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Solving One Dimensional Elasticity Problem using Finite Element Method

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Abstract
Finite element method was first established by Richard Courant in 1943 to handle specific boundary-value issues for partial differential equations. Since then, the range of potential uses has grown steadily, and today includes nonlinear solid mechanics, fluid structure interactions, turbulent flows in industrial, multicomponent reactive flows, mass transfer in porous media, elastic flows in medical sciences, electromagnetism used this method to approximate the problems. This project discussed a one-dimensional axial loaded bar solved by finite element methods. In finite element formulation, weak form is particularly useful because it leads to a system of algebraic equations that can be solved using numerical techniques. Weak form obtained from integrating the strong form over the approach was first established by Richard Courant in 1943 to handle specific boundary-value issues for Partial Differential Equation [4]. Then has been developed in 1950s, when engineers began employing numerical tools problem domain. Galerkin methods with linear and quadratic shape functions are used to achieve an accurate approximation of the solution. The stiffness matrix and load vector need to be formulated and solved to obtain the solution for a given problem. This project solved a problem using both shape functions starting with 2 elements. The approximate solution then will be compared to the exact solution by increasing the number of elements and nodes that will show in the graph. As a result, when the number of elements is increasing, the value of displacement for quadratic shape functions will give a better approximation compared to linear shape functions.

Keywords: Finite element method; Axial loaded bar; Galerkin method; linear shape function; quadratic shape function.

Introduction

Finite elements used in various applications in partial differential equations such as science and engineering applications. It is involving equations of solids and fluids mechanics, heat transport, and for the propagation of acoustics and electromagnetic waves [2]. Since then, the range of potential uses has grown steadily, and today includes nonlinear solid mechanics, fluid structure interactions, turbulent flows in industrial, multicomponent reactive flows, mass transfer in porous media, elastic flows in medical sciences, electromagnetism used this method to approximate the problems.

The ability of finite elements to interpolate in the approximation of scalar and vector-valued functions, as well as their capacity is to approximate mathematical models. Two key component that contribute to the efficiency of the finite element method given in terms of PDE within a suitable mathematical framework. FEM is a numerical method for doing FEA of any given mathematical concept Any physical phenomenon, such as the behavior of structures or fluids, heat transfer, wave propagation, or the development of biological cells, must be understood and quantified in its entirety using mathematics. In its current form the FE method was formalized by civil engineers. The method was proposed and formulated previously in different manifestations by mathematicians and physicists [2] . When it first came out, the finite element approach held out a lot of potential for the modelling of many mechanical applications in civil and aerospace engineering.

FEM is one of the simplest and most established techniques for solving differential equations is the use of finite difference approximations. It was initially recognised by L. Euler (1707–1783) around 1768 in one dimension, and C. Runge (1856–1927) possibly extended it to dimension two around 1908. In a finite difference technique, approximate finite difference terms are used in place of the differential equations' derivatives. Instead of the differential equation, a sizable algebraic system of equations that can be solved quickly on a computer is provided. The elasticity equations describe how an elastic material moves under a force. An elastic material is one that returns to its original shape after the force is lifted.

Modelling elasticity is useful in manufacturing applications such as suspension cables and nail bending, and biological applications such as weight on bones and tendons. In this project, we will consider the axially loaded bar in elasticity problem for a linear and quadratic shape function. Finite element equations established on the principle of virtual work, which are entirely equivalent to those of internal and boundary equilibrium, strain–displacement compatibility and constraint conditions [5].

In elasticity problem, it contains equilibrium equations relating to stresses, kinematic equations relating to the strains and displacements and the constitutive equations relating to the stresses and strains. Linear elasticity, generalized Hooke's law and stress-strain relations to form an equations of second-order ordinary differential equations. Hooke's law is a law of elasticity developed by the English scientist Robert Hooke in 1660. He mentioned that the displacement of the deformations of an object is directly proportional to the deforming force or load [1]. The concepts of stress and strain can also be used to describe Hooke's law. Stress is the force acting on a unit area of a material as a consequence of an externally applied force. The relative deformation brought on by tension is known as strain. Stress is proportional to strain at relatively low loads. This can be seen clearly in the application of bulk modulus, shear modulus, and Young's modulus.

$$Stress, \sigma = \frac{F}{A}. \tag{1}$$

$$Strain, \varepsilon = \frac{\delta l}{L}. \tag{2}$$

where ε is the strain due to the stress applied, δl is the change in length and L is the original length of the material. The greater the stress, the greater the strain and the proportionality constant in this relation is called the Young modulus [1].
Below is the relationship between stress, strain and Young Modulus:

$$stress = (Young\ Modulus) \times strain. \tag{3}$$

FEM is the most suitable method since the problem is about to solving the elasticity problem [4]. The term finite element approach only appeared in 1960, while the concepts behind finite element analysis actually have a considerably longer history when applied to elasticity problems. The work of Courant from 1943 marks the first attempts to employ piecewise continuous functions defined over triangular domains in the literature on applied mathematics [5].Displacement formulation, stress formulation, or mixed formulation methods can all be used to present the governing equations.

The main purpose of this study is to aiming on solving one dimensional axially loaded elastic problem using finite element method. In FEM, the Galerkin Method is a kind of weighted residual method, where a trial function is assumed as solution [9].The function may not satisfy the differential equation and the boundary conditions exactly. By substituting the trial function in the differential equation and boundary condition, an error known as residual will be produced. This leads to a system of linear equations that can be solved to obtain the coefficients and therefore the solution to the PDE. The Galerkin method is a popular choice for solving PDEs in FEM because it is easy to implement, and it is also a good choice for problems with mixed boundary conditions [12]. In this project, we will solve an axially loaded bar using two different type of shape function which is linear and quadratic shape function.

Problem definition and Formulation

One dimensional structural problem of axial loaded bar fixed at one end shown in Figure 1 with unit length and unit cross sectional are. Approximate model and exact analytical solution will be discussed as the result in this project. [9]

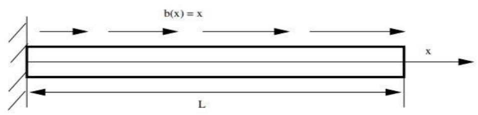


Figure 1 One-dimensional of a bar unit length subject to a linear body force and fixed point $x = 0$

Weak form 1-D bar problem is given as:

$$\int_0^l AE\frac{du}{dx}\frac{dW}{dx}dx = \int_0^l W(x)b(x)dx\,,\ \ 0 \le x \le L \tag{4}$$

with boundary b_0 where $b(x)$ is the parameter that may be functions of coordinate x.The essential and natural boundary conditions are of the form:

$$u|_{x=0} = u_0$$
$$AE\frac{du}{dx}\Big|_{x=L} = 0 \tag{5}$$

In 1-D problems, these boundary regions are the points $x = 0$, and $x = l$. For the problem of axial loaded bar, the primary variable *u* is longitudinal displacement, $b = EA$, where *E* is the elastic modulus and *A* is the cross sectional area. Exact analytical solution for the above problem is given by [8]:

$$u(x) = \frac{a}{6EA}(3L^2x - x^3) \tag{6}$$

Fundamental of Galerkin Weight Residual Method (WRM)

Previous research from [12] mentioned that one dimensional elasticity problem can be solved using Galerkin method because it allows for the efficient and accurate solution by transforming the differential equations into a discrete algebraic system. A numerical method for solving differential equations in the field of finite element analysis is called the Galerkin Weight Residual Method (WRM) where the weight functions satisfy the specified boundary conditions, resulting in a more accurate and efficient numerical solution for one-dimensional elasticity problems [8]. As a result, an algebraic system of equations is created, which can be solved to determine an approximation of the solution. For the approximation to be accurate and to assure convergence, the weight function must meet a number of requirements. Typically, the trial function used to represent the unidentified solution utilized to determine the weight function. The basic concept of WRM is the integration of a function consists of a product of the residual function and a weight function and forcing the integrated value to zero in getting an algebraic function. The guessed solution is expressed in term of shape function, N_i and degree of freedoms**,** u_i instead of interpolation functions where both N_1 and N_2 is the shape function.

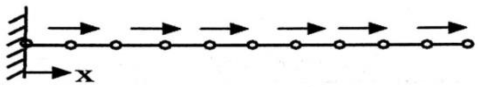


Figure 2 Node generation in a bar

The equivalent representation of a problem statement shown as Table 1:

Table 1: Equivalent representation of problem statement

Differential equation	Weighted residual method	Weak form
$\boldsymbol{AE\frac{d^2u}{dx^2} + b(x) = 0}$ subject to $\boldsymbol{u(0) = 0}$ AE$\boldsymbol{\frac{du}{dx}(L) = 0}$	$\int_0^l w\left(AE\frac{d^2\hat{u}}{dx^2} + b(x)\right)dx = 0$ subject to $u(0) = 0$ AE$\frac{d\hat{u}}{dx}(L) = 0$	$\int_0^l AE\frac{d\hat{u}}{dx}\frac{dW}{dx}dx = \int_0^l W(x)b(x)dx$ subject to $\hat{u}(0) = 0$ $W(0) = 0$

The reason weak form was chosen because the functions need to be approximated by differentiable it once.

Finite element method on one dimensional elasticity problem

There are a few steps of FEM that has being discussed by [10],

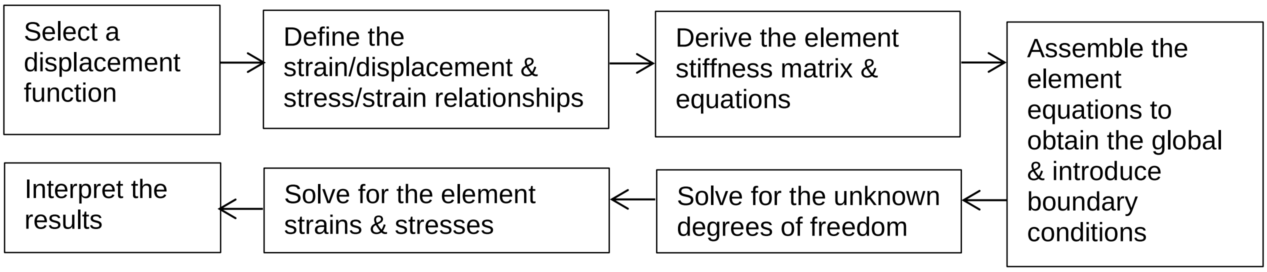


Figure 3 Steps on solving axial loaded bar problem in finite element method

Galerkin method in the FEM is used to approximate the solution of PDEs by discretizing the problem domain and constructing a system of algebraic equations. The global shape function represents the variation of unknown quantity throughout the domain, while the stiffness matrix relates the nodal displacements to internal forces or stresses. The stress and strain are computed from the displacement field and are crucial for analysing the structural behaviour.

Approximate solution using Galerkin method

Consider an elastic bar of variable cross section such as shown in Figure 4. The stiffness matrix and load vector need to be formulated and solved to obtain the solution for a given problem. This project solved a problem using linear and quadratic shape functions with 2 elements.

$(E = 200 \times 10^6, A = 0.04m^2, L = 5m,\ P = 8 \times 10^3 N, q = 2 \times 10^3 m^2)$

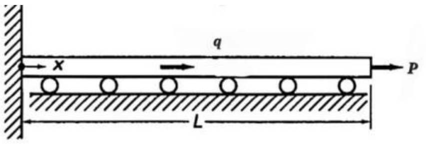


Figure 4 One-dimensional bar

boundary conditions:

$$EA\frac{du(x)}{dx}\bigg|_{x=0} = -F_0 \qquad (7)$$
$$EA\frac{du(x)}{dx}\bigg|_{x=L} = F_L$$

displacement boundary conditions:

$$u|_{x=0} = u_0 \qquad (8)$$
$$u|_{x=L} = u_L$$

One dimensional linear element

Approximate the problem using two linear element and find unknown c_0, c_1 by applying the boundary conditions:

$$u(x) = c_0 + c_1(x) \qquad (9)$$

At $x = 0$,

$$u(0) = u_1 \Rightarrow c_0 = u_1$$

At $x = l$,

$$u(l\,) = u_2 \Rightarrow u(l\,) = c_0 + c_1(l)$$

$$c_1 = \frac{u_2 - u_1}{l}$$

Then the equation becomes

(10)

$$u(x) = u_1 + \left(\frac{u_2 - u_1}{l}\right)(x)$$

$$= u_1\left(1 - \frac{x}{l}\right) + \left(\frac{x}{l}\right)u_2$$

Hence the shape function

$$[N_1 \quad N_2] = \left[\left(1 - \frac{x}{l}\right) \quad \left(\frac{x}{l}\right)\right]. \tag{11}$$

Stiffness matrix,

$$[K^e] = \int_0^l EA[B]^T[B]dx \tag{12}$$

$$[B] = \begin{bmatrix}\frac{dN_1}{dx} & \frac{dN_2}{dx}\end{bmatrix}$$

$$W_1(x) = 1 - \frac{x}{l} \quad , \quad \frac{dW_1}{dx} = -\frac{1}{l}$$

$$W_2(x) = \frac{x}{l} \quad , \quad \frac{dW_2}{dx} = \frac{1}{l}$$

$$\int_0^l AE\frac{d\hat{u}}{dx}\frac{dW_1}{dx}dx = \frac{AE}{L}[1 \quad -1]$$

$$\int_0^l AE\frac{d\hat{u}}{dx}\frac{dW_2}{dx}dx = \frac{AE}{L}[1 \quad -1]$$

Due to the symmetrically, element 1 and element 2 would have the local stiffness matrix

$$[K^e] = \int_0^l EA\begin{bmatrix}\frac{dN_1}{dx} & \frac{dN_2}{dx}\end{bmatrix}^T\begin{bmatrix}\frac{dN_1}{dx} & \frac{dN_2}{dx}\end{bmatrix}dx \tag{13}$$

$$= \int_0^l EA\cdot\frac{1}{l}\begin{bmatrix}-1\\1\end{bmatrix}\cdot\frac{1}{l}[-1 \quad 1]dx$$

$$\frac{EA}{l}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

The component of local load vector, *f* for element 1 and 2 has 2-dimensional vector,

$$\int_0^l W(x)q(x) = \int_0^l q[N]^T dx \tag{14}$$

$$\int_0^l W_1(x)q(x) = \int_0^l W_2(x)q(x) = q\int_0^l \begin{bmatrix}N_1\\N_2\end{bmatrix}dx = \frac{ql}{2}\begin{bmatrix}1\\1\end{bmatrix}$$

Assemble the global stiffness matrix and load vector,

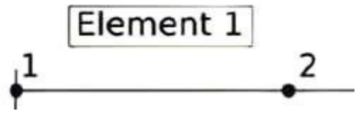


Figure 5 Finite element mesh 1

$$[K^c]\{d^e\} = [f^e_{ext}] + \{F^e_{int}\} \tag{15}$$

$$\frac{AE}{l}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix} = \frac{ql}{2}\begin{Bmatrix}1\\1\end{Bmatrix} + \begin{Bmatrix}-Q_1\\Q_2\end{Bmatrix}$$

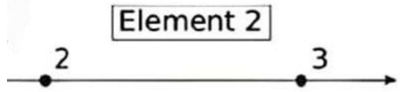


Figure 6 Finite element mesh 2

$$\frac{AE}{l}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\ u_3\end{Bmatrix} = \frac{ql}{2}\begin{Bmatrix}1\\ 1\end{Bmatrix} + \begin{Bmatrix}-Q_2\\ Q_3\end{Bmatrix} \tag{16}$$

$$\frac{AE}{l}\begin{bmatrix}1 & -1 & 0\\ -1 & 1+1 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} = \frac{ql}{2}\begin{bmatrix}1\\ 1+1\\ 1\end{bmatrix} + \begin{bmatrix}Q_0^{(1)}\\ Q_l^{(1)} - Q_l^{(2)}\\ Q_l^2\end{bmatrix}$$

$$\frac{AE}{l}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} = \frac{ql}{2}\begin{bmatrix}1\\ 2\\ 1\end{bmatrix} + \begin{bmatrix}-Q_0^{(1)}\\ 0\\ 0\end{bmatrix}$$

.

Hence the displacement, u and reaction force, Q can be calculated using the equilibrium equation,

$$[K]\{u\} = \{F\}\{Q\} \tag{17}$$

where K stands for the global stiffness matrix, u is the displacement vector, F is the force vector and Q is the reaction force. Therefore, by solving argebraically, we can find the global stiffness matrix.

$$\frac{AE}{l}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\end{bmatrix} = \begin{bmatrix}5\times 10^3\\ 1\times 10^3\\ 5\times 10^3\end{bmatrix} + \begin{bmatrix}-Q_0^{(1)}\\ 0\\ 0\end{bmatrix} \tag{18}$$

$$\frac{AE}{l}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}0\\ u_2\\ u_3\end{bmatrix} = \begin{bmatrix}5\times 10^3\\ 1\times 10^3\\ 5\times 10^3\end{bmatrix} + \begin{bmatrix}-Q_0^{(1)}\\ 0\\ 0\end{bmatrix}$$

$$8\times 10^6\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}0\\ u_2\\ u_3\end{bmatrix} = \begin{bmatrix}5\times 10^3\\ 1\times 10^3\\ 5\times 10^3\end{bmatrix} + \begin{bmatrix}-Q_0^{(1)}\\ 0\\ 0\end{bmatrix}$$

Hence value for $u_2$and u_3,

$$u_1 = 0m, \quad u_2 = 2.0\times 10^3 m \quad u_3 = 3.0\times 10^3 m \tag{19}$$

Thus the reaction force Q,

$$32u_2 = 4 + (Q)$$

$$Q = 1.5\times 10^3\text{N}.$$

From Hooke's Law,

$$\sigma = E\varepsilon \tag{20}$$

where σ is the axial stress, E is the Young Modulus and ε is the axial strain.

$$\sigma = \frac{F}{A} \tag{21}$$

$$\sigma = \frac{1.5\times 10^3}{0.04}$$

$$= 3.75\times 10^3\ N/m^2.$$

$$\varepsilon = \frac{\sigma}{E} = 1.875\times 10^3. \tag{22}$$

To check that the result is correct, we will compare the numerical result against analytical solution of a 1-

D bar problem given by

$$u = \left(\frac{q}{2EA}\right)x^2 + \left(\frac{p+ql}{EA}\right)x \cdot \tag{23}$$

The equation can be written as

$$u = N_1u_1 + N_2u_2 + N_3u_3 \tag{24}$$

with shape function

$$W_1(x) = 1 - \frac{3x}{l} + \frac{2x^2}{l^2} - \frac{x}{l}, \frac{dW_1}{dx} = -\frac{3}{l} + \frac{4x}{l^2}$$

$$W_2(x) = \frac{4x}{l} + \frac{4x^2}{l^2}, \frac{dW_2}{dx} = \frac{4}{l} - \frac{8x}{l^2}$$

$$W_3(x) = -\frac{x}{l} + \frac{2x^2}{l^2}, \frac{dW_2}{dx} = -\frac{1}{l} - \frac{4x}{l^2}$$

Then, the local stiffness matrix and the local nodal forces vectors can be established as:

$$N = \langle N_1 \quad N_2 \quad N_3 \rangle$$

$$B = \langle \frac{dN_1}{dx} \quad \frac{dN_2}{dx} \quad \frac{dN_3}{dx} \rangle \tag{25}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \cdot$$

Hence

$$[B] = \left[-\frac{3}{l} + \frac{4x}{l^2} \quad \frac{4}{l} - \frac{8x}{l^2} \quad -\frac{1}{l} - \frac{4x}{l^2}\right] \cdot \tag{26}$$

If $E$is Young's Modulus and A is the cross sectional area of the element, the local stiffness matrix of the element is a 3×3 matrix and has the form:

$$[K^e] = \int_0^l EA[B]^T[B] \cdot dx \tag{27}$$

$$[K^e] = EA\int_0^l \begin{bmatrix} -\frac{3}{l} + \frac{4x}{l^2} \\ \frac{4}{l} - \frac{8x}{l^2} \\ -\frac{1}{l} - \frac{4x}{l^2} \end{bmatrix} \left[-\frac{3}{l} + \frac{4x}{l^2} \quad \frac{4}{l} - \frac{8x}{l^2} \quad -\frac{1}{l} - \frac{4x}{l^2}\right] \cdot dx$$

$$[K^e] = EA\int_0^l \begin{bmatrix} \left(-\frac{3}{l} + \frac{4x}{l^2}\right)\left(-\frac{3}{l} + \frac{4x}{l^2}\right) & \left(-\frac{3}{l} + \frac{4x}{l^2}\right)\left(\frac{4}{l} - \frac{8x}{l^2}\right) & \left(-\frac{3}{l} + \frac{4x}{l^2}\right)\left(-\frac{1}{l} - \frac{4x}{l^2}\right) \\ \left(\frac{4}{l} - \frac{8x}{l^2}\right)\left(-\frac{3}{l} + \frac{4x}{l^2}\right) & \left(\frac{4}{l} - \frac{8x}{l^2}\right)\left(\frac{4}{l} - \frac{8x}{l^2}\right) & \left(\frac{4}{l} - \frac{8x}{l^2}\right)\left(-\frac{1}{l} - \frac{4x}{l^2}\right) \\ \left(-\frac{1}{l} - \frac{4x}{l^2}\right)\left(-\frac{3}{l} + \frac{4x}{l^2}\right) & \left(-\frac{1}{l} - \frac{4x}{l^2}\right)\left(\frac{4}{l} - \frac{8x}{l^2}\right) & \left(-\frac{1}{l} - \frac{4x}{l^2}\right)\left(-\frac{1}{l} - \frac{4x}{l^2}\right) \end{bmatrix}$$

$K_{11} = EA\int_0^l \left(-\frac{3}{l} + \frac{4x}{l^2}\right)\left(-\frac{3}{l} + \frac{4x}{l^2}\right)dx$ $= EA\left[\frac{9}{l} - \frac{12}{l} + \frac{16}{3l}\right]$ $= \frac{EA}{3l}[7]$	$K_{13} = EA\int_0^l \left(-\frac{3}{l} + \frac{4x}{l^2}\right)\left(-\frac{1}{l} - \frac{4x}{l^2}\right)dx$ $= EA\left[\frac{3}{l} - \frac{8}{l} + \frac{16}{3l}\right]$ $= \frac{EA}{3l}[1] = K_{31}$	$K_{23} = EA\int_0^l \left(\frac{4}{l} - \frac{8x}{l^2}\right)\left(-\frac{1}{l} - \frac{4x}{l^2}\right)dx$ $= EA\left[-\frac{4}{l} + \frac{12}{l} - \frac{32}{3l}\right]$ $= \frac{EA}{3l}[-8] = K_{32}$
$K_{12} = EA\int_0^l \left(-\frac{3}{l} + \frac{4x}{l^2}\right)\left(\frac{4}{l} - \frac{8x}{l^2}\right)dx$ $= EA\left[-\frac{12}{l} + \frac{20}{l} - \frac{32}{3l}\right]$ $= \frac{EA}{3l}[-8] = K_{21}$	$K_{22} = EA\int_0^l \left(\frac{4}{l} - \frac{8x}{l^2}\right)\left(-\frac{3}{l} + \frac{4x}{l^2}\right)dx$ $= EA\left[\frac{16x}{l^2} - \frac{64x^2}{2l^3} + \frac{164x^3}{3l^4}\right]_0^l$ $= \frac{EA}{3l}[16]$	$K_{33} = EA\int_0^l \left(-\frac{1}{l} - \frac{4x}{l^2}\right)\left(-\frac{1}{l} - \frac{4x}{l^2}\right)dx$ $= EA\left[-\frac{1}{l} - \frac{4}{l} + \frac{16}{3l}\right]$ $= \frac{EA}{3l}[7]$

If *E* and *A* constant, the K has the form:

$$[K^e] = \frac{EA}{3l}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix} \tag{28}$$

Similarly from previous step in linear case, the nodal forces vector is a three dimensional vector has the form:

$$\int_0^l W(x)q\,(x) = q[N]dx \tag{29}$$

$$= q\int_0^l \begin{bmatrix} 1 - \frac{3x}{l} + \frac{2x^2}{l^2} \\ \frac{4x}{l} + \frac{4x^2}{l^2} \\ -\frac{x}{l} + \frac{2x^2}{l^2} \end{bmatrix} dx$$

$$\int_0^l W_1(x)q\,(x) = \int_0^l W_2(x)q\,(x) = \int_0^l W_3(x)q\,(x) = ql\begin{bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{bmatrix}$$

Therefore,

$$\frac{AE}{L}\begin{bmatrix} 8 & -9 & 1 \\ -9 & 18 & -9 \\ 1 & -9 & 8 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = ql\begin{bmatrix} 2/3 \\ 5/3 \\ 2/3 \end{bmatrix}. \tag{30}$$

Assemble the global and stiffness matrix and load vector gives

$$\frac{AE}{L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = ql\begin{bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{bmatrix} + \begin{bmatrix} -Q_0^{(1)} \\ 0 \\ 0 \end{bmatrix} \tag{31}$$

To solve this system of equations for linear shape function, we have to apply the essential boundary condition *u=0* at *x=0.* This is equivalent to set $u_1 = 0$.

$$1.6\times 10^6\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = ql\begin{bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{bmatrix} + \begin{bmatrix} -Q_0^{(1)} \\ 0 \\ 0 \end{bmatrix}$$

$$1.6\times 10^6\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}\begin{bmatrix} 0 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1.6\times 10^3 + (-Q) \\ 6.6\times 10^3 \\ 1.6\times 10^3 \end{bmatrix}$$

Then solving for u_2 and u_3,

$$u_1 = 0m, \quad u_2 = 4\times 10^3 m \quad u_3 = 6\times 10^3 m \tag{32}$$

Since $u_1 = 0$ it is clear that the displacement field within an element *e* is given by

Thus the reaction force Q,

$$1.28\times 10^7 u_2 + 1.6\times 10^6 u_3 = 1.6\times 10^3 + (-Q)$$

$$Q = 6.08\times 10^3\ N.$$

From Hooke's Law,

$$\sigma = E\varepsilon \tag{33}$$

where σ is the axial stress, *E* is the Young Modulus and ε is the axial strain.

$$\sigma = \frac{F}{A} \tag{34}$$

$$\sigma = \frac{6.08 \times 10^3}{0.04}$$

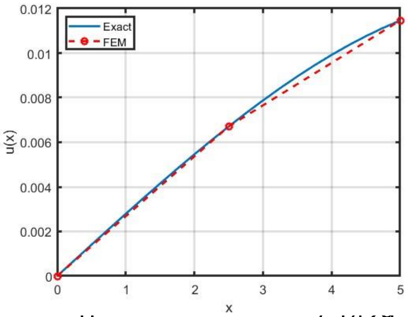
$$= 1.52 \times 10^3\ N/m^2.$$

Therefore the strain,

$$\varepsilon = \frac{\sigma}{E} = 7.6 \times 10^8.$$

Results and discussion

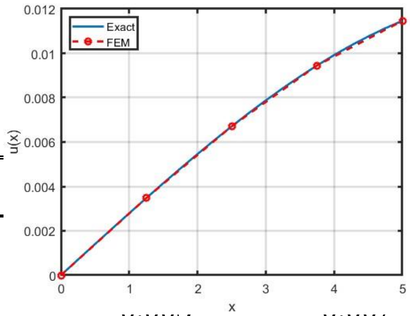
Below is the result of the graph for both shape function.



x	[illegible]	[illegible]	5.0 m
u_{linear}	[illegible]	[illegible]	0.011
u_{exact}	[illegible]	[illegible]	0.011
$\text{Error}_{\text{linear}}$ (%)	0	7.1429	0

Figure 7 Comparison between linear and quadratic approximation and exact result (3-nodes)

Table 2: Comparison between linear approximation and exact solution (2-element, 3-node)



x	0.0 m	[illegible]	[illegible]	3.75 m	5.0 m
$u_{\text{quadratic}}$	0	[illegible]	[illegible]	0.009	0.011
u_{exact}	0	[illegible]	[illegible]	0.009	0.011
$\text{Error}_{\text{quadratic}}$(%)	0	0	0	0	0

Figure 8 Comparison between linear and quadratic approximation and exact result (5-nodes)

Table 3: Comparison between quadratic approximation and exact solution (4-element, 5-node)

x	0.0 m		4.0 m	5.0 m
$u_{\text{quadratic}}$	0	8	0.010	0.011
u_{exact}	0	8	0.010	0.011
$\text{Error}_{\text{quadratic}}$(%)	0		0	0

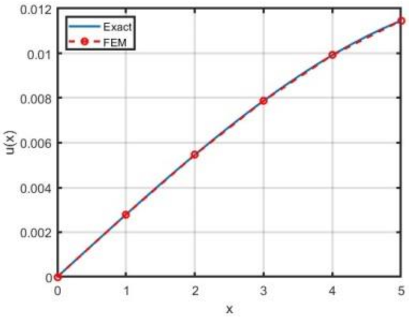


Figure 9 Comparison between linear and quadratic approximation and exact result (6-nodes)

Table 4: Comparison between quadratic approximation and exact solution (5-element, 6-node)

Conclusion

This paper discuss on solving 1-D elasticity problem using Finite Element Method. Weak form is particularly used because it leads to a system of algebraic equations that can be solved using numerical techniques. Weak form obtained from integrating the strong form over the problem domain. The choice between linear and quadratic shape functions depends on the desired level of accuracy.

By increasing the number of elements and nodes, a finer discretization of the bar can be achieved. The additional nodes provide more data points capture the variations on the displacement more accurately. It is also implies reducing the size where smaller elements allow for a more localized approximation of the displacement field within each element. This helps to capture local variations and gradients more precisely .Use more nodes enable the use of higher order polynomials. Higher-order shape functions provide a better approximation of curvature leads to more accurate solutions.

As the mesh is refined, the numerical solution approaches the exact solution more closely. This convergence is essential for obtaining reliable and accurate results.

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