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Prime Power Noncoprime Graph of Noabelian Metabelian of Order at Most 24

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Abstract Graph theory is a branch of mathematics related to the study of graphs. A graph consists of a set of vertices, also known as nodes, and a set of edges, known as arcs or lines, that connect pairs of vertices thru some specific conditions. The study of graph theory originated from Cayley graphs and since then many researches have been that relates graphs and groups. One of the purposes is to investigate the properties of groups thru some graphs. This research focuses on prime-power noncoprime graphs of a group. It is defined as a graph that consists a vertex set that come from the nontrivial elements of the group and two vertices are connected if and only if the greatest common divisor of the order of those two elements is a prime power. The graph is constructed for nonabelian metabelian groups of order at most 24 where a metabelian group is a group G that contains at least a normal subgroup H, such that H and G/H are both abelian. It is obtained that, the constructed graphs of those groups can be categorized into certain type of graphs or can be presented as a graph containing a certain types of subgraphs.

Keywords: prime power noncoprime graph; prime power; nonabelian metabelian; order at most 24

Introduction

Graph theory is a branch of mathematics that deals with the study of graphs. Graph theory provides a framework for analyzing and understanding the properties and characteristics of networks. Graphs are used to represent a wide range of real-world situations and abstract concepts. In this research, the focus is given on nonabelian metabelian group. Metabelian group is a group G having an abelian normal subgroup, H such that the factor group of G and H is also abelian.

Researchers have utilised several ways to investigate the qualities of a group. One of the techniques found is by defining the graph of groups and investigate their properties in terms of corresponding geometric structure.According to the Devi [1] the relation can be investigated by looking at the relationships between its elements or subgroups. This relationship can be thought of as the corresponding defined graph's vertex adjacency. In 1878 Cayley [2] has invented the study on graph theory. He defined and explained the abstract structure of group generated by a set of generator. Dehn [3] in 1987 contributed to the investigation on the properties of the structure.

One of the graphs of group is a prime graph. A prime graph was introduced by William [4] further continued by Ma et al. [5] where the coprime graph was introduced. Sattanathan and Kala defined a graph with the elements of subgroups as it vertices, where two vertices a and b are adjacent if and only if gcd (|a|, |b|) = 1. Mansoori *et al.* [6] and Aghababei *et al.* [7] continue to study the noncoprime graph. Masriani *et al.* [8] found that integers modulo n for a prime power. Zulkifli and Mohd Ali [9] then discovered prime power noncoprime graph.

Definition 1.1: [9] Prime Power Noncoprime Graph

The prime power noncoprime graph of a group G denoted as $\prod_{noncopr}(G)$ is a graph whose vertices are elements of $G \setminus \{e\}$ and two distinct vertices x and y in G area adjacent if and only if $(|x|,|y|) = p^s$ in which $x, y \in G \setminus \{e\}$, p is prime and $s \in N$.

Throughout the research [9], the element of e is excluded and focus only on dihedral groups, quasi-dihedral groups and generalized quartenion groups only. In this research, prime power noncoprime graphs will be denoted as $\Gamma^{noncopr}(G)$ Also, the connection between vertices, x and y denoted as $x \sim y$. Inspired by the study of prime power noncoprime graph, this study extended to prime power noncoprime graph of nonabelian metabelian of order at most 24.

Thus, the first part of this paper covers research's introduction and the second part will discuss on preliminaries for both groups and graph theory which are essential in this study. Results and conclusion will be discussed in section three and four respectively.

Preliminaries

Some fundamental concepts about groups and graphs theory are stated in this section and will be used throughout the research.

Definition 2.1 [10] Metabelian

A group G having an abelian normal subgroup, H such that G/H is also abelian.

Definition 2.2 [11] Dihedral Groups of Degree *n*

Dihedral Groups, D_n in which orden of D_n is *2n* for every $n \in \square$ as well as $n \geq 3$ is expressed as the set of a regular n-gon. Here the dihedral groups D_n can be represented in the following representation:

$$D_n = \langle a, b : a^n = b^2 = e, ba = a^{-1}b \rangle$$

Definition 2.3 [12] Quasi-dihedral Groups

The quasi-dihedral groups, QD_{2n} of order $2n$ where $n \geq 4$ is generated by two elements a and b. The quasi-dihedral groups, QD_{2n} can be represented as follow:

$$QD_{2n} = \langle a, b : a^{2n-1} = b^2 = e, ba = a^{2n-1}b \rangle$$

Definition 2.4 [13] Generalized Quaternion Groups

The group of generalized quaternions, Q_{4n} of order $4n$ where $n \geq 1$ is generated by two elements a and b. The generalized quaternions, Q_{4n} can be represented as follow:

$$Q_{4n} = \langle a, b : a^n = b^2 = e, a^{2n} = e, b^{-1}ab = a^{-1} \rangle$$

Theorem 2.1 [9] Let G be a dihedral group, D_n where n is prime. Then $\Gamma^{noncopr}(G)$ is a disconnected graph with $K_n \cup K_{n-1}$.

Results and discussion

In this section the prime power noncoprime graph of nonabelian metabelian groups order of at most 24 are presented.

In nonabelian metabelian group of order at most 24, consist dihedral group D_n where *n* is prime. The graph for this group constructed by using Theorem 2.1.

Proposition 3.1 Let *G* be a dihedral group, D_3 where $n = 3$ and *n* is prime. Then $\Gamma^{noncopr}(G)$ is a disconnected with $K_3 \cup K_2$

Proof: Let $G = D_3 = \langle a,b \mid a^n = b^2 = e, bab = a^{-1} \rangle$. For $g \in G$ each elements is either 1, 2 or *n*. By the $\Gamma^{noncopr}(G)$, $(|x|,|y|) = p^s$ where *p* is prime and $s \in N$. So, there are two cases that ned to be

considered.

Case 1: Let $x,y \in G \setminus \{e\}$ and $|x| = |y|$. If the order of *x* and *y* are either 2 or *n*, the $(|x| = |y|) = (2,2) = 2$ and $(|x| = |y|) = (n,n) = n$. Therefore, adjacencies exist between each vertex of the same order. This implies that two components exist through the difference between the order of *x* which is 2 and *n*. Hence, a complete subgraph are formed in each component.

Case 2: Let $x,y \in G \setminus \{e\}$ and $|x| = |y|$. Given that the order of each element of *G* is either 2 or *n*. So, there is no adjacencies between two elements *x* and *y* since $(|x| = |y|) = 1 \neq p^s$.

Hence, it can be concluded that $\Gamma^{noncopr}(G)$ is a disconnected graph with two components since they are separated according to the elements' order in *G* given by 2 or *n*. Also, each component produces a complete graph and it is proven by Case 1. Hence, $\Gamma^{noncopr}(G) = K_3 \cup K_2$.

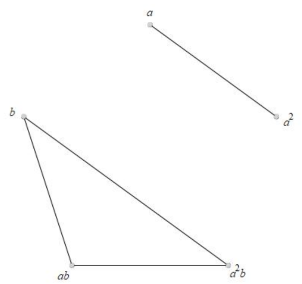


Figure 1 $\Gamma^{noncopr}(D_3)$.

Other than this group, the prime power noncoprime graph constructed using Definition 1.1

Proposition 3.2 Let $G = D_4$ be a nonabelian metabelian group of order 8. Then

$$\Gamma^{noncopr}(G) = K_{2n-1}$$

Proof: Let *G* be a nonabelian metabelian groups of order 8. Then, $V(\Gamma^{noncopr}(G)) = G \setminus \{e\}$, hence $|x| = 2$ or 4.

Let $x \in G$, for $|x| = 2$, then $x \in \{a^2, b, ab, a^2b\}$. For $|x| = 4$ then $x \in \{a, a^3\}$. Let $x,y \in V(\Gamma^{noncopr}(G))$: If $|x| = 2$ and $|y| = 2$ or 4, then $(|x|,|y|) = 2$. Hence, $a^2 \sim b, a^2 \sim ab, a^2 \sim a^2b, a^2 \sim a, a^2 \sim a^3$,

$b \sim ab, b \sim a^2b, b \sim a, b \sim a^3, ab \sim a^2b, ab \sim a, ab \sim a^3$. If $|x| = 4$ and $|y| = 4$ then $(|x|,|y|) = 2^2$. Hence, $a \sim a^3$. Thus, $\Gamma^{noncopr}(G) = K_{2n-1} = K_7$.

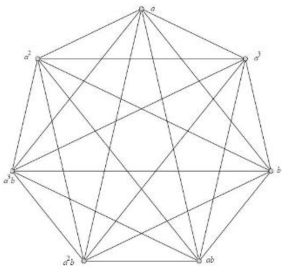


Figure 2 $\Gamma^{noncopr}(D_4)$.

In nonabelian metabelian of order at most 24, the power prime noncoprime graph constructed generally for groups of order 16.

Proposition 3.3 Let *G* be a nonabelian metabelian groups of order 16. Then,

$$\Gamma^{noncopr}(G) = K_{2n-1}$$

Let *G* be a nonabelian metabelian groups of order 16. Then, $V(\Gamma^{noncopr}(G)) = G \setminus \{e\}$, hence $|x| = 2,3,4$ or 8. Since it is multiple of 2, then $(2,2) = 2, (2,4) = 2, (2,8) = 2, (4,4) = 2^2, (4,8) = 2^2$ and $(8,8) = 2^3$. Thus $\Gamma^{noncopr}(G) = K_{2n-1} = K_{15}$.

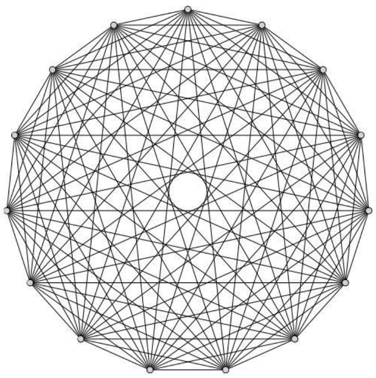
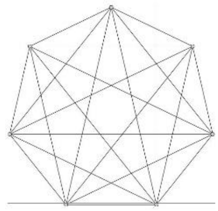
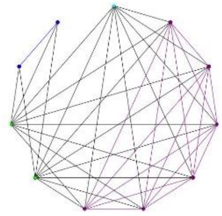
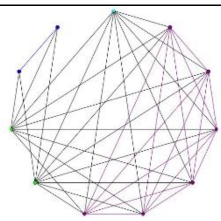
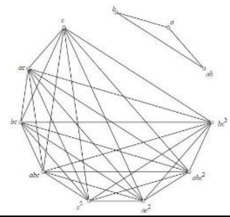
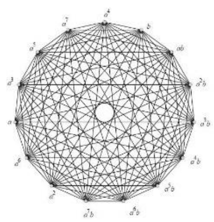


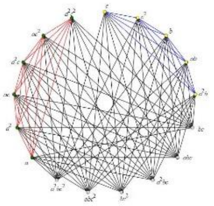
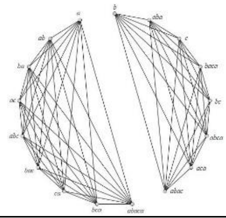
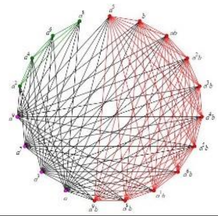
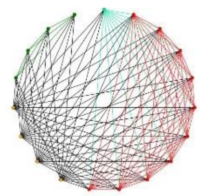
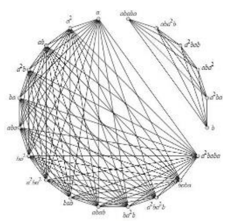
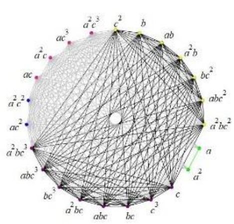
Figure 2 General graph of $\Gamma^{noncopr}(|G| = 16)$.

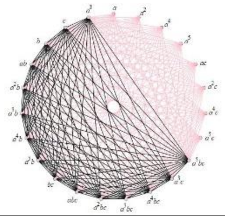
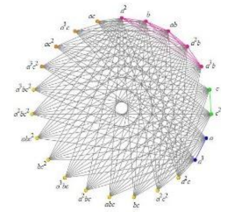
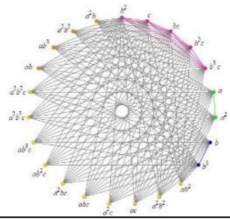
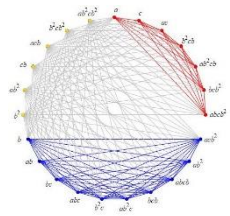
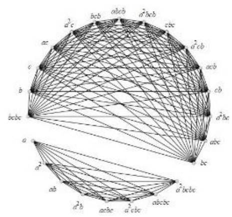
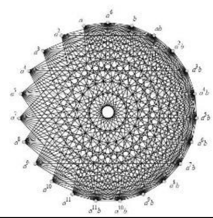
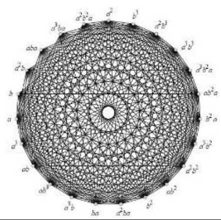
The results of prime power noncoprime graphs are summarized in table 1.

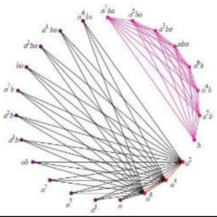
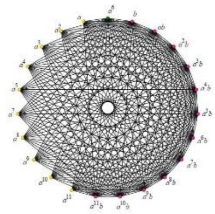
Table 1: Summary of Prime Power Noncoprime Graphs of Nonabelian Metabelian Groups of Order at Most 24.

No	Group	$\|G\|$	Graph
1	D_3	6	$K_3 \cup K_2$

2	D_4	8	
3	Q_4		
4	D_5	10	$K_4 \cup K_5$
5	$\mathbb{Z}_3 \rtimes \mathbb{Z}_4$	12	
6	D_6	12	
7	A_4	12	
8	D_7	14	$K_6 \cup K_7$
9	D_8	16	
10	Quasihedral-16		
11	Q_8		
12	$D_4 \times \mathbb{Z}_2$		
13	$Q \times \mathbb{Z}_2$		
14	Modular-16		
15	B		
16	K		
17	$G_{4,4}$		
18	D_9	18	$K_8 \cup K_9$

19	$S_3 \times \mathbb{Z}_3$		
20	$(\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_2$		
21	D_{10}	20	
22	$Fr_{20} \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$	20	
23	$\mathbb{Z}_4 \rtimes \mathbb{Z}_5$	20	
24	$Fr_{21} \cong \mathbb{Z}_7 \rtimes \mathbb{Z}_5$	21	
25	D_{11}	22	$K_{10} \cup K_{11}$
26	$S_3 \times \mathbb{Z}_4$	24	

27	$S_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	24	
28	$D_4 \times \mathbb{Z}_3$	24	
29	$Q_4 \times \mathbb{Z}_3$	24	
30	$A_4 \times \mathbb{Z}_2$	24	
31	$(\mathbb{Z}_6 \times \mathbb{Z}_2 \rtimes \mathbb{Z}_2)$	24	
32	D_{12}	24	
33	$\mathbf{Z}_2 \times (\mathbf{Z}_3 \rtimes \mathbf{Z}_4)$	24	

34	$\mathbf{Z}_3$ ã $\mathbf{Z}_8$	24	
35	$\mathbf{Z}_3$ ã Q_4	24	

Conclusion

In nonabelian metabelian groups of other at most 24, there are dihedral groups. Prime power noncoprime graph of dihedral group with order 2*n* where *n* is prime are constructed by using Theorem 2.1. The other groups, their prime power noncoprime graphs are constructed by using Definition 1.1.

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