These methods are used to present the solutions of fuzzy system of linear equations and applied those methods to get the solution for fuzzy matrix equation system and determine the types of solutions based on definitions after solve the system.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. INFORMATION AND CONTROL 8 (pp. 338-353). Berkeley, California: Elsevier.

[2] Zadeh, L. A. (1975). The Concept of a Linguistic Variable and its Application to Approximate Reasoning-I. INFORMATION SCIENCES 8 (pp. 199-249). Berkeley, California: Elsevier.

[3] Zadeh, L. A. (1975). The Concept of a Linguistic Variable and its Application to Approximate Reasoning-II. INFORMATION SCIENCES 8 (pp. 301-357). Berkeley, California: Elsevier.

[4] Nahmias, S. (1978). Fuzzy Variables. Fuzzy Sets and Systems 1 (pp. 97-110). Pittsburgh, USA: North-Holland Publishing Company.

[5] Puri, M. L., & Ralescu, D. A. (1983). Differentials of fuzzy functions. Journal of Mathematical Analysis and Applications 91 (pp. 552-558). Berkeley, California: Elsevier.

[6] Goetschel Jr., R., & Voxman, W. (1986). Elementary fuzzy calculus. Fuzzy Sets and Systems 18 (pp. 31-43). Moscow, USA: North-Holland Publishing Company.

[7] Wu, C., & Ming, M. (1991). Embedding problem of fuzzy number space: Part I. Fuzzy Sets and Systems 44 (pp. 33-38). Harbin, China: North-Holland Publishing Company.

[8] Wu, C., & Ming, M. (1992). Embedding problem of fuzzy number space: Part III. Fuzzy Sets and Systems 46 (pp. 281-286). Harbin, China: North-Holland Publishing Company.

[9] Friedman, M., Ming, M., & Kandel, A. (1998). Fuzzy linear system. Fuzzy Sets and Systems 96 (pp. 201-209). Fowler Ave, Tampa, USA: Elsevier

[10] Friedman, M., Ming, M., & Kandel, A. (2000). Fuzzy duality linear system. Fuzzy Sets and Systems 109 (pp. 55-58). Fowler Ave, Tampa, USA: Elsevier

[11] Abbasbandy, S., Otadi, M., & Mosleh, M. (2008). Minimal solution of general dual fuzzy linear systems. **Chaos, Solitons & Fractals 37** (pp. 1113-1124). Tehran, Iran: Elsevier

[12] Ghanbari, M., Allahviranloo, T., & Pedrycz, W. (2022). A straightforward approach for solving dual fuzzy linear systems. Fuzzy Sets and Systems 435 (pp. 89-106). Aliabad Katoul, Iran: Elsevier

[13] Zheng, B., & Wang, K. (2006). General fuzzy linear systems. Applied Mathematics and Computation 181 (pp. 1276-1286). Lanzhou, China: Elsevier

[14] Zheng, B., & Wang, K. (2006). Inconsistent fuzzy linear systems. Applied Mathematics and Computation 181 (pp. 973-981). Lanzhou, China: Elsevier

[15] Ezzati, R. (2011). Solving fuzzy linear systems. Soft computing 15 (pp. 193-197). Karaj, Iran: Springer

[16] Sun, X., & Guo, S. (2019). Solution to General Fuzzy Linear System and Its Necessary and Sufficient Condition. Fuzzy Information and Engineering (pp. 317-327). Liaoning, China: Springer

[17] Ezzati, R.,Khezerloo, S. & Yousefzadeh, A. (2012). Solving Fully Fuzzy Linear System of Equations in General Form. Journal of Fuzzy Set Valued Analysis (pp. 1-11). Karaj, Iran: International Scientific Publications and Consulting Services (ISPACS)

[18] Allahviranloo, T., Salahshour, S, Homayoun-nejad, M., & Baleanu, D. (2013). General Solutions of Fully Fuzzy Linear Systems. Uncertain Dynamical Systems: Analysis and Applications (pp. 1-9). Mobarakeh, Iran: Hindawi Publishing Corporation

[19] Abbasi, F., & Allahviranloo, T. (2022). Solving fully fuzzy linear system: A new solution concept. Information Sciences 589 (pp. 608-635). Amol, Iran: Elsevier

[20] Hooshangian, L., & Hashemzehi, S. (2013). The solution of dual fully fuzzy linear system. 13th Iranian Conference on Fuzzy Systems (IFSC) (pp. 1-5). Tehran, Iran: ResearchGate

[21] Gemawati, S., Nasfianti, L., Mashadi, M., & Hadi, A. (2018). A new method for dual fully fuzzy linear system with trapezoidal fuzzy numbers by QR decomposition. Journal of Physics: Conference Series 1116 (pp. 1-5). Pekanbaru, Riau, Indonesia: IOP publishing

[22] Otadi, M. (2013). A New Method for Solving General Dual Fuzzy Linear Systems. Journal of Mathematical Extension 7 (pp. 63-75). Firoozkooh, Iran: IJMEX. Retrieved from

[23] Zengtai, G., & Xiaobin, G. (2011). Inconsistent fuzzy matrix equations and its fuzzy least squares solutions. Applied Mathematical Modelling 35 (pp. 1456-1469). Lanzhou, China: Elsevier

[24] Rosimah R. & Amidora I. (2023) Proceedings Science and Mathematics 1:1-10

[25] Zumelzu, N., Bedregal, B., Mansilla, E., Bustince, H., & Diaz, R. (2022). Admissible orders on fuzzy numbers. IEEE Transactions on Fuzzy Systems 30 (pp. 4788-4789). Punta Arenas, Chille: IEEE

[26] Bede, B. (2013). Mathematics of Fuzzy Sets and Fuzzy Logic. Redmond, USA: Springer.

[27] Wan Daud, W. S., Ahmad, N., & Malkawi, G.(2018). A new solution of pair matrix equations with arbitrary triangular fuzzy numbers. Turkish Journal of Mathematics 43 (pp. 1196-1197). Perlis, Malaysia: TUBITAK

[28] Das, S., & Chakraverty, S.(2012). Numerical Solution of Interval and Fuzzy System of

Linear Equations. Applications and Applied Mathematics: An International Journal 7 (pp. 336-337). Rourkela, India: AAM

[29] Guo, X., & Shang, D. (2012). Fuzzy Symmetric Solutions of Fuzzy Matrix Equations. Advances in Fuzzy System 2012 (pp. 2-3). Lanzhou, China: Hindawi Publishing Corporation

[30] Allahviranloo, T., & Hashemi, A. (2014). The embedding method to obtain the solution of fuzzy linear systems. International Journal of Industrial Mathematics 6 (pp. 230). Tehran, Iran: IJIM

[31] Otadi, M., & Mosleh, M. (2012). Solution of Fuzzy Matrix Equation System. International Journal of Mathematics and Mathematical Sciences2012 (pp. 5). Firoozkooh, Iran: Hindawi Publishing Corporation

[32] Amrahov, S. E., & Askerzade, I.(2011). Strong Solutions of the Fuzzy Linear Systems. Computer Modelling in Engineering and Sciences 76 (pp. 213-214). Ankara, Turkey: ResearchGate